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ASYMPTOTIC BEHAVIOUR OF OSCILLATORY SOLUTIONS OF n-TH ORDER DIFFERENTIAL EQUATIONS WITH QUASIDERIVATIVES

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his 70^{th} *birthday*

Summary. Sufficient conditions are given under which the sequence of the absolute values of all local extremes of $y^{[i]}$, $i \in \{0, 1, \ldots, n-2\}$ of solutions of a differential equation with quasiderivatives $y^{[n]} = f(t, y^{[0]}, \ldots, y^{[n-1]})$ is increasing and tends to ∞. The existence of proper, oscillatory and unbounded solutions is proved.

MSC 1991: 34C10

I. INTRODUCTION

Consider a nonlinear differential equation

$$y^{[n]} = f(t, y^{[0]}, \ldots, y^{[n-1]}) \text{ in } D \tag{1}$$

where $n \geqslant 3$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $D = \mathbb{R}_+ \times \mathbb{R}^n$, $y^{[i]}$ is the i-th quasiderivative of y defined by

$$y^{[0]} = \frac{y}{a_0(t)},\ y^{[i]} = \frac{1}{a_i(t)}(y^{[i-1]})',\ i = 1, 2, \ldots, n-1,\ y^{[n]} = (y^{[n-1]})', \tag{2}$$

the functions $a_i\colon \mathbb{R}_+ \to (0, \infty)$ are continuous and $f\colon D \to \mathbb{R}$ fulfils the local Carathéodory conditions.

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Throughout the paper the sign hypothesis

$$f(t, x_1, \dots, x_n)x_1 \leqslant 0, \;\; f(t, 0, x_2, \dots, x_n) = 0 \text{ in } D \tag{3}$$

will be assumed.

Let $y\colon [0, b) \to \mathbb{R}$, $b \leqslant \infty$ be continuous, have the quasiderivatives up to the order $n-1$ and let $y^{[n-1]}$ be absolutely continuous. Then y is called a solution of (1) if (1) is valid for almost all $t \in [0, b)$.

A solution y is called non-continuable if either $b = \infty$ or

$$b < \infty \quad \text{and} \quad \limsup_{t \to b} \sum_{i=0}^{n-1} |y^{[i]}(t)| = \infty.$$

Let $y\colon [0, b) \to \mathbb{R}$ be a non-continuable solution of (1). It is called *proper* if $b = \infty$ and $\sup\limits_{\tau \leqslant t < \infty} |y(t)| > 0$ holds for an arbitrary number $\tau \in \mathbb{R}_+$. It is called *singular of the* 1-*st (*2-*nd) kind* if there exists $t^* \in \mathbb{R}_+$ such that $y(t) \equiv 0$ in $[t^*, \infty)$ and $\sup\limits_{\tau \leqslant t < t^*} |y(t)| > 0$ for $\tau \in [0, t^*)$ (if $b < \infty$). It is called *oscillatory* if there exists a sequence of its zeros tending to b (tending to t^*) if y is either proper or singular of the 2-nd kind (if y is singular of the 1-st kind).

A great effort has been exerted to the study of proper oscillatory solutions of the differential equation of the n-th order

$$y^{(n)} = f(t, y', \dots, y^{(n-1)}) \qquad \text{in } D \tag{4}$$

provided (3) is valid. The function $f\colon D \to \mathbb{R}$ is considered to fulfil the local Carathéodory conditions. Thus (4) is a special case of (1).

In the case $n = 2$ many authors have studied the following problem for proper oscillatory solutions for special types of (4) and (3): Let $\{\tau_k\}_1^\infty$ be the sequence of the absolute values of all local extremes of y (of y') on $\mathbb{R}_+$. When is $\{|y(\tau_k)|\}_1^\infty$ ($\{|y'(\tau_k)|\}_1^\infty$) monotone? The basic condition, among many others, is the monotonicity of f with respect to the independent variable, see Bihari [8], Belohorec [7], Das [10], Bobrowski [9], Foltyńska [11] and Bartušek [1].

In the case $n = 3$ the situation in this direction is described in [2]. The sequence $\{y'(\tau_k)|\}_1^\infty$ is increasing without any additional assumptions on f.

Similar situation occurs in the case $n = 4$, see [3, 4]. Without any additional assumptions on f the sequence $\{|y'(\tau_k)|\}_1^\infty$ is either increasing to ∞ or decreasing to zero for $k \to \infty$.

The first goal of the present paper consists in a generalization of the above mentioned results to equation (1): To establish the existence of a set of proper oscillatory solutions of (1) for which the sequence of the absolute values of all local extremes of $y^{[i]}$, $i \in \{0,1,2,\dots,n-2\}$ is increasing.

This problem is interesting not only by itself but may be used for studying the existence of proper oscillatory solutions of (2) with a given asymptotic behaviour. Thus the second goal consists in deriving sufficient conditions for the existence of a (proper) oscillatory solution y the quasiderivative $y^{[i]}$, $i \in \{0,1,\dots,n-3\}$ of which is unbounded. This problem has not yet been solved for (1) and new results will be obtained also for (4).

Notation. If $b_i \in C^0(I)$, then $I^0(t) \equiv 1$,

$$I^k(t,b_1,\dots,b_k) = \int_0^t b_1(s) I^{k-1}(s,b_2,\dots,b_k)\,\mathrm{d}s,\ t \in I.$$

Put $a_{nj+i}(t) = a_i(t)$, $j \in \{-1,0,1\}$, $i \in \{1,\dots,n-1\}$, $\mathbb{N} = \{1,2,\dots\}$.

We will assume the following hypotheses (not all simultaneously):

(H1) Let one of the following assumptions hold:

(a) Let either $a_1/a_2 \in C^1(\mathbb{R}_+)$ for $n=3$ or $a_1, a_2 \in C^1(\mathbb{R}_+)$, $\frac{a_3}{a_1} \in C^2(\mathbb{R}_+)$ for $n=4$ or for $n>4$ let there exist an index $l \in \{1,2,\dots,n-4\}$ such that a_{l+j}, $j=1,2$ are absolutely continuous and a'_{l+j}, $j=1,2$ are locally bounded from below;

(b)

$$|f(t,x_1,\dots,x_n)| \leqslant A(t) g(|x_1|) \text{ in } \mathbb{R}_+ \times [-\varepsilon,\varepsilon]^n, \tag{5}$$

where $\varepsilon > 0$, $A \in L_{\mathrm{loc}}(\mathbb{R}_+)$, $g \in C^0[0,\varepsilon]$, $g(0)=0$, $g(x)>0$ for $x>0$, $\int_0^\varepsilon \frac{\mathrm{d}t}{g(t)} = \infty$;

(H2) $|f(t,x_1,\dots,x_n)| \leqslant b(t)\omega\Big(\sum_{i=1}^n |x_i|\Big)$ in D, where $b \in L_{\mathrm{loc}}(\mathbb{R}_+)$, $\omega \in C^0(\mathbb{R}_+)$, $\omega(x) > 0$ for $x>0$, $\int_1^\infty \frac{\mathrm{d}t}{\omega(t)} = \infty$;

(H3) let

$$a_n(t) h(|x_1|) \leqslant |f(t,x_1,\dots,x_n)| \quad \text{in } D,$$

where $a_n \in L_{\mathrm{loc}}(\mathbb{R}_+)$, $a_n \geqslant 0$, $h \in C(\mathbb{R}_+)$, $h(0)=0$, $h(x)>0$ for $x>0$, h is nondecreasing, and let one of the following assumptions hold:

(i) $h(x) = x^\lambda$; $0<\lambda<1$,

$$\int_0^\infty a_{i+1}(s_{i+1}) \int_0^{s_{i+1}} a_{i+2}(s_{i+2}) \int_0^{s_{i+2}} \dots \int_0^{s_{n-1}} a_n(s_n) \Big[\int_0^{s_n} a_{n+1}(s_{n+1}) \dots \int_0^{s_{i+n-1}} a_{i+n}(s_{i+n})\,\mathrm{d}s_{i+n} \dots \mathrm{d}s_{n+1}\Big]^\lambda \mathrm{d}s_n \dots \mathrm{d}s_{i+1} = \infty,\ i=0,1,\dots,n-1;$$

(ii) $h(x) = x$, $\int_0^\infty a_i(t)\,\mathrm{d}t = \infty$, $i = 1, 2, \ldots, n-1$,

$$\limsup_{t\to\infty} I^1(a_{n-1}) \int_t^\infty \frac{I^{n-1}(s, a_1, \ldots, a_{n-1})}{I^1(s, a_{n-1})} a_n(s)\,\mathrm{d}s > 1;$$

(iii) $\int_0^\infty a_i(t)\,\mathrm{d}t = \infty$, $i = 1, 2, \ldots, n$.

(H4) $|f(t, x_1, \ldots, x_n)| \leqslant d(t) f_1(|x_1|)$ in D, where $d \in C^0(\mathbb{R}_+)$, $d \geqslant 0$, $f_1 > 0$ is nondecreasing in $(0, \infty)$, $f_1 \in C^0(\mathbb{R}_+)$.

In the sequel, a special type of (1) will be studied:

$$y^{[n]} = a_n(t) g(y^{[0]}), \tag{6}$$

where $a_n \in L_{\mathrm{loc}}(\mathbb{R}_+)$, $a_n \geqslant 0$, $g \in C^2(\mathbb{R})$, $g(0) = 0$, $g(x) < 0$ and $g(-x) = -g(x)$ for $x > 0$, $g' \leqslant 0$ in $\mathbb{R}$, $g'' \geqslant 0$ in $\mathbb{R}_+$, a_n is nondecreasing, $a_1 \in C^1(\mathbb{R}_+)$, $a_1' \geqslant 0$ in $\mathbb{R}_+$.

II. Asymptotic behaviour of the set of oscillatory solutions of (1)

In this section, properties of solutions of (1) that fulfil the Cauchy initial conditions

$$\begin{gathered} l \in \{0, 1, \ldots, n-1\},\ \sigma \in \{-1, 1\},\ \sigma y^{[i]}(0) > 0 \text{ for } i = 0, 1, \ldots, l-1, \\ \sigma y^{[l]}(0) \leqslant 0,\ \sigma y^{[j]}(0) < 0 \text{ for } j = l+1, \ldots, n-1 \end{gathered} \tag{7}$$

will be investigated.

The structure of oscillatory solutions that fulfil (7) is described in [6].

Definition. Let $y\colon J = [0, b) \to \mathbb{R}$; $b \leqslant \infty$ be a solution of (1) for which there exist a number $j \in \{0, 1, \ldots, n-1\}$ and sequences $\{t_k^i\}$, $\{\bar{t}_k^{n-1}\}$, $i \in \{0, 1, \ldots, n-1\}$, $k \in \{k_i, k_i + 1, \ldots\}$ such that

(i) $k_i = 0$ for $i \leqslant j$, $k_i = 1$ for $i > j$, $\lim_{k\to\infty} t_k^0 = b$;

(ii) $0 \leqslant t_{k-1}^0 < t_k^{n-1} \leqslant \bar{t}_k^{n-1} < t_k^{n-2} < \ldots < t_k^1 < t_k^0$;

$$\begin{gathered} y^{[i]}(t_k^i) = 0,\ i = 0, 1, \ldots, n-2,\ y^{[i]} \neq 0 \text{ otherwise in } J, \\ y^{[n-1]}(t) = 0 \text{ for } t \in [t_k^{n-1}, \bar{t}_k^{n-1}],\ y^{[n-1]} \neq 0 \text{ otherwise in } J; \end{gathered}$$

(iii) if we put $t_{-1}^0 = 0$, $t_0^i = 0$ for $i > j$ then

$$\begin{aligned} y^{[i]}(t) y^{[0]}(t) &> 0 \text{ for } t \in (t_{k-1}^0, t_k^i),\ i = 0, 1, \ldots, n-1, \\ &< 0 \text{ for } t \in (t_k^i, t_k^0),\ i = 0, 1, \ldots, n-2, \\ &< 0 \text{ for } t \in (\bar{t}_k^{n-1}, t_k^0),\ i = n-1, \end{aligned}$$

$k = 0, 1, \ldots$. The set of all such solutions will be denoted by $O(b)$.

Remark. It is evident that if $y \in O(\infty)$, then y is proper. If $y \in O(b)$, $b < \infty$ and y is non-continuable, then y is singular of the 2-nd kind.

A relation between solutions of (1) satisfying the Cauchy initial conditions (7) and the set $O(b)$ was described in [6].

Proposition 1. [6] *Let $y\colon [0,b) \to \mathbb{R}$ be a non-continuable oscillatory solution of* (1) *that fulfils* (7), *and let* (H1) *be valid.*

(i) *Then $y \in O(b)$ and y is either proper or singular of the* 2*-nd kind.*

(ii) *If* (H2) *is valid, then $b = \infty$ and y is proper.*

The following theorem states sufficient conditions under which proper oscillatory solutions $y \in O(\infty)$ exist.

Proposition 2. [6] *Let* (H1), (H2) *and* (H4) *hold. Then a solution y of* (1) *that fulfils the Cauchy initial conditions* (7), *is oscillatory, proper and $y \in O(\infty)$.*

Remark. Note that the assumptions (H3), (i) and (5) cannot be valid simultaneously.

Lemma 1. *Let $y\colon J = [t_1, t_2] \to \mathbb{R}$, $t_1 < t_2$ be a solution of* (1).

(a) *If $j \in \{1, 2, \dots, n\}$, $y^{[j]}(t) \geqslant 0$ ($\leqslant 0$) in J, then $y^{[j-1]}$ is nondecreasing (nonincreasing) in J.*

(b) *If $j \in \{1, 2, \dots, n\}$, $y^{[j]}(t) > 0$ (< 0) in J, then $y^{[j-1]}$ is increasing (decreasing) in J.*

(c) *If $y^{[0]}(t) \geqslant 0$ ($\leqslant 0$) in J, then $y^{[n-1]}$ is nonincreasing (nondecreasing) in J.*

Proof follows directly from (2), or see [6].

Lemma 2. *Let $y \in O(b)$, $b \leqslant \infty$, $i \in \{0, 1, \dots, n-2\}$, $a_{i+1} \in C^1(\mathbb{R}_+)$.*

(a) *If $a'_{i+1}(t) \geqslant 0$ in $\mathbb{R}_+$, then*

$$y^{[i]}(t)\operatorname{sgn} y^{[i]}(t_k^{i+1}) \text{ is concave in } \Delta_k = [t_k^{i+1}, t_k^{i-1}] \tag{8}$$

for $k \geqslant k_{i+1}$.

(b) *If $b < \infty$ and $i < n-2$, then there exists $\bar{k} \in \mathbb{N}$ such that* (8) *holds for $k \geqslant \bar{k}$.*

Proof. (a) Let $k \geqslant k_{i+1}$. Put $t_0 = t_k^{i+1}$, $t_1 = t_k^{i-1}$, $J = [0, b)$. Without loss of generality, suppose that $\operatorname{sgn} y^{[i]}(t_0) > 0$. The opposite case can be dealt with similarly.

It follows from $y \in O(b)$ and Lemma 2 that

$$\begin{gathered} y^{[i+1]}(t_0) = 0, y^{[i+1]}(t) < 0 \text{ in } (t_0, t_1], \\ y^{[i+2]}(t) \leqslant 0 \text{ in } \Delta_k, y^{[i+1]} \text{ is decreasing in } \Delta_k. \end{gathered} \tag{9}$$

Further using (2),

$$(10)\qquad (y^{[i]}(t))'' = (a_{i+1}(t)y^{[i+1]}(t))' = a_{i+1}(t)a_{i+2}(t)y^{[i+2]}(t) + a'_{i+1}(t)y^{[i+1]}(t).$$

Then according to (9), (10) we have $(y^{[i]}(t))'' \leqslant 0$ and the statement is valid in this case.

(b) Let $b < \infty$, $i \leqslant n-3$ be valid. By virtue of $y \in O(b)$ and Lemma 1

$$(11)\qquad y^{[i+3]}(t) < 0, y^{[i+2]} \text{ is nonincreasing in } \Delta_k.$$

If $a'_{i+1} \equiv 0$ in J, then the statement follows from (10) and (9). Let $\bar{k} \geqslant 2$ be such that

$$(12)\qquad 0 < b - t_{\bar{k}}^{i+1} \leqslant \frac{\min\limits_{s\in J} a_{i+1}(s)a_{i+2}(s)}{\max\limits_{s\in J} a_{i+2}(s) \max\limits_{r\in J} |a'_{i+1}(r)|}.$$

Consider $k \geqslant \bar{k}$. It follows from (11) that for $t \in \Delta_k$ we have

$$(13)\qquad \begin{aligned} 0 \leqslant -y^{[i+1]}(t) &= -y^{[i+1]}(t) + y^{[i+1]}(t_0) = -\int_{t_0}^{t} (y^{[i+1]}(t))'\,\mathrm{d}t \\ &= -\int_{t_0}^{t} a_{i+2}(t)y^{[i+2]}(t)\,\mathrm{d}t \leqslant -\max_{s\in J} a_{i+2}(s)(b - t_{\bar{k}}^{i+1})y^{[i+2]}(t). \end{aligned}$$

Finally, this together with (10), (9) and (12) implies

$$\begin{aligned} (y^{[i]}(t))'' &\leqslant a_{i+1}(t)a_{i+2}(t)y^{[i+2]}(t) - \max_{r\in J} |a'_{i+1}(r)| y^{[i+1]}(t) \\ &\leqslant y^{[i+2]}(t)[a_{i+1}(t)a_{i+2}(t) - \min_{s\in J} a_{i+1}(s)a_{i+2}(s)] \leqslant 0. \end{aligned}$$

This completes the proof. □

The following Kolmogorov-Horny type or Hardy inequality is very useful. The proof is similar to the case without quasiderivatives, see [11].

Lemma 3. *Let $\Delta = [t_1, t_2] \subset \mathbb{R}$, $t_1 < t_2$. Let $b_i > 0$, $i = 0, 1, \ldots, n$ and let Z be continuous such that the quasiderivatives $Z^{[i]}$ defined by*

$$Z^{[0]} = \frac{Z}{b_0},\ Z^{[i]} = \frac{1}{b_i(t)}(Z^{[i-1]})',\ i = 1, 2, \ldots, n$$

are continuous for $i = 1, \ldots, n-1$ and $Z^{[n]} \in L_{\mathrm{loc}}(\Delta)$. Suppose that $Z^{[i]}$, $i = 1, 2, \ldots, n-1$ have a zero in Δ and there exists a constant C such that

$$\max_{t\in\Delta} \frac{b_{i+1}(t)}{b_i(t)} \leqslant C,\ i = 1, 2, \ldots, n-1.$$

Denote

$$\nu_i = \max_{t\in\Delta}|Z^{[i]}(t)|,\ i = 0,1,\dots,n-1,\ \nu_n \geqslant |Z^{[n]}(t)|$$

a.e. *in* Δ. *Then*

$$\nu_i \leqslant (2\sqrt{C})^{i(n-i)}\nu_0^{\frac{n-i}{n}}\nu_n^{\frac{i}{n}},\ i = 0,1,\dots,n.$$

Proof. Let $i \in \{1,2,\dots,n-1\}$. Let $\Delta_1 \subset \Delta$, $\Delta_1 = [\tau,\tau_1]$ be the smallest interval such that

$$\max_{t\in\Delta_1}|Z^{[i]}(t)| = \nu_i,\ \min_{t\in\Delta_1}|Z^{[i]}(t)| = 0.$$

Then function $Z^{[i]}$ does not change its sign in Δ_1 and

$$\begin{aligned}\nu_i^2 &= 2\int_{\Delta_1}|Z^{[i]}(t)(Z^{[i]}(t))'|\,\mathrm{d}t = 2\int_{\Delta_1}|Z^{[i]}(t)Z^{[i+1]}(t)b_{i+1}(t)|\,\mathrm{d}t\\ &\leqslant 2\nu_{i+1}\int_{\Delta_1}\frac{b_{i+1}(t)}{b_i(t)}|(Z^{[i-1]}(t))'|\,\mathrm{d}t \leqslant 4C\nu_{i+1}\nu_{i-1}.\end{aligned}$$

From this and using the mathematical induction we can easily prove that

$$\nu_i \leqslant (2\sqrt{C})^i\nu_0^{\frac{1}{i+1}}\nu_{i+1}^{\frac{i}{i+1}},\ i = 1,2,\dots,n-1,$$

$$\nu_i \leqslant (2\sqrt{C})^{i(n-i)}\nu_0^{\frac{n-i}{n}}\nu_n^{i/n},\ i = 1,\dots,n-1.$$

□

Lemma 4. *Let* $y \in O(b)$, $b \leqslant \infty$, $i \in \{1,2,\dots,n-2\}$, $k \in \{k_{i+1},k_{i+1}+1,\dots\}$, *let* $y^{[i]}(t)\operatorname{sgn}y^{[i]}(t_k^{i+1})$ *be concave in* $[t_k^{i+1},t_k^{i-1}]$. *Let either* $\beta \in [1,2]$, $\gamma = \frac{\beta}{2}$ *and* $y^{[i+1]}(t)\operatorname{sgn}y^{[i+1]}(t_k^{i+2})$ *is concave in* $\Delta_1 = [t_k^{i+1},t_k^i]$ *or* $\beta = \gamma = 1$. *If*

$$\gamma \leqslant \frac{\min\limits_{t\in\Delta_1}a_i(t)}{\max\limits_{t\in\Delta_2}a_i(t)}\cdot\frac{\min\limits_{t\in\Delta_2}a_{i+1}(t)}{\max\limits_{t\in\Delta_1}a_{i+1}(t)},\ \Delta_2 = [t_k^i,t_k^{i-1}], \tag{14}$$

then $\sqrt{\beta}|y^{[i]}(t_k^{i+1})| < |y^{[i]}(t_{k+1}^{i+1})|$.

Proof. Put $t_0 = t_k^{i+1}$, $t_1 = t_k^i$, $t_2 = t_k^{i-1}$, $\overline{\Delta}_1 = t_0 - t_1$, $\overline{\Delta}_2 = t_2 - t_0$, $\delta_1 = (t_0,t_1)$, $\delta_2 = (t_1,t_2)$ and suppose, without loss of generality, that $y^{[i]}(t_0) > 0$. Then $y \in O(b)$ and Lemma 1 yields

$$\begin{cases} y^{[i-1]}(t) > 0 \text{ in } \Delta_1\cup\delta_2, y^{[i-1]}(t_2) = 0,\\ y^{[i-1]} \text{ is increasing (decreasing) in } \Delta_1 \text{ (in } \Delta_2),\\ y^{[i]}(t) > 0\ (<0) \text{ in } \delta_1 \text{ (in } \delta_2),\ y^{[i]}(t_1) = 0,\\ y^{[i+1]}(t_0) = 0, y^{[i+1]}(t) < 0 \text{ in } \delta_1\cup\Delta_2,\\ y^{[i]} \text{ and } y^{[i+1]} \text{ are decreasing in } \Delta_1\cup\Delta_2.\end{cases} \tag{15}$$

The statement will be proved indirectly. Thus, suppose that $\sqrt{\beta}y^{[i]}(t_0) \geqslant |y^{[i]}(t_{k+1}^{i+1})|$. As $y \in O(b)$, we get

$$\sqrt{\beta}y^{[i]}(t_0) > |y^{[i]}(t_2)|. \tag{16}$$

As $y^{[i]}$ is concave in Δ_1, $y^{[i]}$ is above the secant going through the points $P_1 = [t_0, y^{[i]}(t_0)]$ and $P_2 = [t_1, 0]$. Thus $\int_{\Delta_1} y^{[i]}(t)\,dt \geqslant P$ where $P = \frac{\bar{\Delta}_1}{2} y^{[i]}(t_0)$ is the area of the triangle $P_1P_2P_3$, $P_3 = [t_0, 0]$. Similarly, it follows from (15) and from the concavity of $y^{[i]}$ in Δ_2 that $\int_{\Delta_2} |y^{[i]}(t)|\,dt \leqslant \bar{P}$, where $\bar{P}$ is the area of the triangle $[t_1, 0]$, $[t_2, 0]$, $[t_2, |y^{[i]}(t_2)|]$, $\bar{P} = \frac{\bar{\Delta}_2}{2}|y^{[i]}(t_2)|$. From this and from (15) we have

$$y^{[i-1]}(t_1) = -\int_{\Delta_2} (y^{[i-1]}(t))'\,dt = -\int_{\Delta_2} a_i(t) y^{[i]}(t)\,dt \leqslant \frac{\bar{\Delta}_2}{2}|y^{[i]}(t_2)| \max_{s\in\Delta_2} a_i(s),$$
$$y^{[i-1]}(t_1) > y^{[i-1]}(t_1) - y^{[i-1]}(t_0) = \int_{\Delta_1} a_i(t) y^{[i]}(t)\,dt \geqslant \frac{\bar{\Delta}_1}{2} y^{[i]}(t_0) \min_{s\in\Delta_1} a_i(s).$$

Thus, combining these two results and (16), we obtain

$$\frac{\bar{\Delta}_2}{\bar{\Delta}_1} > \frac{1}{\sqrt{\beta}} \frac{\min\limits_{s\in\Delta_1} a_i(s)}{\max\limits_{s\in\Delta_2} a_i(s)}. \tag{17}$$

Further, (15) yields

$$\begin{aligned} |y^{[i]}(t_2)| &= -\int_{\Delta_2} (y^{[i]}(t))'\,dt \\ &= -\int_{\Delta_2} a_{i+1}(t) y^{[i+1]}(t)\,dt \geqslant \bar{\Delta}_2 |y^{i+1]}(t_1)| \min_{s\in\Delta_2} a_{i+1}(s). \end{aligned} \tag{18}$$

Let $\beta = \gamma = 1$. Then, similarly,

$$y^{[i]}(t_0) = -\int_{\Delta_1} a_{i+1}(t) y^{[i+1]}(t)\,dt \leqslant \bar{\Delta}_1 |y^{[i+1]}(t_1)| \max_{i\in\Delta_1} a_{i+1}$$

and according to (18), (17) and (16)

$$1 > \frac{|y^{[i]}(t_2)|}{y^{[i]}(t_0)} \geqslant \frac{\min\limits_{s\in\Delta_1} a_i(s) \min\limits_{s\in\Delta_2} a_{i+1}(s)}{\max\limits_{s\in\Delta_2} a_i(s) \max\limits_{s\in\Delta_1} a_{i+1}(s)}.$$

The contradiction to (14) proves the statement in this case.

Let $\beta \in [1,2]$, $\gamma = \frac{\beta}{2}$ and let $y^{[i+1]}(t)$ sgn $y^{[i+1]}(t)$ sgn (t_k^{i+2}) be concave in Δ_1. Then, by means of (15) and by $y \in O(b)$ we have $y^{[i+1]}(t_k^{i+2}) > 0$ and thus $y^{[i+1]}$ is concave in Δ_1. From this and from (15) we conclude that

$$y^{[i]}(t_0) = -\int_{\Delta_1} a_{i+1}(t)y^{[i+1]}(t)\,\mathrm{d}t \leqslant \frac{\overline{\Delta}_1}{2}|y^{[i+1]}(t_1)| \max_{s\in\Delta_1} a_{i+1}(s)$$

holds which, together with (18), gives

$$\begin{aligned}\sqrt{\beta} > \frac{|y^{[i]}(t_2)|}{y^{[i]}(t_0)} &\geqslant \frac{2\overline{\Delta}_2}{\overline{\Delta}_1}\frac{\min_{s\in\Delta_2} a_{i+1}(s)}{\max_{s\in\Delta_1} a_{i+1}(s)}\\ &\geqslant \frac{2}{\sqrt{\beta}}\frac{\min_{s\in\Delta_2} a_{i+1}(s)}{\max_{s\in\Delta_1} a_{i+1}(s)}\frac{\min_{s\in\Delta_1} a_i(s)}{\max_{s\in\Delta_2} a_i(s)}.\end{aligned} \tag{19}$$

This inequality contradicts (14). □

In the sequel, proper oscillatory solutions $y \in O(\infty)$ will be studied. Proposition 2 gives criteria for the existence of such solutions.

Theorem 1. *Let $y \in O(\infty)$, $i \in \{1,\dots,n-2\}$, $a_{i+1} \in C^1(\mathbb{R}_+)$, $a'_{i+1}(t) \geqslant 0$ in $\mathbb{R}_+$.*

(a) *If a_i is nonincreasing in $\mathbb{R}_+$, then the sequence $\{|y^{[i]}(t_k^{i+1})|\}$, $k \in \{k_{i+1}, k_{i+1}+1,\dots\}$ of all local extremes of $y^{[i]}$ in $\mathbb{R}_+$ is increasing.*

(b) *If $i \leqslant n-3$, $a'_{i+2}(t) \geqslant 0$ and*

$$\frac{\limsup_{t\to\infty} a_i(t)}{\liminf_{t\to\infty} a_i(t)} < 2, \tag{20}$$

then there exists $\bar{k}$ such that $\{|y^{[i]}(t_k^{i+1})|\}$, $k \geqslant \bar{k}$ is increasing.

P r o o f. It is evident that the assumptions of Lemma 3, with the exception of (14), are fulfilled (use Lemma 1, too).

(a) The statement follows directly from Lemma 4 for $\beta = \gamma = 1$.

(b) Put $\beta = 1$, $\gamma = \frac{1}{2}$ in Lemma 4; according to (20) there exists $\bar{k}$ such that

$$\frac{\min_{t\in\Delta_2} a_{i+1}(t)\min_{t\in\Delta_1} a_i(t)}{\max_{t\in\Delta_1} a_{i+1}(t)\max_{t\in\Delta_2} a_i(t)} = \frac{\min_{t\in\Delta_2} a_i(t)}{\max_{t\in\Delta_1} a_i(t)} \geqslant \frac{1}{2}, \quad t \geqslant t_{\bar{k}}^i$$

and thus (14) is fulfilled. □

Theorem 2. *Let $y \in O(\infty)$, $i \in \{1, \ldots, n-3\}$, $a_{i+1} \in C^1(\mathbb{R}_+)$, $a_{i+2} \in C^1(\mathbb{R}_+)$, $a'_{i+1}(t) \geqslant 0$, $a'_{i+2}(t) \geqslant 0$ in $\mathbb{R}_+$. Let either a_i be nonincreasing in $\mathbb{R}_+$ or let $\lim\limits_{t\to\infty} a_i = A$, $0 < A < \infty$ hold. Then $y^{[i]}$ is unbounded.*

Proof. The assumptions of Lemmas 2 and 4 are fulfilled.

(a) Let a_i be nonincreasing. If $\beta = 2$, $\gamma = 1$ is chosen, then (14) is valid and

$$\sqrt{2}|y^{[i]}(t_k^{i+1})| < |y^{[i]}(t_{k+1}^{i+1})| \tag{21}$$

holds. Thus

$$\lim_{k\to\infty} |y^{[i]}(t_k^{i+1})| = \infty. \tag{22}$$

(b) Let $\lim\limits_{t\to\infty} a_i = A$ exist, $0 < A < \infty$ and let $\varepsilon > 0$, $0 < \beta = 2 - \varepsilon$, $\gamma = 1 - \varepsilon/2$. Then there exists $\bar{k}$ such that

$$\min_{t\in\Delta_1} a_i(t) / \max_{t\in\Delta_2} a_i(t) \geqslant 1 - \varepsilon/2, \ k \geqslant \bar{k}$$

and (14) evidently holds. Thus

$$\sqrt{2-\varepsilon}|y^{[i]}(t_k^{i+1})| < |y^{[i]}(t_{k+1}^{i+1})|, \tag{23}$$

and (22) is valid. □

Remark. (a) Note that, under the assumptions of Theorem 2, the statement of Theorem 1 is valid and thus the sequence $\{|y^{[i]}(t_k^{i+1})|\}$ of the local extremes of $|y^{[i]}|$ is increasing for all admissible k.

(b) The inequalities (21) and (23) give an estimate from below for the speed of the increase of the sequence of all absolute values of local extremes of $y^{[i]}$.

The following results are consequences of the above Theorems 1 and 2 and Lemma 3 and give sufficient conditions for the existence of unbounded proper oscillatory solutions of (1).

Theorem 3. *Let $i \in \{1, \ldots, n-2\}$, $a_{i+1} \in C^1(\mathbb{R}_+)$, $a'_{i+1}(t) \geqslant 0$ in $\mathbb{R}_+$ and let either a_i be nonincreasing in $\mathbb{R}_+$ or $i \leqslant n-3$, $a'_{i+2}(t) \geqslant 0$ and (20) be valid. Let*

$$\frac{a_{j+1}(t)}{a_j(t)} \leqslant C < \infty, \ j = 1, 2, \ldots, n-2 \text{ in } \mathbb{R}_+, \tag{24}$$

(H1) and (H4) hold with $d(t) = a_{n-1}(t)$. Then every proper oscillatory solution of (1) fulfilling the Cauchy initial conditions (7) does not tend to zero for $t \to \infty$.

If, moreover, (H2) *and* (H3) *hold then every solution of* (1) *fulfilling* (7) *is proper oscillatory and does not tend to zero for $t \to \infty$.*

Proof. Let y be a proper oscillatory solution of (1) for which (7) holds. According to Proposition 1, $y \in O(\infty)$ and the statement of Theorem 1 holds. Thus there exist k_0 and C_3 such that

$$|y^{[i]}(t_k^{i+1})| \geqslant C_3 > 0, \ k \geqslant k_0. \tag{25}$$

Consider the differential equation equivalent to (1)

$$\frac{1}{a_{n-1}(t)} y^{[n]}(t) = \frac{1}{a_{n-1}(t)} f(t, y^{[0]}, \dots, y^{[n-1]}).$$

As the assumptions of Lemma 3 are fulfilled in $\Delta_k = [0, t_k^{i+1}]$ with

$$\nu_n = \max_{t \in \Delta_k} f_1(|y(t)|),$$

we get, using (25),

$$0 < C_3 \leqslant \nu_i \leqslant K \nu_0^{\frac{n-i}{n}} \nu_n^{\frac{i}{n}} \leqslant K \nu_0^{\frac{n-i}{n}} [\max_{t \in \Delta_k} f_1(|y(t)|)]^{\frac{i}{n}}.$$

The statement follows from the assumption that $f_1 > 0$ is nondecreasing in Δ_k, $k \geqslant k_0$. The rest of the statements of the theorem follow from Proposition 2.

Theorem 4. *Let $n \geqslant 4$, $i \in \{1, \dots, n-3\}$, $a_{i+j} \in C^1(\mathbb{R}_+)$, $a'_{i+j}(t) \geqslant 0$ in $\mathbb{R}_+$, $j = 1, 2$, $\lim\limits_{t \to \infty} a_i = A$, $0 < A < \infty$ hold. Further, let* (24), (H1) *and* (H4) *be valid with $d(t) = a_{n-1}(t)$. Then every proper oscillatory solution of* (1) *fulfilling* (7) *is unbounded in $\mathbb{R}_+$. If, moreover,* (H2) *and* (H3) *hold then every solution of* (1) *fulfilling* (7) *is proper, oscillatory and unbounded in $\mathbb{R}_+$.*

Proof is similar as that of the previous theorem, only Th. 2 must be used instead of Th. 1. □

Let us turn our attention to oscillatory singular solutions of the 2-nd kind from the set $O(b)$, $b < \infty$. According to Proposition 1 such solutions may exist if (H1) and (7) hold.

Theorem 5. *Let $y \in O(b)$, $b < \infty$, $i \in \{1, 2, \dots, n-2\}$, $a_{i+1} \in C^1(\mathbb{R}_+)$, $a_{i+2} \in C^1(\mathbb{R}_+)$.*

(a) *Let either* $i \in \{1, 2, \dots, n-4\}$ *or* $i = n-3$ *and* $a'_{n-1}(t) \geqslant 0$ *in* $\mathbb{R}_+$. *Then* y *is oscillatory singular of the 2-nd kind and there exists* $\bar{k}$ *such that the sequence* $\{|y^{[i]}(t_k^{i+1})|\}$, $k \geqslant \bar{k}$ *is increasing and* $\lim_{k\to\infty} |y^{[i]}(t_k^{i+1})| = \infty$.

(b) *Let either* $i = n-3$ *or* $i = n-2$ *and let* a_i *be nonincreasing,* $a'_{i+1} \geqslant 0$ *in* $\mathbb{R}_+$. *Then* y *is oscillatory singular of the 2-nd kind and there exists* $\bar{k}$ *such that* $\{|y^{[i]}(t_k^{i+1})|\}$, $k \geqslant \bar{k}$ *is increasing.*

Proof. It is evident that the assumptions of Lemma 4, with the exception of (14), are fulfilled for $\varepsilon > 0$, $0 < \beta = 2 - \varepsilon$, $\gamma = \beta/2$ (for $\beta = \gamma = 1$) in case (a) (in case (b)) (use Lemma 2, too). The validity of (14):

Case (a): As $b < \infty$, there exists $\bar{k}$ such that for $k \geqslant \bar{k}$

$$\frac{\min\limits_{t\in\Delta_1} a_i(t) \min\limits_{t\in\Delta_2} a_{i+1}(t)}{\max\limits_{t\in\Delta_2} a_i(t) \max\limits_{t\in\Delta_1} a_{i+1}(t)} \geqslant \frac{\min\limits_{t\in[\tau,b)} a_i(t)a_{i+1}(t)}{\max\limits_{t\in[\tau,b)} a_i(t)a_{i+1}(t)} \geqslant 1 - \varepsilon/2 = \gamma$$

holds where $\tau = t_{\bar{k}}^{i+1}$. Thus (14) is valid.

Case (b): (14) directly follows from the assumptions posed on a_i, a_{i+1}. The statement follows from Lemma 4; as $|y^{[i]}(t_k^{i+1})|$ is increasing and $\lim_{k\to\infty} t_k^{i+1} = b < \infty$, y must be singular of the 2-nd kind. □

Theorem 6. *Let* $y \in O(b)$, $b < \infty$, *let* (H4) *be valid,* $a_{i+j} \in C^1(\mathbb{R}_+)$, $j = 1, 2$ *and let either* $i \in \{1, 2, \dots, n-4\}$ *or* $i = n-3 \geqslant 1$ *and* $a'_{n-1}(t) \geqslant 0$ *in* $\mathbb{R}_+$. *Then* y *is oscillatory singular of the 2-nd kind and unbounded in* $[0, b)$.

Proof is similar to that of Theorem 3. □

II. Asymptotic behaviour of solutions of (4) and (6)

In Section I sufficient conditions are given for the sequence of the absolute values of local extremes of $y^{[i]}$, $i \geqslant 1$ to be increasing. The same result for $y^{[0]}$ is stated in this section but for equation (6) only.

Lemma 5. *Let* $y \in O(b)$, $b \leqslant \infty$ *be a solution of* (6). *Then* $g(y^{[0]}(t))$ *sgn* $y^{[0]}(t_k^1)$ *is convex in* $\Delta_k = [t_k^1, t_k^0]$ *for* $k \in \{k_1, k_1 + 1, \dots\}$.

Proof. Put $t_0 = t_k^1$, $t_1 = t_k^0$, $J = [0, b)$. Without loss of generality, suppose that sgn $y^{[0]}(t_0) > 0$.

It follows from $y \in O(b)$ and Lemma 1 that $y^{[0]} \geqslant 0$, $y^{[1]}(t_0) = 0$, $y^{[1]}(t) < 0$ in $(t_0, t_1]$, $y^{[2]} < 0$ in Δ_k. Then (2) and the assumptions concerning function g yield

$$\begin{aligned}(g(y^{[0]}(t)))'' &= g''(y^{[0]}(t))[a_1(t)y^{[1]}(t)]^2 \\ &\quad + g'(y^{[0]}(t))[a_1'(t)y^{[1]}(t) + a_2(t)a_1(t)y^{[2]}(t)] \geqslant 0.\end{aligned}$$

□

Theorem 7. *Let $y \in O(b)$, $b \leqslant \infty$ be a solution of (6). Let either $a_2'(t) \geqslant 0$ in $\mathbb{R}_+$ or $b < \infty$, $n \geqslant 4$ be satisfied. Then there exists $\bar{k}$ such that*

$$|y^{[0]}(t_k^1)| < |y^{[0]}(t_{k+1}^1)|, \quad k \geqslant \bar{k}.$$

Thus, y is proper (singular of the 2-nd kind) if $b = \infty$ (if $b < \infty$). Moreover, in the case $a_2' \geqslant 0$ we can put $\bar{k} = k_1$.

Proof. According to Lemma 7 the function $g(y^{[0]}(t))$ sgn $y^{[0]}(t_k^1)$ is convex in $[t_k^1, t_k^0]$ for $k \geqslant k_1$. Similarly, it follows from Lemma 2 that there exists $\bar{k}$ such that $y^{[1]}(t)$ sgn $y^{[1]}(t_k^2)$ is concave in $[t_k^1, t_k^0]$ for $k \geqslant \bar{k}$; at the same time, we can put $\bar{k} = k_1$ if $a_2' \geqslant 0$.

Let $k \geqslant \bar{k}$. Put $\Delta_1 = [t_k^1, t_k^0]$, $t_0 = t_k^1$, $t_1 = t_k^0$, $t_2 = t_{k+1}^{n-1}$, $\delta_1 = (t_k^1, t_k^0)$, $\delta_2 = (t_k^0, t_{k+1}^{n-1})$, $\Delta_2 = [t_k^0, t_{k+1}^{n-1}]$, $\overline{\Delta}_1 = t_k^0 - t_k^1$, $\overline{\Delta}_2 = t_{k+1}^{n-1} - t_k^0$. Without loss of generality, suppose that $y^{[0]}(t_k^1) > 0$. Then, according to $y \in O(b)$ and Lemma 1, we have

$$\tag{26} \begin{cases} y^{[0]}(t) > 0 \ (< 0) \text{ in } \delta_1 \ (\text{in } \delta_2), y^{[0]}(t_1) = 0, \\ y^{[1]}(t_0) = 0, y^{[1]}(t) < 0 \text{ in } \delta_1 \cup \Delta_2, \\ y^{[0]}, y^{[1]} \text{ are decreasing in } \Delta_1 \cup \Delta_2, \\ y^{[n-1]}(t) < 0 \text{ in } \Delta_1 \cup \delta_2, y^{[n-1]}(t_2) = 0, \\ y^{[n-1]} \text{ is nonincreasing (nondecreasing) in } \Delta_1 \ (\text{in } \Delta_2). \end{cases}$$

The statement will be proved indirectly. Thus, let us suppose that $|y^{[0]}(t_1)| \geqslant |y^{[0]}(t_{k+1}^1)|$. Then (26) and $y \in O(b)$ yield

$$\tag{27} y^{[0]}(t_1) > |y^{[0]}(t_2)|.$$

It follows from (26) that

$$\begin{aligned}|y^{[n-1]}(t_1)| = y^{[n-1]}(t_2) - y^{[n-1]}(t_1) = \int_{\Delta_2} y^{[n]}(t)\,\mathrm{d}t = \int_{\Delta_2} a_n(t)g(y^{[0]}(t)) \\ \leqslant \overline{\Delta}_2 g(y^{[0]}(t_2))a_n(t_1)\end{aligned}$$

holds. As $g(y^{[0]}(t))$ is convex in Δ_1, we have

$$|y^{[n-1]}(t_1)| > -y^{[n-1]}(t_1) + y^{[n-1]}(t_0) = -\int_{\Delta_1} y^{[n]}(t)\,\mathrm{d}t = -\int_{\Delta_1} a(t)g(y^{[0]}(t))\,\mathrm{d}t$$
$$\geqslant \frac{\overline{\Delta}_1}{2}|g(y^{[0]}(t_0))|a_n(t_1).$$

Thus, according to (27) and $g(x) = -g(-x)$ we conclude

$$\frac{\overline{\Delta}_2}{\overline{\Delta}_1} > \frac{1}{2}\frac{|g(y^{[0]}(t_0))|}{g(y^{[0]}(t_2))} = \frac{1}{2}\frac{|g(|y^{[0]}(t_0)|)|}{|g(|y^{[0]}(t_2)|)|} \geqslant \frac{1}{2}. \tag{28}$$

In the same way as in Lemma 4 the inequality

$$1 > 2\frac{\overline{\Delta}_2}{\overline{\Delta}_1}\frac{\min\limits_{s\in\Delta_2} a_1(s)}{\max\limits_{s\in\Delta_1} a_1(s)} = 2\frac{\overline{\Delta}_2}{\overline{\Delta}_1} \tag{29}$$

can be proved, see the first two inequalities (19), $\beta = 1$. The contradiction of (28) to (29) proves the statement. □

Theorem 8. *Let $a_2' \geqslant 0$. Let $\frac{a_1}{a_2} \in C^1(\mathbb{R}_+)$ ($\frac{a_3}{a_1} \in C^2(\mathbb{R}_+)$) if $n = 3$ ($n = 4$) holds. Let* (H3) *hold where $f(t, x_1, \ldots, x_n) \equiv a_n(t)g(x_1)$. Then every solution y of* (6) *satisfying the Cauchy initial conditions* (7) *is oscillatory proper. Moreover, the sequence of the absolute values of all local extremes of $y^{[0]}$ in $\mathbb{R}_+$ is increasing.*

Proof. The statement is a consequence of Proposition 2 and Theorem 7 because hypotheses (H1) and (H2) are valid. □

Remark. Sufficient conditions, under which the sequence of the absolute values of local extremes tends to ∞, can be obtained from Theorem 4.

In the rest of this section some consequences of Theorems 1, 2 and 4 for equation (4) are given.

Corollary 1. *Let y be an oscillatory solution of* (4) *that fulfils* (7). *Then*

(a) *the sequence of the absolute values of all local extremes of $y^{(i)}$ is increasing for $i \in \{1, 2, \ldots, n-2\}$;*

(b) *$y^{(j)}$, $j = 1, 2, \ldots, n-3$ are unbounded;*

(c) *y is unbounded if* (H4) *holds with $d \equiv 1$.*

Corollary 2. *Let y be a solution of* (4) *that fulfils* (7). *Let there exist functions $\omega \in C^0(\mathbb{R}_+)$, $a \in L_{\mathrm{loc}}(\mathbb{R}_+)$ and $h \in C^0(\mathbb{R}_+)$ such that $\omega > 0$ in $(0, \infty)$, $\int_1^\infty \frac{\mathrm{d}t}{\omega(t)} = \infty$, $a \geqslant 0$, $h > 0$ in $(0, \infty)$, h is non-decreasing and*

$$a(t)h(|x_1|) \leqslant |f(t, x_1, \ldots, x_n)| \leqslant \omega(|x_1|).$$

Further, let one of the following assumptions hold:

(i) $h(x) = x^\lambda$, $0 < \lambda < 1$, $\int_0^\infty t^{(n-1)\lambda} a(t)\,\mathrm{d}t = \infty$;

(ii) $h(x) = x$, $\limsup_{t\to\infty} t \int_t^\infty t^{n-2} a(t)\,\mathrm{d}t > 1$;

(iii) $\int_0^\infty a(t)\,\mathrm{d}t = \infty$.

Then y is proper, oscillatory and unbounded.

Remark. The results of Corollary 1, (a), (b) and of Theorem 7 are new, even for the linear equation

$$y^{(n)} = a(t)y, \quad a \leqslant 0. \tag{30}$$

If either n is odd or the integer part of $n/2$ is odd, then Corollary 2 generalizes results concerning the existence of proper, oscillatory and unbounded solutions of (4) obtained in [5] and [13] (for the linear case (30)).

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