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BOUNDED CONVERGENCE THEOREM AND INTEGRAL OPERATOR FOR OPERATOR VALUED MEASURES

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1. INTRODUCTION

Let $\mathcal{P}_0$ be a δ-ring of subsets of a nonempty set $\boldsymbol{\Omega}$. Let X and Y be Banach spaces and $L(X,Y)$ the Banach space of all bounded linear operators from X to Y.

A set function $m\colon \mathcal{P}_0 \to L(X,Y)$ is called an *operator valued measure countably additive in the strong operator topology* if for every $x \in X$ the set function $E \to m(E)x$ is a countably additive vector measure.

From now on, m will denote an operator valued measure countably additive in the strong operator topology.

We denote by $\mathfrak{S}(\mathcal{P}_0)$ the smallest σ-ring containing $\mathcal{P}_0$. By a $\mathcal{P}_0$-*simple* function on $\boldsymbol{\Omega}$ with values in X we mean a function of the form

$$f = \sum_{i=1}^{r} x_i \chi_{E_i}$$

where $x_i \in X$, $E_i \in \mathcal{P}_0$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, $i,j = 1,2,\ldots,r$. Its integral is defined in the standard way.

For a function $f\colon \boldsymbol{\Omega} \to X$ and a set $A \subset \boldsymbol{\Omega}$, put

$$\|f\|_A = \sup_{x\in A} |f(t)|,$$

where $|f(t)|$ denotes the norm of $f(t)$. By $\mathfrak{B}(\boldsymbol{\Omega}, X)$ we mean the Banach space of all bounded functions $f\colon \boldsymbol{\Omega} \to X$ with the supremum norm.

For each $E \in \mathfrak{S}(\mathcal{P}_0)$, the *semivariation* $\hat{m}(E)$ of the measure m is defined by

$$\hat{m}(E) = \sup \left| \sum_{i=1}^{n} m(E_i)x_i \right|$$

where the supremum is taken over all finite and measurable partitions of $E \in \mathfrak{S}(\mathcal{P}_0)$ and all finite families $\{x_i\}_{i=1}^n \subset X$ with $\|x_i\| \leqslant 1$ for $i = 1, 2, \ldots, n$. From the definition, we note that $\hat{m}$ is monotone and countably subadditive.

For a δ-ring $\mathcal{P}_0$, $\mathcal{P}_1$ will denote the class of those sets from $\mathfrak{S}(\mathcal{P}_0)$ which have finite semivariation. Put $\mathcal{P} = \mathcal{P}_0 \cap \mathcal{P}_1$.

Elements of $\mathcal{P}$ will be called *integrable* sets. A $\mathcal{P}$-simple integrable function on $\mathbf{\Omega}$ with values in X will be called a *simple integrable* function. The set of all simple integrable functions will be denoted by $\mathfrak{T}_s$.

A function $f \colon \mathbf{\Omega} \to X$ is called *measurable* if there is a sequence of simple integrable functions (f_n) such that $\lim\limits_{n\to\infty} f_n(t) = f(t)$ for each $t \in \mathbf{\Omega}$. A measurable function $f\colon \mathbf{\Omega} \to X$ is called *integrable* if there exists a sequence of simple integrable functions (f_n) converging m-almost everywhere to f for which the integrals $\int_{\cdot} f_n \,\mathrm{d}m$, $n = 1, 2, \ldots$ are uniformly countably additive on $\mathfrak{S}(\mathcal{P})$. In that case, the integral of the function f on the set $A \in \mathfrak{S}(\mathcal{P})$ is defined by

$$\int_A f \,\mathrm{d}m = \lim_{n\to\infty} \int_A f_n \,\mathrm{d}m.$$

It was shown in [2, Theorem 16] that if there exists a sequence of integrable functions (f_n) which converges m-almost everywhere to f and the limit $\lim\limits_{n\to\infty} \int_A f_n \,\mathrm{d}m \in Y$ exists for each $A \in \mathfrak{S}(\mathcal{P})$, then f is integrable and

$$\int_A f \,\mathrm{d}m = \lim_{n\to\infty} \int_A f_n \,\mathrm{d}m.$$

This integral, called the Dobrakov integral, was introduced by I. Dobrakov in [2].

For a measurable function g and $E \in \mathfrak{S}(\mathcal{P})$, the L_1-norm $\hat{m}(g, E)$ of g on E is a nonnegative not necessarily finite number defined by

$$\hat{m}(g, E) = \sup\left\{\left|\int_E f \,\mathrm{d}m\right| : f \in \mathfrak{T}_s, |f(t)| \leqslant |g(t)| \text{ for each } t \in E\right\}.$$

The L_1-norm of the function g is defined by

$$\hat{m}(g, \mathbf{\Omega}) = \sup_{E \in \mathfrak{S}(\mathcal{P})} \hat{m}(g, E).$$

All terms not defined in this paper can be found in [2], [3] and [4].

In this paper, we prove the bounded convergence theorem for the Dobrakov integral, and we study the operator on $\mathfrak{B}(\mathbf{\Omega})$ represented by the Dobrakov integral, where $\mathfrak{B}(\mathbf{\Omega})$ is the space of all bounded measurable scalar valued functions with the usual supremum norm on $\mathbf{\Omega}$.

We start with an analogue of Bartle's Bounded Convergence Theorem [1, Theorem II.4.1].

Theorem 2.1. *Let (f_n) be a bounded sequence of integrable functions in $\mathfrak{B}(\Omega, X)$ which converges m- almost everywhere to a measurable function f. Let $F = \bigcup_{n=0}^{\infty} \{t \in \Omega \colon |f_n(t)| > 0\}$, where $f_0 = f$. Suppose that for each $\varepsilon > 0$ there exists a set $E \in \mathcal{P}$ with $\hat{m}(F - E) < \varepsilon$ such that (f_n) converges uniformly to f on E. Then f is integrable and $\int_A f \,\mathrm{d}m = \lim_{n\to\infty} \int_A f_n \,\mathrm{d}m$ for each $A \in \mathfrak{S}(\mathcal{P})$.*

Proof. Suppose $\|f_n\|_\Omega \leqslant K$ for all n. Let $\varepsilon > 0$ be given. Then there exists a set $E \in \mathcal{P}$ with $\hat{m}(F - E) < \varepsilon$ such that (f_n) converges uniformly to f on E.

For each $A \in \mathfrak{S}(\mathcal{P})$, we have

$$\begin{aligned}
&\overline{\lim_{n,p}} \left| \int_A f_n \,\mathrm{d}m - \int_A f_p \,\mathrm{d}m \right| = \overline{\lim_{n,p}} \left| \int_{A\cap F} (f_n - f_p) \,\mathrm{d}m \right| \\
&\leqslant \overline{\lim_{n,p}} \left\{ \left| \int_{A\cap(F-E)} (f_n - f_p) \,\mathrm{d}m \right| + \left| \int_{A\cap F\cap E} (f_n - f) \,\mathrm{d}m \right| \right. \\
&\quad \left. + \left| \int_{A\cap F\cap E} (f - f_p) \,\mathrm{d}m \right| \right\} \\
&\leqslant 2K\hat{m}(A \cap (F - E)) + \overline{\lim_{n}} \|f_n - f\|_E \hat{m}(E) + \overline{\lim_{p}} \|f - f_p\|_E \hat{m}(E) \\
&\leqslant 2K\hat{m}(F - E) \\
&< 2K\varepsilon.
\end{aligned}$$

Thus the limit $\lim_{n\to\infty} \int_A f_n \,\mathrm{d}m \in Y$ exists. By [2, Theorem 6], f is integrable and $\int_A f \,\mathrm{d}m = \lim_{n\to\infty} \int_A f_n \,\mathrm{d}m$ for each $A \in \mathfrak{S}(\mathcal{P})$. □

Corollary 2.2. *Let (f_n) be a bounded sequence of integrable functions in $\mathfrak{B}(\Omega, X)$ which converges m-almost everywhere to a measurable function f. Let $F = \bigcup_{n=0}^{\infty} \{t \in \Omega \colon |f_n(t)| > 0\}$, where $f_0 = f$. Suppose that for each $\varepsilon > 0$ there exists a set $E \in \mathfrak{S}(\mathcal{P})$ with $\hat{m}(F - E) < \varepsilon$ such that (f_n) is a Cauchy sequence in the L_1-norm on E. Then f is integrable and $\int_A f \,\mathrm{d}m = \lim_{n\to\infty} \int_A f_n \,\mathrm{d}m$ for each $A \in \mathfrak{S}(\mathcal{P})$.*

Proof. Let $\varepsilon > 0$ be given. Then there exists a set $E \in \mathfrak{S}(\mathcal{P})$ with $\hat{m}(F-E) < \varepsilon$ such that (f_n) is a Cauchy sequence in the L_1-norm on E. Suppose $\|f\|_\Omega \leqslant K$ for all n. Then the desired result follows immediately from the next relation:

$$\begin{aligned}
&\overline{\lim_{n,p}}\left|\int_A f_n \,\mathrm{d}m - \int_A f_p \,\mathrm{d}m\right| \\
&\leqslant \overline{\lim_{n,p}}\left|\int_{A\cap(F-E)} (f_n - f_p)\,\mathrm{d}m\right| + \overline{\lim_{n,p}}\left|\int_{A\cap F\cap E} (f_n - f_p)\,\mathrm{d}m\right| \\
&\leqslant 2K\hat{m}(A\cap(F-E)) + \overline{\lim_{n,p}}\hat{m}(f_n - f_p, A\cap F\cap E) \\
&\leqslant 2K\hat{m}(F-E) + \overline{\lim_{n,p}}\hat{m}(f_n - f_p, E) \\
&< 2K\varepsilon
\end{aligned}$$

for each $A \in \mathfrak{S}(\mathcal{P})$. □

Corollary 2.3. *Let (f_n) be a bounded sequence of integrable functions in $\mathfrak{B}(\boldsymbol{\Omega}, X)$ which converges m-almost everywhere to a measurable function f. If $\hat{m}$ is continuous on $\mathfrak{S}(\mathcal{P})$ (i.e., if $E_n \in \mathfrak{S}(\mathcal{P})$, $E_n \searrow \emptyset$, $n = 1, 2, \ldots$, then $\lim_{n\to\infty} \hat{m}(E_n) = 0$), then f is integrable and $\int_A f \,\mathrm{d}m = \lim_{n\to\infty} \int_A f_n \,\mathrm{d}m$ for each $A \in \mathfrak{S}(\mathcal{P})$.*

Proof. Let $F = \bigcup_{n=0}^{\infty} \{t \in \boldsymbol{\Omega} \colon |f_n(t)| > 0\}$, where $f_0 = f$. Then $F \in \mathfrak{S}(\mathcal{P})$. Let $\hat{m}$ be continuous on $\mathfrak{S}(\mathcal{P})$. Then the measure m is countably additive in the uniform operator topology on $\mathfrak{S}(\mathcal{P})$ [3, Proof of Lemma 2]. By Egoroff-Lusin's Theorem [2], there is a set $N \in \mathfrak{S}(\mathcal{P})$ and a nondecreasing sequence of sets $F_k \in \mathcal{P}$, $k = 1, 2, \ldots$, with $\bigcup_{n=0}^{\infty} F_k = F - N$ such that N is a m-zero set and on each F_k the sequence (f_n) converges uniformly to the function f. Since $\hat{m}$ is continuous on $\mathfrak{S}(\mathcal{P})$, for each $\varepsilon > 0$ we can select F_k such that $\hat{m}(F - F_k) < \varepsilon$. The desired result now follows immediately from Theorem 2.1. □

3. Operator on $\mathfrak{B}(\boldsymbol{\Omega})$

By $\mathfrak{L}_1\mathfrak{M}(m)$ or $\mathfrak{L}_1\mathfrak{T}(m)$ we denote the set of all measurable or integrable functions g, respectively, with $\hat{m}(g, \boldsymbol{\Omega}) < \infty$. By $\mathfrak{L}_1\mathfrak{T}_s(m)$ we denote the closure in the L_1-norm of the set of all simple integrable functions $\mathfrak{T}_s$ in $\mathfrak{L}_1\mathfrak{M}(m)$. By $\mathfrak{L}_1(m)$ we denote the set of all functions $g \in \mathfrak{L}_1\mathfrak{M}(m)$ whose L_1-norms $\hat{m}(g, \cdot)$ are continuous on $\mathfrak{S}(\mathcal{P})$. It is well-known [3, Theorem 4] that

$$\mathfrak{L}_1(m) \subset \mathfrak{L}_1\mathfrak{T}_s(m) \subset \mathfrak{L}_1\mathfrak{T}(m) \subset \mathfrak{L}_1\mathfrak{M}(m).$$

If $f \in \mathfrak{B}(\boldsymbol{\Omega})$ and $g \in \mathfrak{L}_1\mathfrak{T}(m)$, then fg is integrable [2, Theorem 4]. For $g \in \mathfrak{L}_1\mathfrak{T}(m)$ we consider the operator $T\colon \mathfrak{B}(\boldsymbol{\Omega}) \to Y$ defined by $Tf = \int fg\,\mathrm{d}m$. It is easy to show that the operator T is bounded and $\|T\| \leqslant \hat{m}(g, \boldsymbol{\Omega})$.

Theorem 3.1. *Let $g \in \mathfrak{L}_1\mathfrak{T}(m)$ and $F = \{t \in \boldsymbol{\Omega}\colon |g(t)| > 0\}$. Define $T\colon \mathfrak{B}(\boldsymbol{\Omega}) \to Y$ by $Tf = \int fg\,\mathrm{d}m$. Then T is compact if and only if for each $\varepsilon > 0$ there exists $E_\varepsilon \in \mathfrak{S}(P)$ with $\hat{m}(g, F - E_\varepsilon) < \varepsilon$ such that the operator T_ε defined by $T_\varepsilon f = \int_{E_\varepsilon} fg\,\mathrm{d}m$ is compact.*

Proof. Suppose that T is compact. Since g is measurable, $F \in \mathfrak{S}(\mathcal{P})$. By taking $E_\varepsilon = F$ for each $\varepsilon > 0$ it follows that $T_\varepsilon = T$ and T_ε is compact.

To prove the converse, let $\varepsilon > 0$. Then there exists $E_\varepsilon \in \mathfrak{S}(\mathcal{P})$ with $\hat{m}(g, F - E_\varepsilon) < \varepsilon$ such that T_ε is compact.

Let U be the unit ball of $\mathfrak{B}(\boldsymbol{\Omega})$. Then $\{\int_{E_\varepsilon} fg\,\mathrm{d}m\colon f \in U\}$ is relatively compact. For $f \in U$ we have

$$\left|\int_{\boldsymbol{\Omega}-E_\varepsilon} fg\,\mathrm{d}m\right| = \left|\int_{F-E_\varepsilon} fg\,\mathrm{d}m\right| \leqslant \hat{m}(fg, F - E_\varepsilon) \leqslant \hat{m}(g, F - E_\varepsilon) < \varepsilon.$$

It follows easily that

$$\{Tf\colon f \in U\} = \left\{\int_{E_\varepsilon} fg\,\mathrm{d}m + \int_{\Omega-E_\varepsilon} fg\,\mathrm{d}m\colon f \in U\right\}$$

is totally bounded by 2ε-balls. Hence T is compact. □

In paticular, if $g \in \mathfrak{L}_1\mathfrak{T}_s(m)$, then we can prove that the operator T in Theorem 3.1 is compact.

Theorem 3.2. *Let $g \in \mathfrak{L}_1\mathfrak{T}_s(m)$ and let $T\colon \mathfrak{B}(\boldsymbol{\Omega}) \to Y$ be the linear operator defined by $Tf = \int fg\,\mathrm{d}m$. Then T is compact.*

Proof. Since $g \in \mathfrak{L}_1\mathfrak{T}_s(m)$, there exists a sequence (g_n) of simple integrable functions such that (g_n) converges to g in the L_1-norm in $\mathfrak{L}_1\mathfrak{M}(m)$. Define the operator $T_n\colon \mathfrak{B}(\boldsymbol{\Omega}) \to Y$ by $T_n f = \int fg_n\,\mathrm{d}m$. Since each g_n has a finite range, T_n is a finite rank continuous linear operator.

For $f \in \mathfrak{B}(\boldsymbol{\Omega})$, we have

$$|(T - T_n)f| = \left|\int f(g - g_n)\,\mathrm{d}m\right| \leqslant \hat{m}(f(g - g_n), \boldsymbol{\Omega}) \leqslant \|f\|_{\boldsymbol{\Omega}}\hat{m}(g - g_n, \boldsymbol{\Omega}).$$

Hence $\|T - T_n\| \leqslant \hat{m}(g - g_n, \Omega)$. Since (g_n) converges to g in the L_1-norm and each T_n is compact, T is compact. □

Now proceeding like in the proof of Theorem 3.2, we get the following corollary.

Corollary 3.3. *Let g, $g_n \in \mathfrak{L}_1\mathfrak{T}(m)$ $(n = 1, 2, \ldots)$. Let T, $T_n\colon \mathfrak{B}(\mathbf{\Omega}) \to Y$ be operators defined by $Tf = \int fg\,\mathrm{d}m$ and $T_nf = \int fg_n\,\mathrm{d}m$, respectively. If each T_n is compact and g_n converges to g in the L_1-norm, then T is compact.*

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