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TOPOLOGICAL PROPERTIES OF THE SOLUTION SET OF A CLASS OF NONLINEAR EVOLUTIONS INCLUSIONS

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Abstract. In the paper we study the topological structure of the solution set of a class of nonlinear evolution inclusions. First we show that it is nonempty and compact in certain function spaces and that it depends in an upper semicontinuous way on the initial condition. Then by strengthening the hypothesis on the orientor field $F(t,x)$, we are able to show that the solution set is in fact an R_δ-set. Finally some applications to infinite dimensional control systems are also presented.

Keywords: R_δ-set, homotopic, contractible, evolution triple, evolution inclusion, compact embedding, optimal control

MSC 1991: 34G20, 49A20

1. Introduction

A subset of a metric space is an R_δ-set if it is the intersection of a decreasing sequence of nonempty, compact absolute retracts. It was Yorke [15], who first proved that the Cauchy problem $\dot{x}(t) = f\big(t, x(t)\big)$ a.e., $x(0) = x_0$, with a continuous vector field $f\colon T \times \mathbb{R}^n \to \mathbb{R}^n$ has a solution set which is an R_δ-set of $C(T, \mathbb{R}^n)$. His result was subsequently extended to differential inclusions (i.e. multivalued differential equations) by Himmelberg-Van Vleck [6] and DeBlasi-Myjak [2] for differential inclusions in $\mathbb{R}^n$ and by Papageorgiou [11] and Deimling-Rao [3] for differential inclusions in Banach spaces.

The purpose of this paper is to establish such a topological regularity for the solution set of a class of nonlinear evolution inclusions. Evolution inclusions involve unbounded operators, which are precluded by the formulation of Papageorgiou [11] and Deimling-Rao [3]. Hence evolution inclusions model partial differential equations with multivalued terms and play an important role in optimal control and mathematical physics; see Papageorgiou [12] and Chang [1].

2. Preliminaries

Let X, Y be two Hausdorff topological spaces and f, $g\colon X \to Y$. We say that f, g are "homotopic", if there exists $h\colon [0,1] \times X \to Y$ continuous such $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in X$. A function $f\colon X \to Y$ homotopic to a constant map is said to be "null-homotopic".

A Hausdorff topological space C is said to be "contractible", if the identity map $i_C\colon C \to C$ is null-homotopic. So there exists $h\colon [0,1] \times C \to C$ continuous and $x_0 \in C$ such that $h(0,x) = x$ and $h(1,x) = x_0$ for all $x \in C$. It is easy to check that a contractible space is path connected and so a fortiori connected.

A set C in a metric space is said to be an "absolute retract", if it can replace $\mathbb{R}$ in Tietze's extension theorem; i.e. for every metric space Y and closed $A \subseteq Y$, each continuous function $f\colon A \to C$ admits a continuous extension $\hat{f}\colon Y \to C$. Evidently an absolute retract is contractible. Indeed let $Y = [0,1] \times C$, $A = \{0,1\} \times C$ and $f(0,x) = x$, $f(1,x) = x_0$ on C. Thus an R_δ-set is the intersection of compact, contractible sets. Hyman [7] proved that the converse is also true; i.e. if C has such a representation, then it is an R_δ-set. An R_δ-set is therefore nonempty, compact and connected (in fact, also acyclic). But an R_δ-set need not be path connected. Consider the following set:

$$A = \big(\{0\} \times [0,1]\big) \cup \big[\{t,x\}\colon x = \sin 1/t, t \in (0,1)\big].$$

This set is R_δ, but not path connected (there is no path joining (0,0) to the point $(\frac{1}{\pi}, 0)$).

Let (Ω, Σ) be a measurable space and X a separable Banach space. We will be using the following notations:

$$P_{f(c)}(X) = \big\{A \subseteq X\colon \text{ nonempty, closed, (and convex)}\big\}$$
$$\text{and } P_{(\omega)k(c)}(X) = \big\{A \subseteq X\colon \text{ nonempty, (weakly-) compact, (and convex)}\big\}.$$

A multifunction $F\colon \Omega \to P_f(X)$ is said to be measurable, if the $\mathbb{R}_+$-valued function $\omega \to d\big(x, F(\omega)\big) = \inf\big\{\|x - z\|\colon z \in F(\omega)\big\}$ is measurable for every $x \in X$. If there is a σ-finite measure $\mu(\cdot)$ defined on Σ and Σ is μ-complete (or more generally without requiring the presence of $\mu(\cdot)$, when Σ is closed under the Souslin operation), then the above definition of measurability is equivalent to saying that $\operatorname{Gr} F = \big\{(\omega, x) \in$

$\Omega \times X: x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X (graph measurability). For further details we refer to the survey paper of Wagner [14].

Let $F: \Omega \to P_f(X)$ be a measurable multifunction and let $1 \leqslant p \leqslant \infty$. By S_F^p we will denote the set of selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^p(\Omega, X)$; i.e. $S_F^p = \{f \in L^p(\Omega, X): f(\omega) \in F(\omega) \mu - a.e.\}$. This set may be empty. It is easy to check using Aumann's selection theorem (see Wagner [14], theorem 5.10), that $S_F^p \neq \emptyset$ if and only if $\omega \to \inf\{\|x\|: x \in F(\omega)\} \in L_+^p$. Furthermore, S_F^p is closed in $L^p(\Omega, X)$ and it is convex if and only if for μ-almost all $\omega \in \Omega$, $F(\omega)$ is convex.

Let Y, Z be two Hausdorff topological spaces. A multifunction $G: Y \to 2^Z \setminus \{\emptyset\}$ is said to be upper semicontinuous ($u.s.c$), if for all $C \subseteq Z$ nonempty closed, the set $G^-(C) = \{y \in Y: G(y) \cap C \neq \emptyset\}$ is closed in Y. If $G(\cdot)$ is $u.s.c.$ with closed values and Z is regular, then $\operatorname{Gr} G = \{(y, z) \in Y \times Z: z \in G(y)\}$ is closed. The converse is true if $\overline{G(Y)}$ is compact in Z.

On $P_f(X)$, we can define a generalized metric, known in the literature as Hausdorff metric, by setting

$$h(A, B) = \max\left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right]$$

where $d(a, B) = \inf\{\|a - b\|: b \in B\}$ and $d(b, A) = \inf\{\|b - a\|: a \in A\}$. Then $(P_f(X), h)$ is a complete generalized metric space and $P_{fc}(X)$ is a closed subset of it. A multifunction $H: X \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from X into $(P_f(X), h)$.

The mathematical setting of our problem will be the following: Let $T = [0, r]$ and H a separable Hilbert space. Let X be a dense subspace of H carrying the structure of a separable reflexive Banach space, which embeds into H continuously. Identifying H with its dual (pivot space), we have that $X \to H \to X^*$, with all embeddings being continuous and dense. We will also assume that they are compact. Such a triple of spaces is known in the literature as "evolution triple" (see Zeidler [16]; sometimes the name "Gelfand triple" is also used). To have a concrete example in mind, let Z be a bounded domain in $\mathbb{R}^n$ and let $m \in \mathbb{N}$. Set $X = H_0^m(Z)$, $H = L^2(Z)$ and $X^* = H^{-m}(Z)$. From the Sobolev embedding theorem, we have that (X, H, X^*) is an evolution triple with all embeddings being compact. By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of X (resp. of H, X^*). Also by $(\cdot,\cdot)$ we will denote the inner product in H and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X, X^*). The two are compatible in the sense that $\langle\cdot,\cdot\rangle\big|_{X \times H} = (\cdot,\cdot)$. Let

$$W(T) = \{x \in L^2(T, X): \dot{x} \in L^2(T, X^*)\}.$$

In the definition, the derivative is understood in the sense of vector valued distributions. When furnished with the norm $\|x\|_{W(T)} = \left[\|x\|_{L^2(X)}^2 + \|\dot{x}\|_{L^2(X^*)}^2\right]^{1/2}$, the space $W(T)$ becomes a Banach space, which is clearly separable and reflexive. It is well known that $W(T) \to C(T, H)$ continuously (see Zeidler [16], proposition 23.23, p. 422). So every element in $W(T)$ after possible modification on a Lebesgue null set is equal to a continuous function form T into H. In addition, since we assumed that $X \to H$ compactly, we have that $W(T) \to L^2(T, H)$ compactly (see Zeidler [16], p. 450). Note that if X is a Hilbert space (like in the example), then $W(T)$ is a Hilbert space too.

We will be studying the solution set of the following evolution inclusion defined on T and the evolution triple (X, H, X^*):

$$\text{(1)} \qquad \left\{ \begin{array}{c} \dot{x}(t) + A\big(t, x(t)\big) \in F\big(t, x(t)\big) a.e. \\ x(0) = x_0 \end{array} \right\}$$

3. Existence theorem

In the section, we present an existence theorem for Cauchy problem (1). For this, we will need the following hypotheses on the data:

$H(A)$: $A \colon T \times X \to X^*$ is an operator s.t.

(1) $t \to A(t, x)$ is measurable,

(2) $x \to A(t, x)$ is hemicomtinous, monotone (i.e. for all x, y, $z \in X$, the $\mathbb{R}$-valued function $\lambda \to \langle A(t, x + \lambda y), z \rangle$ is continuous on $[0, 1]$ (hemicontinuity) and for all x, $y \in X$, we have $\langle A(t, x) - A(t, y), x - y \rangle \geqslant 0$ (monotonocity)),

(3) $\langle A(t, x), x \rangle \geqslant c\|x\|^2$ a.e. with $c > 0$,

(4) $\|A(t, x)\|_* \leqslant \alpha_1(t) + \beta_1(t)\|x\|$ *a.e.* with $\alpha_1(\cdot) \in L^2_+$, $\beta_1(\cdot) \in L^\infty_+$.

$H(F)$: $F \colon T \times H \to P_{fc}(H)$ is a multifunction s.t.

(1) $t \to F(t, x)$ is a measurable,

(2) $x \to F(t, x)$ has a sequentially closed graph in $H \times H_w$, where H_w denotes the Hilbert space H endowed with the weak topology (i.e. $\operatorname{Gr} F(t, \cdot) = \{(x, y) \in H \times H \colon y \in F(t, x)\}$ is sequentially closed in $H \times H_w$),

(3) $|F(t, x)| = \sup\{|y| \colon y \in F(t, x)\} \leqslant \alpha_2(t) + \beta_2(t)|x|$ *a.e.* with $\alpha_2(\cdot) \in L^2_+$, $\beta_2(\cdot) \in L^\infty_+$.

By a solution of (1), we mean a function $x(\cdot) \in W(T)$ such that

$$\dot{x}(t) + A\big(t, x(t)\big) = f(t) \; a.e., \quad x(0) = x_0 \in H$$

with $f \in L^2(T, H)$, $f(t) \in F\big(t, x(t)\big)$ a.e. (i.e. $f \in S^2_{F(\cdot, x(\cdot))}$). We will denote the solution set of (1) by $S(x_0)$. So $S(x_0) \subseteq W(T)$.

Theorem 3.1. *If hypotheses $H(A)$, $H(F)$ hold and $x_0 \in H$, then $S(x_0)$ is a nonempty and weakly subset of $W(T)$.*

Proof. We will start by obtaining some a priori bounds for the elements in $S(x_0)$. So let $x(\cdot) \in S(x_0)$. Then by definition, we can find $f \in S^2_{F(\cdot, x(\cdot))}$ s.t.

$$\begin{aligned} \dot{x}(t) + A\big(t, x(t)\big) &= f(t) \text{ a.e., } x(0) = x_0 \\ \text{hence } \big\langle \dot{x}(t), x(t)\big\rangle + \big\langle A\big(t, x(t)\big), x(t)\big\rangle &= \big(f(t), x(t)\big) \text{ a.e.} \\ \text{and so } \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|x(t)\big|^2 + c\big\|x(t)\big\|^2 &\leqslant \big|f(t)\big| \cdot \big|x(t)\big| \text{ a.e.} \end{aligned}$$

On the right-hand side, apply Cauchy's inequality with $\varepsilon > 0$ and also note that there exists $\beta > 0$, $|\cdot| \leqslant \beta\|\cdot\|$, since by hypothesis X embeds into H continuously. So we have

$$\big|f(t)\big| \cdot \big|x(t)\big| \leqslant \beta\big|f(t)\big| \cdot \big\|x(t)\big\| \leqslant \frac{\varepsilon\beta}{2}\big|f(t)\big|^2 + \frac{\beta}{2\varepsilon}\big\|x(t)\big\|^2.$$

Let $\varepsilon = \frac{\beta}{2c}$. Then we have:

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|x(t)\big|^2 &\leqslant \frac{\beta^2}{4c}\big|f(t)\big|^2 \leqslant \frac{\beta^2}{4c}\Big(\alpha_2(t) + \beta_2(t)\big|x(t)\big|\Big)^2 \\ &\leqslant \frac{\beta^2\alpha_2(t)^2}{2c} + \frac{\beta^2\beta_2(t)^2}{2c}\big|x(t)\big|^2 \\ \text{hence } \big|x(t)\big|^2 &\leqslant |x_0|^2 + \frac{\beta^2}{c}\|\alpha\|_2^2 + \frac{\beta^2}{c}\|\beta_2\|_\infty^2 \int_0^t \big|x(s)\big|^2\,\mathrm{d}s. \end{aligned}$$

Invoking Gronwall's inequality, we deduce that there exists $M_1 > 0$ *s.t.* for all $x \in S(x_0)$ and all $t \in T$, we have

$$\big|x(t)\big| \leqslant M_1.$$

Using this bound, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big|x(t)\big|^2 + c\big\|x(t)\big\|^2 \leqslant M_1\big|f(t)\big| \text{ a.e.} \\ &\text{hence } 2c\int_0^r \big\|x(t)\big\|^2\,\mathrm{d}t \leqslant |x_0|^2 + 2M_1\int_0^r \big|f(t)\big|\,\mathrm{d}t \\ &\text{and so } 2c\int_0^r \big\|x(t)\big\|^2\,\mathrm{d}t \leqslant |x_0|^2 + 2M_1\int_0^r \alpha_2(t)\,\mathrm{d}t + \|\beta_2\|_\infty M_1^2 r. \end{aligned}$$

Hence there exists $M_2 > 0$ *s.t.* for all $x \in S(x_0)$, we have

$$\|x\|_{L_2(X)} \leqslant M_2. \tag{2}$$

Finally, let $h \in L^2(T, X)$. We have

$$\begin{aligned}\Big|\int_0^r \langle \dot{x}(s), h(s)\rangle \,\mathrm{d}s\Big| &\leqslant \Big|\int_0^r \langle -A(s, x(s)), h(s)\rangle \,\mathrm{d}s\Big| + \Big|\int_0^r (f(s), h(s))\,\mathrm{d}s\Big| \\ &\leqslant \int_0^r \big\|A(s, x(s))\big\|_* \big\|h(s)\big\|\,\mathrm{d}s + \beta\int_0^r \big|f(s)\big|\cdot\big\|h(s)\big\|\,\mathrm{d}s \\ &\leqslant \int_0^r \big(\alpha_1(s) + \beta_1(s)\big\|x(s)\big\| + \beta\alpha_2(s) + \beta\cdot\beta_2(s)M_1\big)\big\|h(s)\big\|\,\mathrm{d}s.\end{aligned}$$

Applying the Cauchy-Schwartz inequality, we get that there exists $M_3 > 0$ s.t.

$$\begin{gathered}\big|((\dot{x}, h))_0\big| \leqslant M_3 \cdot \|h\|_{L^2(X)} \\ \text{thus } \|\dot{x}\|_{L^2(X^*)} \leqslant M_3.\end{gathered} \tag{3}$$

(here $((\dot{x}, h))_0 = \int_0^r \langle \dot{x}(t), h(t)\rangle \,\mathrm{d}t$; i.e. the duality brackets for the pair $\big(L^2(T, X)$, $L^2(T, X^*)\big)\big)$.

From (2) and(3) above, we see that $S(x_0)$ is a bounded subset of $W(T)$, hence it is relatively sequentially w-compact and since $W(T) \to L^2(T, H)$ compactly (see section 2), we get that $S(x_0)$ is relatively compact in $L^2(T, H)$.

Let $p_{M_1}: H \to H$ be the M_1-radial retraction map and consider the new orientor field $\hat{F}(t, x) = F\big(t, p_{M_1}(x)\big)$; i.e.

$$\hat{F}(t, x) = \begin{cases} F(t, x), & \text{if } |x| \leqslant M_1, \\ F\Big(t, \frac{M_1 x}{|x|}\Big), & \text{if } |x| \geqslant M_1. \end{cases}$$

Clearly $t \to \hat{F}(t, x)$ is measurable (see hypothesis $H(F)$ (1)), whole since $p_{M_1}(\cdot)$ is nonexpansive, we can easily check that $\hat{F}(t, \cdot)$ has a graph which is sequentially closed in $H \times H_w$. Finally, note that

$$\big|\hat{F}(t, x)\big| \leqslant \alpha_2(t) + \beta_2(t)M_1 = \varphi(t) \ a.e., \text{ with } \varphi(\cdot) \in L^2_+.$$

Let $V = \big\{h \in L^2(T, H)\colon \big|h(t)\big| \leqslant \varphi(t) \ a.e.\big\}$. We know hat V equipped with the relative weak $L^2(T, H)$-topology is compact, metrizable. Let $h \in V$ and consider the following evolution equation

$$\left\{\begin{aligned} \dot{x}(t) + A\big(t, x(t)\big) &= h(t) \text{ a.e.} \\ x(0) &= x_0 \end{aligned}\right\}.$$

From theorem 30.A, p. 771 of Zeidler [16], we know that the above Cauchy problem has unique solution $p(h)(\cdot) \in W(T)$. We will show that the map $h \to p(h)$ is sequentially weakly continuous from V into $W(T)$. To this end, assume that $h_n \xrightarrow{w} h$

in V and let $x_n = p(h_n)$. From the a priori bounds established earlier in the proof, we have that $\{x_n\}_{n\geqslant 1}$ is relatively sequentially weakly compact in $W(T)$. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W(T)$ and $\dot{x}_n \xrightarrow{w} z$ in $L^2(T, X^*)$. It is easy to see that $z = \dot{x}$. Note that given $u \in L^2(T, X)$, for every $n \geqslant 1$, we have

$$((\dot{x}_n, u))_0 + \big((\hat{A}(x_n), u)\big)_0 = ((h_n, u))_0$$

where $((\cdot,\cdot))_0$ denotes the duality brackets for the pair $\big(L^2(T,X), L^2(T,X^*)\big)$ (recall that $L^2(T,X)^* = L^2(T,X^*)$), and $\hat{A}\colon L^2(T,X) \to L^2(T,X^*)$ is defined by $\hat{A}(x)(\cdot) = A\big(\cdot, x(\cdot)\big)$ (i.e. $\hat{A}(\cdot)$ is the Nemitsky (superposition) operator corresponding to $A(t,x)$). Observe that

$$((\dot{x}_n, u))_0 \to ((\dot{x}, u))_0$$

and $((h_n, u))_0 = (h_n, u)_{L^2(H)} \to (h, u)_{L^2(H)} = ((h_n, u))_0$.

Also for every $n \geqslant 1$, we have

$$\big\langle \dot{x}_n(t), x_n(t) - x(t)\big\rangle + \big\langle A(t, x_n), x_n(t) - x(t)\big\rangle = \big(h_n(t), x_n(t) - x(t)\big) \text{ a.e.}$$

$$\text{so } \big((\hat{A}(x_n), x_n - x)\big)_0 = (h_n, x_n - x)_{L^2(H)} - ((\dot{x}_n, x_n - x))_0.$$

Using the integration by parts rule for elements of $W(T)$ (see Zeidler [16], proposition 23.23 (iv), p. 423), we have

$$((\dot{x}_n, x_n - x))_0 = \frac{1}{2}\big|x_n(b) - x(b)\big|^2 + ((\dot{x}, x_n - x))_0$$

$$\text{hence } \big((\hat{A}(x_n), x_n - x)\big)_0 = (h_n, x_n - x)_{L^2(H)} - \frac{1}{2}\big|x_n(b) - x(b)\big|^2 - ((\dot{x}, x_n - x))_0.$$

But recall that $W(T)$ embeds into $L^2(T,H)$ compactly and into $C(T,H)$ continuously. Hence $(h_n, x_n - x)_{L^2(H)} \to 0$ and $\big|x_n(b) - x(b)\big| \to 0$ as $n \to \infty$. Furthermore, since $x_n \xrightarrow{w} x$ in $W(T)$, $((\dot{x}, x_n - x))_0 \to 0$. Therefore, finally we have

$$\big((\hat{A}(x_n), x_n - x)\big)_0 \to 0 \text{ as } n \to \infty.$$

The operator $\hat{A}(\cdot)$ is clearly monotone and hemicontinuous (since A is) and so it has property $(\underline{M})$ (see Zeidler [16], pp. 583–584). Since by hypothesis $H(A)$ (4) $\big\{\hat{A}(x_n)\big\}_{n\geqslant 1}$ is bounded in $L^2(T, X^*)$, we may assume that $\hat{A}(x_n) \xrightarrow{w} v$ in $L^2(X^*)$. Then because of property $(\underline{M})$, we have that $v = \hat{A}(x)$; i.e.

$$\hat{A}(x_n) \xrightarrow{w} \hat{A}(x) \text{ in } L^2(X^*),$$

$$\text{hence } \big((\hat{A}(x_n), u)\big)_0 \to \big((\hat{A}(x), u)\big)_0,$$

$$\text{so } ((\dot{x}, u))_0 + \big((\hat{A}(x), u)\big)_0 = ((h, u))_0 \text{ for all } u \in L^2(X),$$

$$\text{thus } \dot{x}(t) + A\big(t, x(t)\big) = h(t) \text{ a.e.}, x(0) = x_0$$

$$\text{and finally } x = p(h).$$

Hence, we have established the sequential weak continuity of $p\colon V \to W(T)$.

Next consider the multifunction $R\colon V \to 2^V$ defined by

$$R(h) = S^2_{\hat{F}\left(\cdot,p(h)(\cdot)\right)}.$$

First let us show that $R(\cdot)$ has nonempty values. Indeed let $\{s_n\}_{n\geqslant 1}$ be simple functions *s.t.* $s_n(t) \xrightarrow{s} p(h)(t)$ a.e. in H. Because of the measurability of $\hat{F}(\cdot, x)$, we have that $t \to \hat{F}\big(t, s_n(t)\big)$ is measurable and so through a simple application of Aumann's selection theorem, we can find $g_n \in L^2(T, H)$, $g_n(t) \in \hat{F}\big(t, s_n(t)\big)$ a.e. Note that $\left|g_n(t)\right| \leqslant \varphi(t)$ a.e. and recall that $\varphi(\cdot) \in L^2_+$. So by passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} g$ in $L^2(T, H)$. Using theorem 3.1 of [10], we have

$$g(t) \in \overline{\operatorname{conv}}\, w - \overline{\lim}\{g_n(t)\}_{n\geqslant 1} \subseteq \overline{\operatorname{conv}}\, w - \overline{\lim}\hat{F}\big(t, s_n(t)\big) \text{ a.e.}$$

But since $\hat{F}(t,\cdot)$ has a graph which is sequentially weakly closed in $H \times H_w$, we can easily check that $w - \overline{\lim}\hat{F}\big(t, s_n(t)\big) = \{y \in H\colon y = w - \lim y_{n_k}, y_{n_k} \in \hat{F}\big(t, s_{n_k}(t)\big), n_1 < n_2 < \ldots < n_k < \ldots\} \subseteq F\big(t, p(h)(t)\big)$ a.e. hence $g \in S^2_{\hat{F}\left(\cdot,p(h)(\cdot)\right)}$ and so $R(h) \neq \emptyset$. It is easy to see that in fact, for every $h \in V$, $R(h) \in P_{f_c}(V)$. We claim that $R\colon V \to P_{f_c}(V)$ is *u.s.c.*, when V equipped with the relative weak $L^2(T, H)$-topology, for which it is a compact, metrizable space. Knowing this fact, to establish the upper semicontinuity of $R(\cdot)$, it is enough to show that $\operatorname{Gr} R$ is closed in $V \times V$ equipped with the product weak topology (for which it is compact and metrizable). So let $[h_n, f_n] \in \operatorname{Gr} R$, $[h_n, f_n] \xrightarrow{w\times w} [h, f]$ in $V \times V$. Then $p(h_n) \xrightarrow{w} p(h)$ in $W(T) \Rightarrow p(h_n) \xrightarrow{s} p(h)$ in $L^2(T, H)$ and so by passing to a subsequence if necessary, we may assume that $p(h_n)(t) \xrightarrow{s} p(h)(t)$ a.e. in H. Using as before, theorem 3.1 of [10] and the fact that $\operatorname{Gr}\hat{F}(t,\cdot)$ is sequentially closed in $H \times H_w$, we get

$$f(t) \in \overline{\operatorname{conv}}\, w - \overline{\lim}\hat{F}\big(t, p(h_n)(t)\big) \subseteq \hat{F}\big(t, p(h)(t)\big) \text{ a.e.}$$

hence $f \in R(h)$.

Therefore $\operatorname{Gr} R$ is closed in $V \times V$ with the relative product weak topology and so $R(\cdot)$ is *u.s.c.* as claimed.

So we can apply the Kakutani-KyFan fixed point theorem to get $x \in R(x)$. As in the beginning of the proof, we can get that $\left|x(t)\right| \leqslant M_1 \Rightarrow \hat{F}\big(t, x(t)\big) = F\big(t, x(t)\big) \Rightarrow S(x_0) \neq \emptyset$. Finally, we will show hat $S(x_0)$ is weakly closed in $W(T)$ and since we already know that it is bounded, it is weakly compact. Since in separable, reflexive Banach spaces (as is $W(T)$), bounded sets endowed with the relative weak topology are relatively compact and metrizable, we can work with sequences. So let $\{x_n\}_{n\geqslant 1} \subseteq S(x_0)$ and assume $x_n \xrightarrow{w} x$ in $W(T)$. Then by definition $x_n = p(f_n)$ with $f_n \in$

$S^2_{F(\cdot,x_n(\cdot))}$. From earlier parts of the proof, we have that $|f_n(t)| \leqslant \varphi(t)$ a.e. Hence we may assume that $f_n \xrightarrow{w} f$ in $L^2(T,H)$ thus $p(f_n) \xrightarrow{w} p(f)$ in $W(T)$. So $x = p(f)$; i.e. $S(x_0)$ is weakly closed, hence weakly compact in $W(T)$. □

In fact from the previous proof, we easily get the following continuous dependence result:

Theorem 3.2. *If hypotheses $H(A)$ and $H(F)$ hold, then the multifunction $S\colon H \to P_{wk}(W(T))$ is u.s.c. from H into $W(T)_w$.*

Proof. Let $C \subseteq W(T)$ be a weakly closed set. Let $S^-(C) = \{z \in H\colon S(z) \cap C \neq \emptyset\}$. Let $\{z_n\}_{n\geqslant 1} \subseteq S^-(C)$ and assume that $z_n \xrightarrow{s} z$ in H. Take $x_n \in S(z_n) \cap C$. From the proof of theorem 3.1, we can easily see that $\{x_n\}_{n\geqslant 1}$ is bounded in $W(T)$ and so we may assume that $x_n \xrightarrow{w} x$ in $W(T)$. Then $x \in C$ and as in the proof of theorem 3.1, we can get $x \in S(z)$ hence $x \in S(z) \cap C$ and so $z \in S^-(C) \Rightarrow S(\cdot)$ is *u.s.c.* as claimed. □

Recalling the $W(T)$ embeds into $L^2(T,H)$ compactly, form theorems 3.1 and 3.2 above we get:

Theorem 3.3. *If hypotheses $H(A)$ and $H(F)$ hold, then for every $x_0 \in H$, $S(x_0)$ is a nonempty compact subset of $L^2(T,H)$ and furthermore, the solution multifunction $S : H \to P_k(L^2(T,H))$ is u.s.c.*

If $x_0 \in X$ (smooth initial datum), then from Papageorgiou [9] we know that $S(x_0)$ is compact in $C(T,H)$. So we have:

Theorem 3.4. *If hypotheses $H(A)$ and $H(F)$ hold, then for every $x_0 \in X$, $S(x_0)$ is a nonempty compact subset of $C(T,H)$ and the solution multifunction $S\colon X \to P_k(C(T,H))$ is u.s.c.*

Remark. If for every $x_0 \in H$, $S(x_0)$ is a singleton, then from theorem 3.2 (resp. 3.3 and 3.4), we have that the solution map is continuous form H into $W(T)_w$ (resp. continuous form H into $L^2(T,H)$ and continuous from X into $C(T,H)$). Note that if $F(t,x)$ is single valued and locally Lipschitz in the x-variable, then we can easily check that $S(x_0)$ is a singleton.

In this section, by strengthening our hypothesis on the orientor field $F(t,x)$, we can establish the topological regularity of the solution set $S(x_0)$.

The stronger hypothesis on $F(t,x)$ is the following:

$H(F)_1$: $F\colon T \times H \to P_{fc}(H)$ is a multifunction s.t.

(1) $t \to F(t,x)$ is measurable,

(2) $x \to F(t,x)$ is h-continuous,

(3) $|F(t,x)| = \sup\{|y|\colon y \in F(t,x)\} \leqslant \alpha_2(t) + \beta_2(t)|x|$ *a.e* with α_2, $\beta_2 \in L^\infty_+$.

Theorem 4.1. *If hypotheses $H(A)$, $H(F)_1$ hold and $x_0 \in H$, then $S(x_0)$ is an R_δ-set of $L^2(T,H)$.*

Proof. From Rybinski [13], we know that we can find $f\colon T \times H \to H$ a Caratheodery function (i.e. $f(\cdot,x)$ measurable, $f(t,\cdot)$ continuous) s.t. for all $(t,x) \in T \times H$, $f(t,x) \in F(t,x)$. From the Scorza-Dragoni theorem (see Himmelberg [5], we know that given $\varepsilon > 0$, we can find $C_\varepsilon \subseteq T$ closed *s.t.* $f|_{C_\varepsilon \times H}$ is continuous and $\lambda(T \setminus C_\varepsilon) < \varepsilon$, with $\lambda(\cdot)$ being the Lebesgue measure on T. Also from the proof of theorem 3.1, we know that by considering if necessary $\hat{F}(t,x)$ instead of $F(t,x)$, we may assume that $|\hat{F}(t,x)| \leqslant m$ with $m = \|\alpha_2\|_\infty + \|\beta_2\|_\infty M_1$. Next choose $D_\varepsilon \subseteq T \setminus C_\varepsilon$ closed and set

$$f_1^\varepsilon(t,x) = \begin{cases} f(t,x), & \text{if } t \in C_\varepsilon, \\ 0, & \text{if } t \in D_\varepsilon. \end{cases}$$

Clearly, since $C_\varepsilon \cap D_\varepsilon = \emptyset$, $f_1^\varepsilon(\cdot,\cdot)$ is a continuous map from $(C_\varepsilon \cup D_\varepsilon) \times H$ into H. Apply Dugundji's extension theorem (see Dugundji [4], theorem 6.1, p. 188), and get a function $f_2^\varepsilon\colon T \times H \to H$ continuous *s.t.* $f_2^\varepsilon|_{C_\varepsilon \cup D_\varepsilon} = f_1^\varepsilon$ and $\|f_2^\varepsilon(t,x)\| \leqslant m$ for all $(t,x) \in T \times H$. Now use the Lasota-Yorke [8] approximation result to get $f_3^\varepsilon\colon T \times H \to H$ a locally Lipschitz map s.t. for all $(t,x) \in T \times H$, we have

$$|f_2^\varepsilon(t,x) - f_3^\varepsilon(t,x)| < \varepsilon.$$

Note that we have

$$|f(t,x) - f_3^\varepsilon(t,x)| \leqslant |f(t,x) - f_2^\varepsilon(t,x)| + |f_2^\varepsilon(t,x) - f_3^\varepsilon(t,x)| \leqslant 2m + \varepsilon$$

while for $t \in C_\varepsilon$, since by construction $f(t,x) = f_1^\varepsilon(t,x) = f_2^\varepsilon(t,x)$ on $C_3 \times H$, we have

$$|f(t,x) - f_3^\varepsilon(t,x)| < \varepsilon.$$

Then define $\hat{F}_\varepsilon : T \times H \to P_{fc}(H)$ by

$$\hat{F}_\varepsilon(t,x) = F(t,x) + B(\varepsilon) + \chi_{T\setminus C_\varepsilon}(t)B(2m+\varepsilon)$$

with $B(\varepsilon) = \{y \in H : |y| \leqslant \varepsilon\}$ and $B(2m+\varepsilon) = \{y \in H : |y| \leqslant 2m+\varepsilon\}$. Clearly $\hat{F}_\varepsilon(t,x)$ is a Caratheodory multifunction; i.e. $\hat{F}_\varepsilon(\cdot,x)$ is measurable and $\hat{F}_\varepsilon(t,\cdot)$ is h-continuous. Observe that since on $C_\varepsilon \times H$, $|f(t,x) - f_3^\varepsilon(t,x)| < \varepsilon$, we have $f_3^\varepsilon(t,x) \in \hat{F}_\varepsilon(t,x)$. Also on $(T \setminus C_\varepsilon) \times H$ we have $|f(t,x) - f_3^\varepsilon(t,x)| \leqslant 2m+\varepsilon$ and $f_3^\varepsilon(t,x) \in \hat{F}_\varepsilon(t,x)$.

Now let $\varepsilon_n = \frac{1}{n}$ and set $\hat{F}_n(t,x) = \hat{F}_{\varepsilon_n}(t,x)$ and $C_n = C_{\varepsilon_n}$. We have:

$$\begin{aligned} h(\hat{F}_n(t,x), F(t,x)) &= h(F(t,x) + B(\tfrac{1}{n}) + \chi_{T\setminus C_n}(t)B(2m+\tfrac{1}{n}), F(t,x)) \\ &\leqslant |B(\tfrac{1}{n})| + \chi_{T\setminus C_n}(t)|B(2m+\tfrac{1}{n})| \leqslant \tfrac{1}{n} + \chi_{T\setminus C_n}(t)(2m+\tfrac{1}{n}), \end{aligned}$$

hence $\theta_n(t) = \sup_{x\in H} h(\hat{F}_n(t,x), F(t,x)) \leqslant \frac{1}{n} + \chi_{T\setminus C_n}(t)(2m+\frac{1}{n}) = \eta_n(t)$.

Recall that $\lambda(T\setminus C_n) < \frac{1}{n}$. So $\eta_n \xrightarrow{\lambda} 0$ (here $\xrightarrow{\lambda}$ denotes convergence in the Lebesgue measure). Note that $\theta_n(\cdot)$ is measurable, since $x \to h(\hat{F}_n(t,x), F(t,x))$ is continuous and so the supremum over H is the same as the supremum over a countable dense subset of H. Hence, since $t \to h(\hat{F}_n(t,x), F(t,x))$ is measurable, we get that $\theta_n(\cdot)$ is measurable too and $\theta_n \xrightarrow{\lambda} 0$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $\theta_n(t) \to 0$ *a.e.* So we have

$$h(\hat{F}_n(t,x), F(t,x)) \to 0 \text{ } a.e. \text{ (uniformly in } x \in H)$$

and the exceptional Lebesque null set is independent of $x \in H$.

Now consider the following multivalued Cauchy problem:

$$\left\{ \begin{array}{c} \dot{x}(t) + A(t,x(t)) \in \hat{F}_n(t,x(t)) \text{ } a.e. \\ x(0) = x_0 \end{array} \right\}.$$

Denote its solution set by $S_n(x_0)$. Because of the convergence of the $\hat{F}_n$'s to F proved above, we can easily check that

$$S(x_0) = \bigcap_{n\geqslant 1} S_n(x_0).$$

From Theorem 3.3 we know that for every $n \geqslant 1$, $S_n(x_0)$ is nonempty and compact in $L^2(H)$. Note that $f_3^{\frac{1}{n}}(t,x) \in \hat{F}_n(t,x)$ and by construction $f_3^{\frac{1}{n}}(\cdot,\cdot)$ is locally Lipschitz. So for $\tau \in [0,r)$ and $y \in H$, the Cauchy problem

$$\left\{ \begin{array}{c} \dot{z}(t) + A(t,z(t) = f_3^{\frac{1}{n}}(t,z(t)) \text{ } a.e. \text{ on } [\tau,r] \\ z(\tau) = y \end{array} \right\}$$

has a unique solution $z_n(\cdot, \tau, y)\colon [\tau, r] \to H$ belonging in $W([\tau, r])$ (see the remark following theorem 3.4). Now let $x \in S_n(x_0)$. Then following Yorke [15], for each $\alpha \in [0,1]$, we set

$$w_n(\lambda, x)(t) = \begin{cases} x(t), & \text{for } t \in \big[0, (1-\lambda)r\big], \\ z_n\big(t, (1-\lambda)r, x((1-\lambda)r)\big), & \text{for } t \in \big[1-\lambda)r, r\big]. \end{cases}$$

Then because of theorem 3.3 (see also the remark following theorem 3.4), we have that $w_n\colon [0,1] \times S_n(x_0) \to S_n(x_0)$ is continuous and $w_n(0,x) = x$, while $w_n(1,x) = z_n$. So $w_n(\cdot,\cdot)$ is a null-homotopy for the set $S_n(x_0)$, with base point z_n. Hence $S_n(x_0)$ is contractible and so by Hyman's theorem [7], we finally have that $S(x_0)$ is an R_δ-set in $L_2(T, H)$. □

If we assume that we have smooth initial datum, then using theorem 3.4 we can have the following stronger version of theorem 4.1:

Theorem 4.2. *If hypotheses $H(A)$ and $H(F_1)$ hold and $x_0 \in X$, then $S(x_0)$ is an R_δ-set in $C(T, H)$.*

5. An application to control systems

Consider the following infinite dimensional control system:

$$(4) \qquad \left\{ \begin{array}{c} \dot{x}(t) + A\big(t, x(t)\big) = f\big(t, x(t)\big)u(t) \text{ a.e.} \\ x(0) = x_0 \\ u(t) \in U(t) \text{ a.e.}, u(\cdot) \text{ is measurable} \end{array} \right\}.$$

In this section, the space X in the evolution triple (X, H, X^*) is a separable Hilbert space. Also the control space is modelled by a separable Banach space Y. In what follows $\mathcal{L}(Y, H)$ is the Banach space of all bounded linear operators from Y into H.

We will assume the following concerning the data of (4):

$H(f)$: $f\colon T \times H \to \mathcal{L}(Y, H)$ is a map s.t.

(1) $t \to f(t,x)u$ is measurable,

(2) $x \to f(t,x)$ is continuous from H into $\mathcal{L}(Y,H)$ with the operator norm topology,

(3) $\|f(t,x)\|_{\mathcal{L}} \leqslant \alpha_2(t) + \beta_2(t)|x|$ a.e. with $\alpha_2, \beta_2 \in L^\infty_+$.

$H(U)$: $U\colon T \to P_{wkc}(Y)$ is a measurable multifunction *s.t.* $U(t) \subseteq W \in P_{wkc}(Y)$ a.e.

A control function $u\colon T \to H$ is said to be "admissible" if $u(\cdot)$ is measurable and $u(t) \in U(t)$ a.e. Every admissible control generates a nonempty set of admissible trajectories (theorem 3.1). Let $S(x_0)$ be the set of all admissible trajectories generated by all possible admissible controls. We define

$$R(t) = \{x(t)\colon x \in S(x_0)\}$$

the reachable set at time $t \in T$ of system (4); i.e. $R(t) = S(x_0)(t) \subseteq H$.

Theorem 5.1. *If hypotheses $H(A)$, $H(f)$, $H(U)$ hold and $x_0 \in H$, then for every $t \in T$, $R(t)$ is a nonempty, compact and connected subset of H.*

Proof. Let $F\colon T \times H \to P_{wkc}(H)$ be defined by

$$F(t,x) = f(t,x)U(t) = \bigcup_{u \in U(t)} f(t,x)u.$$

Let $u_n\colon T \to Y$ $n \geqslant 1$, be measurable functions *s.t.* $U(t) = \overline{\{u_n(t)\}}_{n\geqslant 1}$ for all $t \in T$. Such a sequence exists since by hypothesis $H(U)$, $U(\cdot)$ is a measurable multifunction (see theorem 4.2 of Wagner [14]). Then for every $v \in H$, we have

$$d(v, F(t,x)) = \inf_{n\geqslant 1} |v - f(t,x)u_n(t)|,$$
$$\text{hence } t \to d(v, F(t,x)) \text{ is measurable,}$$
$$\text{and so } t \to F(t,x) \text{ is a measurable multifunction.}$$

Also note that for every x, $y \in H$, we have

$$h(F(t,x), F(t,y)) \leqslant |W| \|f(t,x) - f(t,y)\|_{\mathcal{L}}$$

where $|W| = \sup\{\|u\|\colon u \in W\}$. So because of hypothesis $H(f)$ (2), we have that $F(t,\cdot)$ is h-continuous.

Finally, because of hypothesis $H(f)$ (3), we have:

$$|F(t,x)| \leqslant \hat{\alpha}_2(t) + \hat{\beta}_2(t)|x| \text{ a.e.}$$

with $\hat{\alpha}_2 = |W|\alpha_2$, $\hat{\beta}_2 = |W|\beta_2 \in L^\infty_+$. So the multifunction $F(t,x)$ satisfies hypothesis $H(F)_1$.

Consider the following evolution inclusion:

$$\left\{\begin{array}{c} \dot{x}(t) + A(t, x(t)) \in F(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array}\right\}.$$

A straightforward application of Aumann's selection theorem tells us that the solution set of the above multivalued Cauchy problem, is equal to $S(x_0)$. So from theorem 3.4, we have that $S(x_0)$ is an R_δ-set in $C(T,H)$, hence in particular, it is nonempty, compact and connected in $C(T,H)$. Since the evaluation at $t \in T$ map is continuous on $C(T,H)$ (see Dugundji [4], theorem 2.4, p. 260), we get that $R(t)$ is nonempty, compact and connected in H. □

Also if we are given a cost functional $J\colon C(T,H) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ to be minimized over $S(x_0)$, then we can have the following theorem, provided $J(\cdot)$ satisfies the hypothesis,

$H(J)$: $J\colon C(T,H) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is $l.s.c.$

Theorem 5.2. *If hypotheses $H(A)$, $H(f)$, $H(U)$ and $H(J)$ hold and $x_0 \in H$, then there exists $\hat{x} \in S(x_0)$ s.t.$J(\hat{x}) = \inf\left[J(x)\colon x \in S(x_0)\right]$.*

In particular, if $J(x) = \eta\big(x(r)\big)$, with $\eta\colon H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ $l.s.c.$, we have a terminal cost (Meyer) optimal control problem.

Now let Z be a bounded domain in $\mathbb{R}^N$. Consider the following nonlinear distributed parameter optimal control problem:

$$(5)\qquad \left\{\begin{array}{c} J(x) = \displaystyle\int_Z l\big(z, x(r,z)\big)\,\mathrm{d}z \to \inf = m \\ s.t. \dfrac{\partial x}{\partial t} - \displaystyle\sum_{i=1}^{N} \Big(\eta(|\operatorname{grad} x|^2)\frac{\partial x}{\partial z_i}\Big) = f\big(t,z,x(t,z)\big)u(t,z) \\ x\big|_{T\times\Gamma} = 0, x(0,z) = x_0(z), \big\|u(t,\cdot)\big\|_2 \leqslant M, u(\cdot,\cdot) \text{ is measurable} \end{array}\right\}.$$

We will need the following hypotheses on the data of (5):

$H(\eta)$: $\eta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and there exist γ, $\delta > 0$ such that:
$0 \leqslant \eta(\lambda^2) \leqslant \gamma$ for all $\lambda > 0$ and $\eta(\lambda^2)\lambda - \eta(\mu^2)\mu \geqslant \delta(\lambda - \mu), \lambda \geqslant \mu \geqslant 0$.

$H(f)_1$: $f\colon T \times Z \times \mathbb{R} \to \mathbb{R}$ is a function s.t.
(1) $(t,z) \to f(t,z,x)$ is measurable,
(2) $x \to f(t,z,x)$ is continuous,
(3) $\big|f(t,z,x)\big| \leqslant \beta(t,z)$ a.e. with $\beta(\cdot,\cdot) \in L^\infty(T \times Z)_+$.

$H(l)$: $l\colon Z \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is an integrand s.t.
(1) $(z,x) \to l(z,x)$ is measurable,
(2) $x \to l(z,x)$ is $l.s.c.$,
(3) $\varphi(z) - M|x| \leqslant l(z,x)$ a.e. with $\varphi \in L^1(Z)$, $M > 0$.

Remark. A simple example of a function satisfying $H(\eta)$ is $\eta(t) = 1 + (t+1)^{-1/2}$.

In this case, $H = L^2(Z)$, $X = H_0^1(Z)$ and $X^* = H^{-1}(Z)$. We know that (X, H, X^*) is an evolution triple with all embeddings being compact (Sobolev embedding theorem).

Let $a\colon X \times X \to \mathbb{R}$ be the Dirichlet form defined by

$$a(x,y) = \int_Z \eta(|\operatorname{grad} x|^2)(\operatorname{grad} x, \operatorname{grad} y)_{\mathbb{R}^N}\, \mathrm{d}z, x, y \in H_0^1(Z).$$

Then using hypothesis $H(\eta)$ and recalling that $\left(\int_Z \sum_{i=1}^{N} \left|\frac{\partial x}{\partial z_i}\right|^2 \mathrm{d}z\right)^{1/2}$ is an equivalent norm on $H_0^1(Z)$, we get that

$$\left|a(x,y)\right| \leqslant c\|x\| \cdot \|y\|, c > 0.$$

So there exists $A\colon X \to X^*$, a generally nonlinear operator s.t.

$$\langle Ax, y\rangle = a(x,y) \text{ for all } x, y \in H_0^1(Z).$$

Also if $\zeta(s) = \frac{1}{2}\int_0^{s^2} \eta(t)\,\mathrm{d}t$ and $\xi(v) = \zeta(|v|)$, $v \in \mathbb{R}^N$, we have

$$a(x,y) = \int_Z \left(\xi'(\operatorname{grad} x), \operatorname{grad} y\right)_{\mathbb{R}^N} \mathrm{d}z$$

and using hypotheses $H(\eta)$, we have

$$a(x, x-y) - a(y, x-y) \geqslant \|x-y\|^2$$
$$\text{and so } \langle Ax - Ay, x - y\rangle \geqslant \delta\|x-y\|^2$$

which shows that $A(\cdot)$ is strongly monotone and coercive (since $A(0) = 0$). Furthermore, it is easy to see that $A(\cdot)$ is continuous. So we have satisfied hypothesis $H(A)$.

Let $Y = L^2(Z)$ and $W = \{u \in Y\colon \|u\|_2 \leqslant M\} \in P_{wkc}(Y)$. Also let $\hat{f}\colon T \times H \to \mathcal{L}(Y, H)$ be defined by $\hat{f}(t,x)u(\cdot) = f(t,\cdot,x(\cdot))u(\cdot)$ for all $(x,u) \in H \times Y$. Using hypothesis $H(f)_1$, we can check that $\hat{f}(t,x)$ satisfies $H(f)$. Hence the dynamics of (5) have the following equivalent evolution equation form:

$$\left\{\begin{array}{c} \dot{x}(t) + Ax(t) = \hat{f}(t, x(t))u(t) \\ x(0) = x_0, u(t) \in W \text{ a.e.}, u(\cdot) = \text{ measurable} \end{array}\right\}.$$

We know that a trajectory of this evolution equation (hence of (5) too), belongs in $C(T, H)$. So the integral $\int_Z l(z, x(r,z))\,\mathrm{d}z$ makes sense. Furthermore, using $H(l)$ we can check that $J(\cdot)$ is *l.s.c.* on $C(T, H)$. Since the set of trajectories of (5) is compact in $C(T, H)$ (theorem 3.4), we have the following existence result for problem (5):

Theorem 5.3. *If hypotheses* $H(\eta)$, $H(f)_1$, $H(l)$ *hold and* $x_0(\cdot) \in L^2(Z)$, *then there exists* $x \in L^2(T, H_0^1(Z)) \cap C(T, L^2(Z))$ *with* $\frac{\partial x}{\partial t} \in L^2(T, H^{-1}(Z))$, *which is a trajectory of* (5) *and* $J(x) = m$.

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