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A NOTE ON PRODUCTS OF FRÉCHET SPACES

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In the paper we continue our investigation of products of Fréchet spaces by means of double convergence. Definitions of double π-sequences, ϱ-sequences, σ-sequences as well as double convergence $\lim x_{mn} = x$ and other definitions, propositions and examples can be found in [2]. It is supposed that the continuum hypothesis is true.

Definition 1. A Fréchet space is called compactible (Fréchet compactible, Hausdorff Fréchet compactible) if it can be embedded in a compact (Fréchet compact Hausdorff Fréchet compact) space.

Notice that the space of countable ordinal numbers with the usual convergence is a compactible space but not Fréchet compactible. The Hausdorff Fréchet space X consisting of isolated points x_{mn} and a ϱ-point $x = \lim x_{mn}$ is Fréchet compactible but not Hausdorff Fréchet compactible because no compact Hausdorff Fréchet space contains a ϱ-point [2].

Definition 2. Let $(X, \mathcal{L}, \lambda)$ be a Fréchet space. Let $\lim x_{mn} = x$. The double sequence $\langle x_{mn} \rangle$ is called a strong sequence if no subsequence of $\langle x_{mn} \rangle_n$, $m \in N$, $\mathcal{L}$-converges to the point x.

Definition 3. Let (T, u) be a topological space. A point x of T is called unaccessible if there is an uncountable system $\{S_\xi, \xi < \omega_\alpha\}$ of infinite countable sets $S_\xi \subset T$ such that the following conditions are satisfied:

(1) $x \notin uS_\xi$, $\xi < \omega_\alpha$; (2) If U is a neighborhood of the point x then there is ξ with $S_\xi \subset U$; (3) If $\xi_i < \omega_\alpha$, $i \in N$, are ordinal numbers, then there are subsets $T_{\xi_i} \subset S_{\xi_i}$, such that $S_{\xi_i} - T_{\xi_i}$ are finite sets and $x \notin u\bigl(\bigcup\limits_{i=1}^{\infty} T_{\xi_i}\bigr)$, $i \in N$. A point $x \in T$ is called accessible if it is not unaccessible. A topological space is accessible if each point of it is accessible.

Examples. Let T be an uncountable set consisting of a point x and points $x_{\xi n}$, $n \in N$, $\xi < \omega_1$. Points $x_{\xi n}$ are isolated. A subset $U \subset T$ is a neighbourhood of the point x provided that the sets $\{x_{\xi n}, \xi < \omega_1\} - U$, $n \in N$, are finite. Hence, we have a topological space (T, u) in which the sets $S_\xi = \{x_{\xi n}, n \in N\}$, satisfy properties (1), (2), (3) and so x is an unaccessible point.

Let T be a uncountable set. Fix a point $x \in T$. A set $U \subset T$ is a neighborhood of x if $x \in U$ and the set $T - U$ is finite. All other points are isolated. Then we have a topological space (T, u) which is accessible.

Let (T, u) be a first countable space. Suppose that it contains an unaccessible point x. Let $\{v_i, i \in N\}$ be a complete system of neighborhoods of the point x. Let S_ξ be sets satisfying properties (1), (2), (3). Hence, there are ordinal numbers ξ_i with $S_{\xi_i} \subset V_i$. Consequently, (3) does not hold. This is a contradiction. We have shown that first countable spaces are accessible.

Unaccessibility of a point is a topological property. As a matter of fact, if (T, u), (Q, v) are topological spaces, $h\colon T \to Q$ a homeomorphism and $x \in T$ an unaccessible point with respect to $\{S_\xi, \xi < \omega_\alpha\}$ then $h(x)$ is an unaccessible point in the space Q with respect to $\{h(S_\xi), \xi < \omega_\alpha\}$.

Next, we shall use the notion of unaccessibility in Fréchet spaces $(X, \mathcal{L}, \lambda)$. Points of the sets S_ξ will be denoted $x_{\xi j}$, $j \in N$. Then, in view of (3), we have the following implication: if $\xi_i < \omega_\alpha$, $i \in N$, are ordinal numbers, then $\langle x_{\xi_i j}\rangle$ is a double sequence no double subsequence of which converges to x.

Lemma. *Let $(X, \mathcal{L}, \lambda)$ be a Fréchet space. If $x \in X$ is a ϱ-point then it is unaccessible.*

Proof. Let x be a ϱ-point in X. Let $\langle a_{mn}\rangle$ be a one-to-one ϱ-sequence of points $a_{mn} \neq x$ converging to x. Let $\{\langle K_j^\xi\rangle_j, \xi < \omega_1\}$ be a well-ordered almost disjoint system of increasing sequences of positive integer numbers K_j^ξ such that, if a function $f\colon N \to N$ is given, then there is ξ with $K_j^\xi > f(j)$, $j \in N$. Let U be a neighborhood of the point x. Since $x = \mathcal{L} - \lim\limits_n a_{mn}$, $m \in N$, there is a function g: $N \to N$ such that the points a_{mn}, $n > g(m)$, $m \in N$, belong to U. There is ξ with numbers $k_m^\xi > g(m)$. Consequently $a_{mk_m^\xi} \in U$, $m \in N$. Denote $x_{\xi m} = a_{mk_m^\xi}$ and $S_\xi = \{x_{\xi n}, n \in N\}$. Then $\langle x_{\xi n}\rangle_n$ is a cross-sequence in $\langle a_{mn}\rangle$. We have proved that (1), (2) in Definition 3 are satisfied.

Now, consider a double sequence $\langle x_{\xi_i} j\rangle$, where $\langle \xi_i\rangle$ is a one-to-one sequence of countable ordinals. Denote $z_{ij} = x_{\xi_i j}$. Since $\langle x_{\xi_i j}\rangle_j = \langle a_{jk_j^{\xi_i}}\rangle_j$, each sequence $\langle z_{ij}\rangle_j$ is a cross-sequence in $\langle a_{mn}\rangle$. There is a function $h\colon N \to N$ such that no point z_{ij}, $j > h(i)$, belongs to the set $\{a_{mn}, m < i, n \in N\}$. Consequently, $x \notin \lambda\{z_{ij}, j >$

$h(i), i \in N\}$. We deduce that no double subsequence of $\langle z_{ij} \rangle$ converges to the point x. Hence, also (3) is satisfied. $\square$

Proposition 1. *Let $(X, \mathcal{L}_1, \lambda_1)$ be a Fréchet space and $(Y, \mathcal{L}_2, \lambda_2)$ an accessible Fréchet space. The topological product $(X \times Y, w)$ is not a Fréchet space if and only if there is a ϱ-point in X or there are points $x \in X$, $y \in Y$ which are coupled by a σ-sequence $\langle x_{mn} \rangle$ in X and a strong σ-sequence $\langle y_{mn} \rangle$ in Y.*

P r o o f. Suppose that the topological product $(X \times Y, w)$ is not Fréchet. Denote $(X \times Y, \mathcal{L}_{12}, \lambda_{12})$ the convergence product of the spaces X and Y. There is a point $(a, b) \in X \times Y$ and a set $Z \subset X \times Y$ such that $(a, b) \in wZ$, $(a, b) \notin \lambda_{12} Z$. Hence, we can assume that $Z \cap (X \times b) = \emptyset = Z \cap (a \times Y)$. Denote A resp. B the projection of the set Z into the space X resp. Y. Since $(a, b) \in wZ$, we have $a \in \lambda_1 A$, $b \in \lambda_2 B$ and $a \notin A$, $b \notin B$. Let V be a neighborhood of the point b in the space Y.

Denote $Z_1 = Z \cap (X \times Y)$ and A_1 resp. B_1 the projection of Z_1 into X resp. Y. Hence, $A_1 \subset A$, $B_1 \subset B$, $a \in \lambda_1 A_1$, $b \in \lambda_2 B_1$. Let $\langle x_n \rangle$ be a one-to-one sequence of points $x_n \in A_1$ with $\mathcal{L}_1 - \lim x_n = a$ and $\langle y_n \rangle$ a sequence of points $y_n \in B_1$ such that $(x_n, y_n) \in Z_1$. The sequences $\langle y_n \rangle$, $\langle (x_n, y_n) \rangle$ and their subsequences will be called special sequences. If $\langle y_{n_i} \rangle$ is a constant subsequence of $\langle y_n \rangle$, the point $s = y_{n_i}$ is called a multiple point. Denote M the set of all multiple points. Evidently, $b \notin M$, because $(a, b) \notin \lambda_{12} Z$. We distinguish two cases.

There is a one-to-one sequence $\langle s_n \rangle$, $s_n \in M$, $\mathcal{L}$-converging to b. Denote $\langle (x_{in^i_j}, y_{in^i_j}) \rangle_j$ a special sequence such that $y_{n^i_j} = s_i$, $j \in N$. Then we have a strong π-sequence $\langle y_{in^i_j} \rangle$ and a ϱ-sequence $\langle x_{in^i_j} \rangle$ because $(a, b) \notin \lambda_{12} Z$.

There is a neighborhood U of b in Y containing no multiple point. We can assume that special sequences $\langle y_n \rangle$, $y_n \in U$, are one-to-one. Consider two possibilities. Either there are special sequences $\langle y_{mn} \rangle_n$, $m \in N$, such that if V is a neighborhood of b, then there is $m \in N$ with $y_{mn} \in V$, $n \in N$. Let $\langle (x_{mn}, y_{mn}) \rangle$, $m \in N$, be the corresponding special sequences. Then $\langle y_{mn} \rangle$ is a strong double π-sequence converging to b and $\langle x_{mn} \rangle$ is a ϱ-sequence converging to a. Or the system of special sequences $\langle y_{\xi n} \rangle$, $y_{\xi n} \in U$ is uncountable and (1), (2) are satisfied. According to Definition 3, there are special one-to-one sequences $\langle y_{mn} \rangle_n$ such that the double sequence $\langle y_{mn} \rangle$ converges to b. The corresponding double sequence $\langle x_{mn} \rangle$ converges to a. In view of Lemma, there is no ϱ-subsequence of $\langle y_{mn} \rangle$. Consequently, with respect to Theorem 2 in [2], there is a π-subsequence $\langle y_{in^i_j} \rangle$ of $\langle y_{mn} \rangle$ and then $\langle x_{in^i_j} \rangle$ is a ϱ-sequence in the space X or there is a σ-subsequence $\langle y_{in^i_j} \rangle$ and then $\langle x_{in^i_j} \rangle$ is a σ-subsequence because $(a, b) \notin \lambda_{12} Z$. We have proved that the points a and b are coupled by σ-sequences. It is clear that the sequence $\langle y_{in^i_j} \rangle$ is strong.

Now, let $x \in X$, $y \in Y$ be points coupled by a σ-sequence $\langle y_{mn} \rangle$ and a strong σ-sequence $\langle y_{mn} \rangle$. Let Z be a set of points (x_{mn}, y_{mn}). Since $\lim x_{mn} = x$, $\lim y_{mn} = y$

from Lemma 4 in [2], it follows $(x, y) \in wZ$. On the other hand $(x, y) \notin \lambda_{12}Z$ because x, y are coupled points and $\langle y_{mn} \rangle$ is strong. If the space X contains a ϱ-point, the space $X \times Y$ is not Fréchet because it is supposed that the space Y is not discrete. $\square$

Proposition 2. *A Hausdorff Fréchet compactible space $(X, \mathcal{L}, \lambda)$ is accessible.*

Proof. Suppose there is an unaccessible point x in X. Let $\langle x_{\xi n} \rangle_n$, $\xi < \omega_\alpha$, be one-to-one sequences of points of X satisfying the properties (1), (2), (3) in the Definition 3. Denote $(X_C, \mathcal{L}_C, \lambda_C)$ a Hausdorff Fréchet compact space in which the space X is embedded. There is a subsequence $\langle x_{n_j} \rangle$ of $\langle x_n \rangle$ such that $\mathcal{L}_C - \lim x_{\xi n_j} = x_\xi$. Since the space X_C is regular we can suppose that each neighborhood of the point x contains at least one point x_ξ, $\xi < \omega_\alpha$. Because $x_\xi \neq x$, there is a one-to-one sequence $\langle x_{\xi_i} \rangle$ $\mathcal{L}_C$-converging to the point x. Denote $z_{ij} = x_{\xi_i n_j}$. Then $\mathcal{L}_C\text{-}\lim\limits_j z_{ij} = x$. Since X_C is Fréchet, in the space $(X_C, \mathcal{L}_C, \lambda_C)$ the double sequence $\langle z_{ij} \rangle$ converges to x. The points x, z_{ij} belong to X and X is embedded in X_C. Therefore the double sequence $\langle z_{ij} \rangle$ converges to x in the space $(X, \mathcal{L}, \lambda)$. This is a contradiction with (3). $\square$

Corollary. *Let $(X, \mathcal{L}_1, \lambda_1)$, $(Y, \mathcal{L}_2, \lambda_2)$ be Hausdorff Fréchet compactible spaces. The topological product $(X \times Y, w)$ is not Fréchet if and only if there are points $x \in X$, $y \in Y$ coupled by a σ-sequence in X and a σ-sequence in Y from which at least one is strong.*

Proof follows immediately from Lemma, Proposition 2 and Proposition 1.

Remark. Constructions of Hausdorff Fréchet compact spaces whose topological product is not Fréchet have been published in [1], [3]. According to Proposition 2 each subspace of a Hausdorff Fréchet compact space is accessible. Hence, [1], [3] yield suitable examples of accessible Fréchet spaces the product of which is not Fréchet.

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