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CONGRUENCES AND IDEALS IN TERNARY RINGS

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Summary. A ternary ring is an algebraic structure $\mathcal{R} = (R; t, 0, 1)$ of type $(3, 0, 0)$ satisfying the identities $t(0, x, y) = y = t(x, 0, y)$ and $t(1, x, 0) = x = (x, 1, 0)$ where, moreover, for any a, b, $c \in R$ there exists a unique $d \in R$ with $t(a, b, d) = c$. A congruence θ on $\mathcal{R}$ is called normal if $\mathcal{R}/\theta$ is a ternary ring again. We describe basic properties of the lattice of all normal congruences on $\mathcal{R}$ and establish connections between ideals (introduced earlier by the third author) and congruence kernels.

Keywords: ternary ring, ideal, congruence, normal congruence, congruence kernel

MSC 1991: 13A15, 08A30

The concept of a ternary field was introduced by M. Hall [5] under a different name and used for the so called coordinatization of projective planes, see [5], [10]. It was generalized to a ternary ring by the third author, see [7]. It forms an algebraic tool for a classification of the so called Klingenberg planes which generalize projective planes, see [7], [8] and [9] for more detail. In these costructions we search for a suitable factorization of the assigned ternary ring. This factorization can be done either by an ideal or a congruence. However, the mutual relationship between these two concepts has not yet been investigated. Moreover, only a little is known on the congruence lattice of a ternary ring. For a bit more complex structure, the so called bi-ternary ring, the ideal theory in the sense of H.-P.Gumm and A.Ursini [4], [11] was already settled by the first two authors in [3]; for the reduct called a semiloop it was done in [2].

Our object is to classify congruences in ternary rings, to describe the congruence lattice and to give a mutual relationship between ideals and congruences for ternary rings.

Definition 1. By a ternary ring we mean an $\mathcal{R} = (R, t, 0, 1)$ of type $(3, 0, 0)$ satisfying the identities

$$t(0, x, y) = y = t(x, 0, y), \tag{1}$$

$$t(1, x, 0) = x = t(x, 1, 0), \tag{1'}$$

where for every a, b, c of R there exists a unique element $c \in R$ such that

$$t(a, b, d) = c. \tag{$*$}$$

Lemma 1. *A ternary ring $R = (R, t, 0, 1)$ is a one element algebra if and only if $0 = 1$.*

Proof. Suppose $0 = 1$ and $x \in R$. By (1), we have $t(0, x, 0) = 0$ and, by (1′), $t(0, x, 0) = x$, thus R is a singleton. The converse assertion is trivial. □

Definition 2. An equivalence θ on R is a congruence of a ternary ring $\mathcal{R} = (R; t, 0, 1)$ if it has the substitution property with respect to t, i.e. if $a_i \theta b_i$ for $i = 1, 2, 3$ implies $t(a_1, a_2, a_3) \theta t(b_1, b_2, b_3)$. A congruence θ on $\mathcal{R}$ is called normal if for each a_1, a_2, b_1, b_2, x, y of R, if $a_1 \theta b_1$, $a_2 \theta b_2$ and $t(a_1, a_2, x) \theta t(b_1, b_2, y)$ then also $x \theta y$.

From now on let ω denote the identical relation and ι the full relation on R, i.e. $\iota = R \times R$ and $x \omega y$ if $x = y$. Clearly, ω and ι are normal congruences on a ternary ring $\mathcal{R}$. Denote by $\operatorname{Con} \mathcal{R}$ the congruence lattice of $\mathcal{R}$ and by $\operatorname{Con}_N \mathcal{R}$ the set of all normal congruences on $\mathcal{R}$. Trivially, ω is the least and ι the greatest element of $\operatorname{Con} \mathcal{R}$.

If $a \in R$ and $\Phi \in Con\mathcal{R}$, denote by $[a]_\Phi$ the congruence class of Φ containing a. Introduce a ternary operation t_Φ in the factor set R/Φ as follows:

$$t_\Phi([a]_\Phi, [b]_\Phi, [c]_\Phi) = [d]_\Phi$$

if $t(a, b, c) = d'$ for some $d' \in [d]_\Phi$.

Theorem 1. *Let $\mathcal{R} = (R; t, 1, 0)$ be a ternary ring and $\Phi \in Con\mathcal{R}$. Then $\mathcal{R}/\Phi = (R/\Phi; t_\Phi, [0]_\Phi, [1]_\Phi)$ is a ternary ring if and only if Φ is normal.*

Proof. Let $\mathcal{R}/\Phi = (R/\Phi; t_\Phi, [0]_\Phi, [1]_\Phi)$ be a ternary ring and $[a]_\Phi$, $[b]_\Phi$, $[c]_\Phi \in \mathcal{R}/\Phi$. Then there exists a unique $[d]_{Phi} \in \mathcal{R}/\Phi$ with

$$t_\Phi([a]_\Phi, [b]_\Phi, [c]_\Phi) = [d]_\Phi. \tag{$**$}$$

If $a_1, b_1 \in [a]_\Phi$, $a_2, b_2 \in [b]_\Phi$ and $t(a_1, a_2, x), t(b_1, b_2, y) \in [c]_\Phi$ for some $x, , y \in R$ then, by (**), also $x, y \in [d]_\Phi$. Hence $a_1 \Phi b_1$, $a_2 \Phi b_2$ and $t(a_1, a_2, x) \Phi t(b_1, b_2, y)$ imply $x \Phi y$, thus Φ is normal.

Conversely, if $\Phi \in Con\mathcal{R}$ in normal then (**) is clearly satisfied and hence $\mathcal{R}/\Phi = (R/\Phi; t_\Phi, [0]_\Phi, [0]_\Phi)$ is a ternary ring again. □

Theorem 2. *Let $\mathcal{R} = (R; t, 1, 0)$ be a ternary ring, $\theta \in Con\mathcal{R}$ and let the factor set $CalR/\theta$ be finite. Then θ is normal.*

Proof. Consider the natural mapping $h\colon R \to R/\theta$ given by $h(a) = [a]_\theta$. Trivially, h is a homomorphism of $\mathcal{R}$ onto an algebra $\mathcal{R}/\theta$ with one ternary and two nullary operations $t_\theta, [0]_\theta, [1]_\theta$ satisfying (1) and (1′). Let us consider the mappings $f_{ab}\colon R/\theta \to R/\theta$ defined as follows:

$$f_{ab}(h(x)) = t_\theta(h(a), h(b), h(x)) \quad \text{for each } a, b, x \text{ of } R.$$

These mappings are surjective. Namely, if $h(c) \in R/\theta$ then $t\big(h(a), h(b), h(x)\big) = h\big(t(a, b, x)\big) = h(c)$, where $c = t(a, b, x)$; by (*) such a unique element x exists. However, R/θ is finite, thus every surjective mapping of R/θ onto itself is a bijection. Thus also (*) is satisfied, i.e. $\mathcal{R}/\theta = (R/\theta, t_\theta, [0]_\theta, [1]_\theta)$ is a ternary ring. By Theorem 1, θ is normal. □

Corollary 1. *For every finite ternary ring $\mathcal{R}$,* $\operatorname{Con} R = \operatorname{Con}_N \mathcal{R}$.

We are going to show that for a non-finite ternary ring $\mathcal{R}$ the assertion of Theorem 2 need not hold in general:

Example. A congruence $\Theta \in \operatorname{Con}\mathcal{L}$ on a loop l is called normal if for every four elements x_1, x_2, y_1, $y_2 \in L$ such that $x_1 \Theta y_1$, $(x_1 + y_1)\Theta(x_2 = y_2)$ also $x_2 \Theta y_2$. As was pointed out e.g. in [1], there exists a loop $\mathcal{L}$ and a congruence Θ on $\mathcal{L}$ which is not normal. Let $\mathcal{L} = (\mathcal{L}; +, 0)$ be such a loop and let $\Theta \in \operatorname{Con}\mathcal{L}$ be not normal.

Choose freely but fix from now on an element $1 \in L$ such that $1 \notin [0]_\Theta$. Since θ is not normal then $\Theta \neq L \times L$, i.e. such an element exists. Introduce a new binary operation denoted by dot as follows:

(1) if $a \notin [1]_\Theta$ and $b \notin [1]_\Theta$ then $a \cdot b = 0$;
(2) if $a \in [1]_\Theta$ and $b \notin [1]_\Theta$ then $a \cdot b = b \cdot a = b$;
(3) if $a, b \in [1]_\Theta$ and $a \neq 1 \neq b$ then $a \cdot b = 1$;
(4) if $a, b \in [1]_\Theta$ and $a = 1$ then $a \cdot b = b \cdot a = b$.

Clearly, the identities

$$0 \cdot x = x \cdot 0 = 0 \text{ and} 1 \cdot x = x \cdot 1 = x$$

hold in $\mathcal{L} = (\mathcal{L}; \cdot, 0)$. Introduce a ternary operation t as follows:

$$t(x, y, z) = x \cdot y + z.$$

It is an easy exercise to check that $\mathcal{R} = (L; t, 0, 1)$ is a ternary ring and, moreover, the foregoing $\Theta \in \operatorname{Con} \mathcal{L}$ satisfies also $\Theta \in \operatorname{Con} \mathcal{R}$.

Hence, there exist elements $x_1, x_2, y_1, y_2 \in L$ such that $x_1 \Theta x_2$, $(x_1 + y_1)\Theta(x_2 + y_2)$ but y_1, y_2 are not congruent mod Θ. Applying the foregoing operation $\cdot$ on L, we obtain $t(x, y, z)$ as before. Hence, $x_1 + y_1 = (1, x_1, y_1)$, $x_2 + y_2 = t(1, X_2, y_2)$, i.e. also $t(1, x_1, y_1)\Theta t(1, x_2, y_2)$, thus Θ is not normal in $\mathcal{R} = (L; t, 0, 1)$.

Remark. Let $\mathcal{R} = (R; t, 0, 1)$ be a ternary ring. Introduce a new ternary operation $q\colon R^3 \to R$ as follows:

$$q(a, b, c) = d \text{ if and only if } t(a, b, d) = c.$$

By $(*)$, q is correctly defined. The algebra $\mathcal{R}^* = (R; t, q, 0, 1)$ satisfying the identifies (1), (1′) and

$$t\big(x, y, q(x, y, z)\big) = z = q\big(x, y, t(x, y, z)\big) \tag{2}$$

is called a bi-ternary ring, see [3].

It is easy to see that (2) implies $(*)$. Hence, the reduct $\mathcal{R} = (R; t, 0, 1)$ of a bi-ternary ring $\mathcal{R}^* = (R; t, q, 0, 1)$ is a ternary ring. Since bi-ternary rings are defined by identities, they form a variety. Hence, every congruence Θ on $\mathcal{R}^*$ is normal congruence on reduct $\mathcal{R}(R; t, 0, 1)$. Moreover, for ideals of bi-ternary rings the ideal theory can be used invent by H. P. Gumm and A. Ursini [4], [11], which is based on the universal algebraic approach. Applying it, we have shown in [3] that there exists a one-to-one correspondence between ideals and congruences of bi-ternary rings, i.e. the variety of all bi-ternary rings is ideal determined, see [3], [4].

2. Congruence lattice of ternary rings

Denote by $\theta \cdot \Phi$ the relational product of two binary relations θ, Φ on $\mathcal{R}$.

Theorem 3. *Let $\mathcal{R} = (R; t, 0, 1)$ be a ternary ring and $\Phi \in \operatorname{Con} \mathcal{R}$ and $\theta \in \operatorname{Con}_N \mathcal{R}$. Then $\theta \cdot \Phi = \Phi \cdot \theta$.*

Proof. Suppose $\Phi \in \operatorname{Con} \mathcal{R}$ and $\theta \in \operatorname{Con}_N \mathcal{R}$ and $a\theta \cdot \Phi b$ for some a, b of R. Then there exists $c \in R$ with $a\theta c$ and $c\Phi b$. By $(*)$ there exist elements $k,s \in R$ such that

$$t(1, c,) = a = t(1, b, s). \tag{i}$$

Since $b\Phi c$ we also have

(ii) $$a = t(1, c, k)\Phi t(1, b, k).$$

However, by (i) and (1′)

$$t(1, c, k) = a = t(1, a, 0)\theta t(1, c, 0).$$

Since θ is normal, this implies $k\theta 0$.

Hence, $t(1, b, k)\theta t(1, b, 0) = b$. Together with (ii) it implies $a\Phi \cdot \theta b$, i.e. $\theta \cdot \Phi \subseteq \Phi \cdot \theta$. It implies also

$$\Phi \cdot \theta = \Phi^{-1} \cdot \theta^{-1} = (\theta \cdot \Phi)^{-1} \subseteq (\Phi \cdot \theta)^{-1} = \theta^{-1} \cdot \Phi^{-1} = \theta \cdot \Phi,$$

thus $\theta \cdot \Phi = \Phi \cdot \theta$. □

Recall from [6] that a lattice $\mathcal{L}$ is Arguesian if it satisfies the identity

$$\bigwedge_{i<3}(x_i \vee y_i) \leqslant \big(x_0 \wedge (x_1 \vee m)\big) \vee \big(y_0 \wedge (y_1 \vee m)\big),$$

where

$$m = (x_0 \vee x_1) \wedge (y_0 \vee y_1) \wedge [\{(x_0 \vee x_2) \wedge (y_0 \vee y_2)\} \vee \{(x_2 \vee x_1) \wedge (y_2 \vee y_1)\}].$$

Hence, every Arguensian lattice is modular.

Theorem 4. *For every ternary ring $\mathcal{R}$, $\mathrm{Con}_N\,\mathcal{R}$ is a complete Arguesian lattice which is a sublattice of $\mathrm{Con}\,\mathcal{R}$.*

P r o o f. It is a routine to show that an arbitrary intersection of normal congruences is a normal congruence. Since also ω, $\iota \in \mathrm{Con}_N\,\mathcal{R}$, this means that $\mathrm{Con}_N\,\mathcal{R}$ is a complete lattice.

By Theorem 3, every two normal congruences permute and thus, by [6], $\mathrm{Con}_N\,\mathcal{R}$ is Arguesian.

In both the lattice $\mathrm{Con}\,\mathcal{R}$ and $\mathrm{Con}_N\,\mathcal{R}$ the meet coincides with set intersection.

It remains to prove that also the operation join coincides in these lattices. Since θ_1, $\theta_2 \in \mathrm{Con}_N\,\mathcal{R}$ are permutable, then $\theta_1 \cdot \theta_2$ is the least congruence containing θ_1 and θ_2. We need only to show that also $\theta_1 \cdot \theta_2$ is normal.

Let a_1, a_2, b_1, b_2, x, $y \in R$ and suppose

$$a_1\theta_1 \cdot \theta_2 b_1, \; a_2\theta_1 \cdot \theta_2 b_2 \quad \text{and} \quad t(a_1, a_2, x)\theta_1 \cdot \theta_2 t(b_1, b_2, y).$$

Then there exist c_1, c_2, $c_3 \in R$ with

$$a_1\theta_1 c_1, c_1\theta_2 b_1,$$
$$a_2\theta_1 c_2, c_2\theta_2 b_2,$$
$$t(a_1, a_2, x)\theta_1 c_3, c_3\theta_2 t(b_1, b_2, y).$$

By $(*)$, there exist a unique $z \in R$ with

$$t(c_1, c_2, z) = c_3,$$

whence

$$t(a_1, a_2, x)\theta_1 t(c_1, c_2, z) \text{ and } t(c_1, c_2, y)\theta_2 t(b_1, b_2, y).$$

Since θ_1, θ_2 are normal, we conclude

$$x\theta_1 z \text{ and } z\theta_2 y,$$

i.e. $x\theta_1 \cdot \theta_2 y$, which proves normality of $\theta_1 \cdot \theta_2$. □

Theorem 5. *Let $\mathcal{R} = (R; t, 0, 1)$ be a ternary ring and a, $b \in R$, $\theta \in \operatorname{Con}\mathcal{R}$. Then*

$$card[0]_\theta \leqslant \operatorname{card}[a]_\theta.$$

If, moreover, θ is normal, then

$$\operatorname{card}[a]_\theta = \operatorname{card}[b]_\theta.$$

P r o o f. For each $a \in R$ define a unary polynomial function $\varphi_a(z) = t(1, a, z)$. By $(1')$, we have

$$\varphi_a(0) = t(1, a, 0) = a.$$

Hence, φ_a induces a mapping of $[0]_\theta$ into $[a]_\theta$. By $(*)$, φ_a is an injection. This proves the first assertion.

Now, suppose $\theta \in \operatorname{Con}_N \mathcal{R}$. If $d \in [a]_\theta$ then, by $(*)$, there exist a unique $c \in R$ with $\varphi_a(c) = t(1, ac,) = d$. By $(1')$ we have $d = t(1, d, 0)$. Using $d \in [a]_\theta$ and normality of θ we conclude from $t(1, a, c) = t(1, d, 0)$ also $c \in [0]_\theta$. Hence, φ_a is also surjective, i.e. it is a bijection. Then $\operatorname{card}[a]_\theta = \operatorname{card}[0]_\theta = \operatorname{card}[b]_t$. □

Corollary 2. *Let θ Φ be normal congruences on a ternary ring $\mathcal{R} = (R; t; 0, 1)$. If $[a]_\theta = [a]_\Phi$ for some $a \in R$ then $\theta = \Phi$.*

It is an easy consequence of Theorem 5 since the mapping $\varphi_a(z) = t(1, a, z)$ is bijection which does not depend on the choice of θ.

Recall that an algebra $\mathcal{A} = (A, F)$ is congruence-uniform if $\operatorname{card}[a]_\theta = \operatorname{card}[d]_t$ for each $\theta \in \operatorname{Con} \mathcal{A}$ and every a, b of A. $\mathcal{A}$ is congruence-regular if $[a]_\theta = [a]\Phi$ implies $\theta = \Phi$ for each $a \in A$ and every two θ, $\Phi \in \operatorname{Con} \mathcal{A}$.

By using Theorem 2, Theorem 5 and Corollary 1, we obtain

Corollary 3. *Every finite ternary ring is congruence-regular and congruence-uniform.*

3. Ideals of ternary rings

The concept of an ideal of a ternary ring occured for the first time in [7]:

Definition 3. Let $\mathcal{R} = (R; t, 0, 1)$ be a ternary ring. For a, $b \in R$ we put $a + b = t(1, a, b)$. A subset $J \subseteq R$ is called an ideal of $\mathcal{R}$ if the following hold:

(I_1) $0 \in J$;

(I_2) if $b = a + r$ for some $r \in J$ then there exists $r' \in J$ with $a = b + r$;

(I_3) for every a, b, c of R and every r_1, r_2, r_3 of J there exists $r \in J$ with

$$t(a + r_1, b + r_2, c + r_3) = t(a, b, c) + r;$$

(I_4) if $t(a, b, y) = t(a, b, x) + r$ for some $r \in J$ then there exists $r' \in J$ with $y = x + r'$.

Remark. If J is an ideal of a ternary ring $\mathcal{R} = (R; t, 0, 1)$ and $a \in R$, r_1, $r_2 \in J$, then $t(a, r_1, r_2) \in J$ and $t(r_1, a, r_2) \in J$. Moreover, if $r \in R$ and $(a + r_1) + r_2 = a + r$ then $r \in J$, see e.g. [7].

Theorem 6. *Let $\mathcal{R} = (R; t, 0, 1)$ be a ternary ring and $J \subseteq R$. The following are equivalent:*

(1) *J is an ideal of $\mathcal{R}$;*

(2) *$0 \in J$ and if $t(a + r_1, b + r_2, c + r) = t(a, b, c) + s$ for some r_1, $r_2 \in J$, then $r \in J$ iff $s \in J$.*

Proof. (1) $\Rightarrow$ (2): For any elements a, b, r_1, r_2, r of R there exists $s \in R$ such that

$$t(a + r_1, b + r_2, c + r) = t(a, b, c) + s = t\bigl(1, (a, b, c), s\bigr).$$

By $(*)$, this "s" is uniquely determined. Suppose $r_1, r_2 \in J$. If $r \in J$, then, by (I_3), we have $s \in J$. If $r' \in J$ then there exists $k_1 \in R$ such that $(a + r_1, b + r_2, c + r') = t(a, b, c) + k_1$ and, by the foregoing part, $k_1 \in J$. By (I_2), there exists $k_3 \in J$ with

$$t(a, b, c) = t(a = r_1, b + r_2, c + r') + k_2),$$

thus also

$$t(a + r_1, b + r_2, c + r) = \big(t(a + r_1, b + r_2, c + r') + k_2\big) + s.$$

Since k_2, $s \in J$, there exists $k_3 \in J$ with

$$\big(t(a + r_1, b + r_2, c + r') + k_3\big) + s = t(a + r_1, b + r_2, c + r') + k_3,$$

see e.g. the foregoing Remark. Hence

$$t(a + r_1, b + r_2, c + r) = t(a + r_1, b + r_2, c + r') + k_3.$$

By (I_4) there exists $k_4 \in J$ with $c + r = (c + r') = c + k$ where $k \in J$, see the foregoing Remark again.

Applying $(*)$ we conclude $r = k$, thus $r \in J$.

(2) $\Rightarrow$ (1): We prove directly (I_2) and (I_4) of the definition. The condition (I_3) follows immediately by $(*)$ and (2).

First we prove that if $(a + r) + r_2 = a + r$ and r_1, $r \in J$ then also $r_2 \in J$. Indeed, we have

$$(a + r_1) + r_2 = t(1, a + r_1, r_2) = t(1 + 0, a + r_1, 0 + r_2) = a + r = t(1, a, 0).$$

By (2) we obtain $r_2 \in J$.

Now, we suppose $b = a + r$ for $r \in J$. By $(*)$ there exists $r' \in R$ with $a = b + r'$. Then $a = (a + r) + r' = a + 0$. Since r, $0 \in J$, we conclude $r' \in J$, thus the condition (I_2) is evident.

Prove (I_4): let $t(a, b, y) = t(a, b, x) + r$ for $r \in J$. By $(*)$ there exists $r' \in R$ with $y = x + r'$. We obtain

$$t(a, b, y) = t(a + 0, b + 0, x + r') = t(a, b, x) + r.$$

Since 0, $r \in J$, (2) implies also $r' \in J$. □

Theorem 7. *Let $\mathcal{R} = (R; t, 0, 1)$ be a ternary ring and θ a binary relation on R. The following are equivalent:*
(1) *θ is a normal congruence on $\mathcal{R}$;*
(2) *$[0]_\theta$ is an ideal of $\mathcal{R}$ and $a\theta b$ if only $b = a + r$ for some $r \in [0]_\theta$;*

Proof. (1) $\Rightarrow$ (2): Suppose $a\theta b$. By $(*)$, there exists $r \in R$ with $b = t(1, a, r) = a + r$. Since $a = t(1, a, 0)$, we conclude $t(1, a, r)\theta t(1, a, 0)$. By (1), θ is normal, thus also $r\theta 0$, i.e. $r \in [0]_\theta$.

Conversely, if $b = a + r$ and $r \in [0]\theta$ then $t(1, a, 0)\theta t(1, a, r)$ whence $a\theta b$. Now, put $J = [0]_t$ and suppose

$$(***) \qquad t(a + r_1, b + r_2, c + r_3) = t(a, b, c) + r.$$

Suppose r_1, r_2, $r_3 \in [0]\theta$. Then $(a + r_1)\theta a$, $(b + r_2)\theta b$, $(c + r_3)\theta c$, i.e. also

$$(****) \qquad t(a + r_1, b + r_2, c + r_3)\theta t(a, b, c).$$

By using $(*)$ and $(***)$ we obtain $r \in [0]_\theta = J$. Suppose r_1, r_2, $r \in J = [0]_\theta$. Then $(a + r_1)\theta a$, $(b + r_2)\theta b$ and $(****)$ give $(c + r_3)\theta c$ since θ is normal. By the first part of this proof, $r_3 \in [0]_\theta$. Applying (2) of Theorem 6, J be an ideal of $\mathcal{R}$.

(2) $\Rightarrow$ (1): Let J be an ideal of $\mathcal{R}$. It is an easy exercise to show that the relation θ defined by

$$a\theta b \text{ if and only if } b = a + r \text{ for some } r \in J$$

is a congruence on $\mathcal{R}$ and $J = [0]_\theta$. It remains to prove that θ is normal. Let $a_i\theta b_i$ for $i = 1, 2, 3$, let x, $y \in R$ and suppose $t(a_1, a_2, x) = a_3$, $t(b_1, b_2, y) = b_3$. Then $b_i = a_i + r_i$ for some $r_i \in [0]_\theta$. By $(*)$, there exists $r \in R$ with $y = t(1, x, r) = x + r$. Hence

$$t(b_1, b_2, y) = t(a_1 + r_1, a_2 + r_2, x + r) = a_3 + r_3.$$

By (2) of Theorem 6, we have $r \in [0]_\theta$ whence

$$t(1, x, 0)\theta t(1, x, r),$$

i.e. $x\theta(x + r)$. Since $x + r = y$, we conclude $x\theta y$. □

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