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ASYMPTOTIC PROPERTIES OF NONOSCILLATORY SOLUTIONS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS OF n-TH ORDER

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1. INTRODUCTION

We consider a neutral differential equation in the form

$$L_n[x(t)+p(t)x(h(t))]+q(t)f(x(g(t)))=b(t), \tag{E}$$

where $L_0z(t)=z(t)$, $L_kz(t)=a_k(t)(L_{k-1}z(t))'$, $k=1,2,\dots,n$, $a_0=a_n=1$, $a_k\in C([0,\infty),(0,\infty))$, $k=1,2,\dots,n-1$, $p,q,h,g,b\in C([0,\infty),\mathbb{R})$, $q(t)\not\equiv 0$ on any half line $[t_0,\infty)$, $t_0\geqslant 0$;

(C_1) $h(t)<t$, $\lim\limits_{t\to\infty}h(t)=\infty$, $\lim\limits_{t\to\infty}g(t)=\infty$, $h(t)$ is an increasing function on $[t_0,\infty)$, $t_0\geqslant 0$;

(C_2) $uf(u)>0$ for $u\neq 0$.

Let $t_0\geqslant 0$ be such that $T=\min\{\inf\limits_{t\geqslant t_0}h(t),\inf\limits_{t\geqslant t_0}g(t)\}\geqslant 0$. A function $x(t)\in C[T,\infty),\mathbb{R})$ is a solution of (E) on $[t_0,\infty)$ if the functions $L_k[x(t)+p(t)\,x(h(t))]$, $0\leqslant k\leqslant n$ exist and are continuous on $[t_0,\infty)$ and $x(t)$ satisfies the equation (E) on $[t_0,\infty)$.

In this paper we will consider only such solutions of the equation (E) that $\sup\{|x(t)|;\,t\geqslant t_x\}>0$ for any $t_x\geqslant t_0$. Such a solution is called nonoscillatory if it is eventually of constant sign for sufficiently large t. Otherwise it is called oscillatory.

Many authors have been studying the oscillatory properties of solutions of neutral differential equations with positive or negative coefficient $q(t)$. Numerous interesting results of this type can be found, for example, in the papers [1–3, 5, 6, 9–11] and in the references given therein. We know only the papers [4, 8] dealing with the case

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when the coefficient $q(t)$ can change the sign. In this paper we extend some results from the papers [4, 7, 8] to the equation (E).

2. Preliminaries

Denote

(1) $h^{[0]}(t) = t$, $h^{[k]}(t) = h(h^{[k-1]}(t))$, $k = 1, 2, \dots$; $h^{[-k]}(t)$ denotes the inverse function to $h^{[k]}(t)$, $n = 1, 2, \dots$.

(2) $P_0(t) = 1, P_{k+1}(t) = P_k(t)p(h^{[k]}(t)), k = 0, 1, 2, \dots$;

(3) $\gamma(t) = \sup\{s \geqslant 0;\ h(s) \leqslant t\}$, $t \geqslant 0$, $\gamma_0(t) = t$, $\gamma_{k+1}(t) = \gamma(\gamma_k(t))$, $k = 0, 1, \dots$.

It is easily verify that $\gamma_k(t_0) < \gamma_{k+1}(t_0)$, $\lim\limits_{t\to\infty} \gamma_k(t_0) = \infty$ if $h(t)$ satisfies (C_1).

Lemma 1. [8; Lemma 1] (a) *Let*

$$u(t) = v(t) + p(t)v(h(t)), \tag{4}$$

where $u, v, p, h \in C([t_0, \infty), \mathbb{R})$, *and* $h(t)$ *satisfies the condition* (C_1).

(b) *Let there exist constants* $p_1, p_2 \in \mathbb{R}$ *such that*

$$|p(t)| \leqslant p_1 < 1, \qquad p(t)\,p(h(t)) \geqslant 0, \tag{5}$$

or

$$p_2 \leqslant p(t) \leqslant 0 \tag{6}$$

holds for $t \geqslant t_0$. *Assume that* $0 < v(t)$, $\liminf\limits_{t\to\infty} v(t) = 0$ *and there exists* $\lim\limits_{t\to\infty} u(t) = L \in \mathbb{R}$.

Then $L = 0$.

A similar lemma we can find in [5, Lemma 1.5.2], in which $h(t) = t - c$, $0 < c \in \mathbb{R}$ and $p(t)$ has a constant sign.

Lemma 2. *Let the assumption* (a) *of Lemma* 1 *be satisfied. Let there exist constants* $p_1, p_3 \in \mathbb{R}$ *such that either* (5) *or*

$$p(t) \leqslant p_3 < -1 \tag{7}$$

holds for $t \geqslant t_0$.

Assume that $0 \leqslant v(t) \leqslant v_0 < \infty$ *and* $\lim\limits_{t\to\infty} u(t) = 0$. *Then* $\lim\limits_{t\to\infty} v(t) = 0$.

P r o o f. i) Let (5) hold. Then the proof is the same as the proof of Lemma 2 [9].

ii) Let (7) hold. Then from (4) with regard to (1) and (7) we obtain

$$v(t) \leqslant \frac{1}{p_2}\,[u(h^{-[1]}(t)) - v(h^{-[1]}(t))], \text{ for } \gamma_1(t) \geqslant t_0.$$

By iteration for sufficiently large t we have

$$\begin{aligned}(8)\quad v(t) \leqslant \frac{1}{p_2}\,u(h^{-[1]}(t)) - \frac{1}{{p_2}^2}\,u(h^{-[2]}(t)) + \dots \\ + (-1)^{n-1}\,\frac{1}{{p_2}^n}\,u(h^{-[n]}(t) + (-1)^n\,\frac{1}{{p_2}^n}\,v(h^{-[n]}(t)) \quad \text{for } \gamma_n(t) \geqslant t_0.\end{aligned}$$

In view of $\lim_{t\to\infty} u(t) = 0$, for any $\varepsilon_1 > 0$ there exists a sufficiently large t_1 such that $|u(t)| < \varepsilon_1$ for any $t \geqslant t_1$. Then from (8) we obtain

$$|v(t)| \leqslant \frac{\varepsilon_1}{|p_2| - 1} + \frac{v_0}{|p_2|^n}.$$

Therefore for any $\varepsilon > 0$ there exist ε_1 and $n = n_0 \in N$ such that $\frac{\varepsilon_1}{|p_2|-1} + \frac{v_0}{|p_2|^n} < \varepsilon$. Then the last two relations imply $\lim_{t\to\infty} v(t) = 0$. □

Lemma 3. *Let the assumption* (a) *of Lemma* 1 *hold and* $|p(t)| \leqslant p_1 < 1$ *for* $t \geqslant t_0 > 0$. *If* $v(t) > 0$ *and* $u(t)$ *is bounded from above* ($v(t) < 0$ *and* $u(t)$ *is bouded from below) on* $[t_0, \infty)$, *then* $v(t)$ *is bouded.*

Proof. Let $v(t) > 0$ and $u(t) \leqslant K < \infty$ for $t \geqslant t_0$. From (4) in view of (1)–(3) we obtain

$$v(t) = u(t) - p(t)v(h(t)) = u(t) - P_1(t)u(h(t)) + P_2(t)v(h^{[2]}(t)), \quad t \geqslant \gamma(t_0).$$

Repeating this argument we find

$$(9)\qquad v(t) = \sum_{k=0}^{m-1} (-1)^k P_k(t)u(h^{[k]}(t)) + (-1)^m P_m(t)v(h^{[m]}(t)),$$
$$t \geqslant \gamma_m(t_0), m = 1, 2, \dots$$

Let $0 < v(t) \leqslant c$ for $t \in [t_0, \gamma(t_0)]$, then $h^{[m]}(t) \in [\gamma_m(t_0), \gamma_{m+1}(t_0)]$ and $0 < v(h^{[m]}(t)) \leqslant c$.

If $|p(t)| \leqslant p_1 < 1$ then by (2) $| P_k(t) |\leqslant p_1^k < 1$ for $t \geqslant \gamma_k(t_0), k = 1, 2, \dots, m$. Then from (9) we have

$$0 < v(t) \leqslant K \sum_{k=0}^{m-1} p_1^k + p_1^m c \leqslant \frac{K}{1 - p_1} + c = K_1 < \infty$$

for $t \in [\gamma_m(t_0), \gamma_{m+1}(t_0)], m = 1, 2, \dots$. The last relation for $t \to \infty$ implies that $0 < v(t) \leqslant K_1$.

Analogously we prove the result if $v(t) < 0$ and $u(t)$ is bounded from below on $[t_0, \infty)$. □

Lemma 4. [7, Lemma 2] *Let $w \in C([t_0,\infty),\mathbb{R})$, $v \in C^1([t_0,\infty),\mathbb{R})$ and let there exist $\lim_{t\to\infty}[w(t)v'(t)+v(t)]$ in the extended real line $\mathbb{R}^{\#}$. Then $\lim_{t\to\infty} v(t)$ exists in $\mathbb{R}^{\#}$.*

Denote

$$A_0(t)=1,\quad A_k(t)=\int_{t_0}^{t}\frac{A_{k-1}(s)}{a_k(s)}\,\mathrm{d}s,\ \text{if}\quad A_k(\infty)=\infty \tag{10}$$

for $k=1,2,\dots,n-1$.

$$A_0(t)=1,\quad A_k(t)=\int_{t}^{\infty}\frac{A_{k-1}(s)}{a_k(s)}\,\mathrm{d}s,\ \text{if}\quad A_k(t_0)<\infty \tag{11}$$

for $k=1,2,\dots,n-1$.

Lemma 5. *Let $k\in\{1,2,\dots,n\}$, $u_k(t)=\int_T^t A_{k-1}(s)(L_{k-1}z(s))'\,\mathrm{d}s$, where $T>t_0$, $L_0z(t),\dots,L_nz(t)$ are continuous functions on $[t_0,\infty)$ and $A_k(t)$, $k=1,2,\dots,n-1$ are defined by (10) or by (11).*

(i) *If*

$$\lim_{t\to\infty}u_k(t)=+\infty\ (-\infty)\ \text{for}\quad k=2,3,\dots,n,$$

then $\lim_{t\to\infty}u_i(t)=+\infty(-\infty)$, $i=1,2,\dots,k-1$.

(ii) *Let $z(t)$ be a bounded function on $[t_0,\infty)$ and let there exist $\lim_{t\to\infty}u_n(t)$, then $\lim_{t\to\infty}z(t)=z_0\in\mathbb{R}$.*

If (10) holds, then in addition $\lim_{t\to\infty}L_iz(t)=0$, $i=1,2,\dots,n-1$.

Proof. We easily prove that the functions $u_k(t)$, $k=1,2,\dots,n-1$ satisfy the differential equation

$$\frac{A_k(t)}{A_k'(t)}u_k'(t)-u_k(t)=\varepsilon\overline{u}_{k+1}(t),\quad t\geqslant T>t_0, \tag{13_k}$$

where $\varepsilon=+1$, or -1 if (10) or (11) holds, respectively,

$$\overline{u}_{k+1}(t)=u_{k+1}(t)+A_k(T)L_{k-1}z(T),\ k=1,2,\dots,n-1. \tag{14}$$

In view of (10) or (11) and (C_1) we have $A_k(t)>0$, $A_k'(t)>0$, $k=1,2,\dots n-1$, for $t\geqslant T>t_0$.

The equation (13_k), $k\in\{1,2,\dots,n-1\}$ can be written in the form

$$\left(\frac{u_k(t)}{A_k(t)}\right)'=\varepsilon\,\frac{A_k'(t)}{A_k^2(t)}\overline{u}_{k+1}(t),\quad t\geqslant T.$$

From the last equation we obtain

$$(15_k)\qquad u_k(t) = \varepsilon\, A_k(t) \int_T^t \frac{A_k'(s)}{A_k^2(s)} \overline{u}_{k+1}(s)\,\mathrm{d}s,$$

$k \in \{1, 2, \dots, n-1\}$.

(i) Let $k \in \{1, 2, \dots, n-1\}$ and $\lim_{t\to\infty} u_{k+1}(t) = \infty$.Then by (14) $\lim_{t\to\infty} \overline{u}_{k+1}(t) = \infty$. Then from (15_k), taking into account (10) or (11), we obtain that $\lim_{t\to\infty} u_k(t) = \infty$. If $k > 1$, we can repeat this process and getting successively that $\lim_{t\to\infty} u_i(t) = \infty$, $i = k-1, \dots, 2, 1$.

(ii) Let $z(t)$ be a bounded on $[T, \infty)$, $T > t_0$. Then $u_1(t) = z(t) - z(T)$ is bounded on $[T, \infty)$. If there exists $\lim_{t\to\infty} u_n(t)$, then in view of (13_{n-1}) and Lemma 3 there exists $\lim_{t\to\infty} u_{n-1}(t)$. If we proceed similarly we successively get that there exist $\lim_{t\to\infty} u_k(t)$, $k = 1, 2, \dots, n-2$. Then with regard to the case (i) and the fact that $u_1(t)$ is bounded, there are b_k: $|b_k| < \infty$ such that $\lim_{t\to\infty} u_k(t) = b_k$, $k = 1, 2, \dots, n$. Therefore from

$$\lim_{t\to\infty} \left[\frac{A_k(t)}{A_k'(t)} u_k'(t) - u_k(t)\right] = \varepsilon[b_{k+1} + A_k(T) L_k z(T)] = \overline{b}_{k+1},$$

$(k = 1, 2, \dots, n-1)$ and (12) we obtain

$$(16)\qquad \lim_{t\to\infty} \frac{A_k(t)}{A_k'(t)} u_k'(t) = \lim_{t\to\infty} A_k(t) L_k z(t) = \overline{b}_{k+1} - b_k = c_k \in \mathbb{R},$$

$$\lim_{t\to\infty} z(t) = b_1 + z(T) = \overline{b}_1.$$

Let (10) hold. Then from (16) in view of (10) we obtain

$$\lim_{t\to\infty} L_k z(t) = 0, \;\; k = 1, 2, \dots, n-1.$$

□

Remark. Denote $q_+(t) = \max\{0, q(t)\}$, $q_-(t) = \max\{0, -q(t)\}$. Then $q(t) = q_+(t) - q_-(t)$, $t \in [t_0, \infty)$.

3. Main results

Theorem 1. *Let* (C_1), (C_2) *and either* (10) *or* (11) *hold. Let there exist constants* $p_1, p_2, p_3 \in \mathbb{R}$ *such that either* (5) *or*

$$p_3 \leqslant p(t) \leqslant p_2 \leqslant -1 \quad \text{for} \quad t \geqslant t_0. \tag{17}$$

In addition we suppose that for $T \geqslant t_0$

$$\int_T^\infty A_{n-1}(t)|b(t)|\,\mathrm{d}t < \infty, \tag{18}$$

and either

$$\int_T^\infty A_{n-1}(t)q_+(t)\,\mathrm{d}t = \infty, \tag{19_1}$$

$$\int_T^\infty A_{n-1}(t)q_-(t)\,\mathrm{d}t < \infty, \tag{20_1}$$

or

$$\int_T^\infty A_{n-1}(t)q_+(t)\,\mathrm{d}t < \infty, \tag{19_2}$$

$$\int_T^\infty A_{n-1}(t)q_-(t)\,\mathrm{d}t = \infty. \tag{20_2}$$

Then every bounded solution of the equation (E) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} z(t) = 0$. *If* (10) *holds then in addition*

$$\lim_{t\to\infty} L_k z(t) = 0, \quad k = 1, 2, \ldots, n-1. \tag{21}$$

P r o o f. Let $x(t)$ be a bounded positive solution of (E) on $[t_0, \infty)$. Without loss of generality we suppose that $x(g(t)) > 0$, $x(h(t)) > 0$ for $t \geqslant t_1 \geqslant t_0$. If $x(t)$ is bounded, then in view of (5) or (17) we have that $z(t)$ is bounded on $[t_1, \infty)$.

Multiplying the equation (E) by $A_{n-1}(t)$ and then integrating from t_1 to t ($> t_1$) we get

$$\begin{aligned} u_n(t) = \int_{t_1}^t A_{n-1}(s) L_n z(s)\,\mathrm{d}s = \int_{t_1}^t A_{n-1}(s)q_-(s)f(x(g(s)))\,\mathrm{d}s \\ - \int_{t_1}^t A_{n-1}(s)q_+(s)f(x(g(s)))\,\mathrm{d}s + \int_{t_1}^t A_{n-1}(s)b(s)\,\mathrm{d}s. \end{aligned} \tag{22}$$

Let (19_1), (20_1) hold. If

$$\int_{t_1}^{\infty} A_{n-1}(s)q_+(s)f(x(g(s)))\,\mathrm{d}s = \infty, \tag{23}$$

then from (22) in view of the boundedness of $x(t)$ (> 0), (C_1), (C_2), (18), and (20_1) we obtain $\lim_{t\to\infty} u_n(t) = -\infty$. Then by Lemma 5 $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the assumption that $z(t)$ is bounded. Therefore

$$\int_{t_1}^{\infty} A_{n-1}(s)q_+(s)f(x(g(s)))\,\mathrm{d}s < \infty. \tag{24}$$

Then (22) with regard to the boundedness of $x(t)$ (> 0), (C_1), (C_2), (18), (20_1) and (24) yields that there exists a $b_n \in \mathbb{R}$ such that $\lim_{t\to\infty} u_n(t) = b_n$. Then in view of Lemma 5, the case (ii) and the boundedness of $z(t)$ there exists a finite $\lim_{t\to\infty} z(t) = b_0 \in \mathbb{R}$.

If (10) holds then in addition $\lim_{t\to\infty} L_k z(t) = 0$, $k = 1, 2, \ldots, n-1$.

From (24) in view of (C_1), (C_2) and (19_1) we have $\liminf_{t\to\infty} x(t) = 0$. Using Lemma 1 we obtain $\lim_{t\to\infty} z(t) = 0$. Now by Lemma 2 we have $\lim_{t\to\infty} x(t) = 0$. Analogously we can prove the result if (19_2), (20_2) hold. □

The following examples are illustrative:

Example 1. Consider the equation

$$(\mathrm{E}_1) \qquad \Big(\mathrm{e}^{-t}\big(x(t) - \frac{\mathrm{e}^{-2\pi}}{2}x(t-2\pi)\big)'\Big)' + \frac{\mathrm{e}^{-(t+\pi)}}{2}\frac{1+\cos t}{2+\cos t}x(t-\pi) = \frac{\mathrm{e}^{-2t}(5-3\sin t)}{2}.$$

The assumptions (5), (10), (18), (19_1), (20_1) of Theorem 1 are satisfied. The equation (E_1) has a nonoscillatory solution $x(t) = \mathrm{e}^{-t}(2-\cos t)$. Then $z(t) = \frac{1}{2}\mathrm{e}^{-t}(2-\cos t)$, $L_1 z(t) = \mathrm{e}^{-t}z'(t)$. We easily see that $x(t)$, $z(t)$, $L_1 z(t)$ tends to 0 as $t \to \infty$.

Example 2. Consider the equation

$$(\mathrm{E}_2) \qquad (\mathrm{e}^{2t}(x(t) - 5\mathrm{e}^{-2}x(t-2))')' - 4\mathrm{e}^{2t+1}(t^2+1)x(t+1) = -t^2\mathrm{e}^t.$$

The assumptions (5), (10), (18), (19_2), (20_2) of Theorem 1 are satisfied. The equation (E_2) has a nonoscillatory solution $x(t) = \mathrm{e}^{-t}$. Then $z(t) = -4\mathrm{e}^{-t}$, $L_1 z(t) = 4\mathrm{e}^{t}$. We easily see that $x(t)$, $z(t)$ tend to 0 as $t \to 0$ and $L_1 z(t)$ tends to ∞ as $t \to \infty$.

We see that if (11) is satisfied then (21) need not hold.

Theorem 2. *Let* (C_1), (C_2), (5), (11), (18) *hold. In we suppose in addition that* $q(t) \geqslant 0$ *on* $[t_0, \infty)$ *and*

$$\int_T^\infty A_{n-1}(t) q(t)\,\mathrm{d}t = \infty, \ T > 0, \tag{25}$$

then every solution of (E) *is either oscillatory or* $\lim_{t\to\infty} z(t) = 0$ *and* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a positive solution of (E) on $[t_0, \infty)$. Without loss of generality we suppose that $x(g(t)) > 0$, $x(h(t)) > 0$ for $t \geqslant t_1 \geqslant t_0$. Multiplying the equation (E) by $A_{n-1}(t)$ and then integrating from t_1 to t we have

$$\begin{aligned} u_n(t) &= \int_{t_1}^t A_{n-1}(s) L_n z(s)\,\mathrm{d}s \\ &= \int_{t_1}^t A_{n-1}(s) b(s)\,\mathrm{d}s - \int_{t_1}^t A_{n-1}(s) q(s) f(x(g(s)))\,\mathrm{d}s. \end{aligned}$$

Then with regard to (C_1), (C_2), (18) and (24), the last equation implies that $u_n(t)$ is bounded from above, i.e.there exist a $T \geqslant t_1$ and a constant $K > 0$ such that $\bar{u}_n(t) = u_n(t) + A_n(T) L_{n-1} z(T) \ < K < \infty$ for $t \geqslant T$. Then using (15_{n-1})and(11) we get

$$\begin{aligned} u_{n-1}(t) &\leqslant \, - K A_{n-1} \int_T^t \frac{A'_{n-1}(s)}{A^2_{n-1}(s)}\,\mathrm{d}s \\ &= K\Big[1 - \frac{A_{n-1}(t)}{A_{n-1}(T)}\Big] \leqslant K, \quad t \geqslant T. \end{aligned}$$

If we repeat this argument $n-2$-times we get that $u_1(t) = z(t) - z(T)$ is bounded from above. Using Lemma 3 we obtain that $x(t)$ is bounded on $[t_0, \infty)$. Now we apply Theorem 1 we obtain that $\lim_{t\to\infty} z(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3. *Let* (C_1), (C_2) *hold and let* $p(t)$ *be a bounded function on* $[t_0, \infty)$. *In addition we suppose that*

$$\int_{t_0}^\infty \frac{\mathrm{d}t}{a_i(t)} = \infty, \quad i = 1, 2, \ldots n-1, \tag{26}$$

$$\int_{t_0}^\infty |b(t)|\,\mathrm{d}t < \infty, \tag{27}$$

and either

$$(28_1) \qquad \int_{t_0}^{\infty} q_+(t)\,\mathrm{d}t = \infty,$$

$$(29_1) \qquad \int_{t_0}^{\infty} q_-(t)\,\mathrm{d}t < \infty,$$

or

$$(28_2) \qquad \int_{t_0}^{\infty} q_+(t)\,\mathrm{d}t < \infty,$$

$$(29_2) \qquad \int_{t_0}^{\infty} q_-(t)\,\mathrm{d}t = \infty.$$

Then every bounded solution of the equation (E) *is either oscillatory or*

$$\liminf_{t\to\infty} |x(t)| = 0 \text{ and } \lim_{t\to\infty} L_k z(t) = 0 \text{ for } k = 1, 2, \ldots, n-1.$$

Proof. Let $x(t)$ be a bounded positive solution of (E) on $[t_0, \infty)$. Without loss of generality we suppose that $x(g(t)) > 0$, $x(h(t)) > 0$ for $t \geqslant t_1 \geqslant T$. Because $p(t)$ and $x(t)$ are bounded on $[t_0, \infty)$ then $z(t)$ is bounded. Integrating the equation (E) from t_1 to t we get

$$(30) \qquad L_{n-1}z(t)(t) - L_{n-1}z(t_1) + \int_{t_1}^{t} q_+(s) f(x(g(s)))\,\mathrm{d}s = \int_{t_1}^{t} b(s)\,\mathrm{d}s + \int_{t_1}^{t} q_-(s) f(x(g(s)))\,\mathrm{d}s.$$

Let (28_1), (29_1) hold. If

$$\int_{t_1}^{\infty} q_+(s) f(x(g(s)))\,\mathrm{d}s = \infty,$$

then from (30) in view of the boundedness of $x(t)$ (> 0), (C_1), (C_2), (27), (28_1) we obtain $\lim_{t\to\infty} L_{n-1}z(t) = -\infty$. In view of (26) the last relation implies $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the fact that z(t) is boundet on $[t_0, \infty)$.

Therefore

$$(31) \qquad \int_{t_1}^{\infty} q_+(s) f(x(g(s)))\,\mathrm{d}s < \infty.$$

From (30) with regard to (C_1), (C_2), (28_1) and the boundedness of $x(t)$ we have $\liminf_{t\to\infty} x(t) = 0$.

In view of (27), (31) and (29_1), (30) implies that there exists a finite $\lim_{t\to\infty} L_{n-1}z(t)$. Now if we use (26) and the boundedness of $z(t)$, we have $\lim_{t\to\infty} L_k z(t) = 0$, $k = 1, 2, \ldots, n-1$.

Analogously we prove the result if (28_2), (29_2) hold. □

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