Czechoslovak Mathematical Journal

Ladislav Nebeský

Geodesics and steps in a connected graph

Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 1, 149–161

Persistent URL: http://dml.cz/dmlcz/127346

Czechoslovak Mathematical Journal, 47 (122) 1997, Praha

GEODESICS AND STEPS IN A CONNECTED GRAPH

LADISLAV NEBESKÝ, Praha

(Received January 4, 1995)

Let G be a connected (finite undirected) graph. By a step in G will mean an ordered triple (u, v, x) of vertices in G with the property that $d(u, v) = 1$ and $d(u, x) = d(v, x) + 1$, where d denotes the distance function of G. The concept of a step is closely related to that of a geodesic (or a shortest path). An axiomatic characterization of the set of all geodesics in a connected graph was given by the present author in [5]. A characterization of the set of all steps in a connected graph will be given here.

The letters g, h, i, j, k, m and n will be reserved for denoting integers.

Let V be a finite nonempty set. We denote by $\Sigma(V)$ the set of all sequences

$$(v_0, \ldots, v_n), \tag{1}$$

where $n \geqslant 0$ and $v_0, \ldots, v_n \in V$.

By a graph we mean here a finite undirected graph with no loops or multiple edges, i.e. a graph in the sense of [1] or [2], for example. If G is a graph, then $V(G)$ and $E(G)$ denote its vertex set and its edge set, respectively. Let $v_0, \ldots, v_n \in V(G)$, where $n \geqslant 0$; we say that (1) is a walk in G if $\{v_i, v_{i+1}\} \in E(G)$ for each i, $0 \leqslant i < n$. Obviously, every walk in G is an element of $\Sigma(V(G))$. By a path in G we mean such a walk (1) in G that the vertices $v_0, \ldots, v_n$ are mutually distinct.

Let G be a connected graph, and let d denote the distance function of G. (Note that in [3] a characterization of the distance function of a connected graph was given.) Obviously, if (1) is a walk in G, then $d(v_0, v_n) \leqslant n$. By a geodesic (or a shortest path) in G we mean such a walk (1) that $d(v_0, v_n) = n$. It is not difficult to see that every geodesic in G is a path. We now introduce the concept of a step in G. By a *step* in G we will mean an ordered triple (u, v, x), where $u, v, x \in V(G)$ and

$$d(u, v) = 1 \text{ and } d(u, x) = d(v, x) + 1. \tag{2}$$

Obviously, (u, v, x) is a step in G if and only if there exists a geodesic (1) in G with the properties that $n \geqslant 1$, $u = v_0$, $v = v_1$ and $x = v_n$. In the present paper a characterization of the set of all steps in a connected graph will be given.

Let V be a finite nonempty set, and let $T \subseteq V^3$. If u, v, $x \in V$, then instead of

$$(u, v, x) \in T \text{ or } (u, v, x) \notin T$$

we will write

$$uv \rightarrow_T x \text{ or } uv \textit{ non } \rightarrow_T x, \text{ respectively.}$$

We denote by $\Gamma(V, T)$ the graph H with $V(H) = V$ and

$$E(H) = \{\{u, v\}\,;\, u, v \in V, u \neq v \text{ and there exists } x \in V \text{ such that } uv \rightarrow_T x \text{ or } vu \rightarrow_T x\}.$$

Proposition 1. *Let V be a finite nonempty set, and let $T \subseteq V^3$. Assume that there exists a connected graph G with the properties that $V(G) = V$ and T is the set of all steps in G. Then $G = \Gamma(V, T)$.*

Proof. Let d denote the distance function of G. Since $V(G) = V(\Gamma(V, T))$, we see that $G = \Gamma(V, T)$ if and only if $E(G) = E(\Gamma(V, T))$.

Consider arbitrary $u, v \in V$.

Let $\{u, v\} \in E(G)$. Then $d(u, v) = 1$. Since $d(v, v) = 0$, we see that (u, v, v) is a step in G. This means that $uv \rightarrow_T v$. Since $u \neq v$, we have $\{u, v\} \in E(\Gamma(V, T))$.

Conversely, let $\{u, v\} \in E(\Gamma(V, T))$. Then $u \neq v$ and there exists $x \in V$ such that $uv \rightarrow_T x$ or $vu \rightarrow_T x$. The fact that (u, v, x) or (v, u, x) is a step in G implies that $d(u, v) = 1$. Hence $\{u, v\} \in E(G)$.

We have $G = \Gamma(V, T)$, which completes the proof. □

Proposition 1 is an introduction to the next theorem, which is the main result of the present paper.

Theorem 1. *Let V be a finite nonempty set, and let $T \subseteq V^3$. Assume that $\Gamma(V, T)$ is connected. Then the following statements* (I) *and* (II) *are equivalent:*

(I) *T is the set of all steps in $\Gamma(V, T)$;*

(II) *T fulfils Axioms* A–H *(for arbitrary $u, v, x, y \in V$):*

A *if $uv \rightarrow_T x$, then $vu \rightarrow_T u$;*

B *if $uv \rightarrow_T x$ and $vu \rightarrow_T y$, then $x \neq y$;*

C *if $uv \rightarrow_T x$ and $xy \rightarrow_T v$, then $xy \rightarrow_T u$;*

D *if $uv \rightarrow_T x$ and $xy \rightarrow_T v$, then $uv \rightarrow_T y$;*

E *if* $uv \to_T x$ *and* $uy \to_T v$, *then* $y = v$;

F *if* $uv \to_T x$, $vu \to_T y$ *and* $xy \to_T y$, *then* $xy \to_T u$;

G *if* $uv \to_T x$ *and* $xy \to_T y$, *then either* $xy \to_T u$ *or* $yx \to_T v$ *or* $uv \to_T y$;

H *if* $u \neq x$, *then there exists* $z \in V$ *such that* $uz \to_T x$.

Combining Theorem 1 with Proposition 1, we get the following result:

Corollary 1. *Let V be a finite nonempty set, and let $T \subseteq V^3$. Then there exists a connected graph G with the properties that $V(G) = V$ and T is the set of all steps in G if and only if $\Gamma(V,T)$ is connected and T fulfils Axioms* A–H *(for arbitrary $u, v, x, y \in V$).*

For the proof of Theorem 1 we will need three remarks and three lemmas.

In Remarks 1–3 and Lemmas 1–3 we will assume that V is a finite nonempty set, $T \subseteq V^3$ and T fulfils Axioms A, B, C, D and H.

Remark 1. Let $u, v, x \in V$ be such that $uv \to_T x$. Axiom B implies that $u \neq v$, and therefore, $\{u, v\} \in E(\Gamma(V,T))$.

Let $u_0, u_1, \ldots, u_n, w_1, \ldots, w_n \in V$, where $n \geqslant 1$, and let

$$u_0u_1 \to_T w_1, \ldots, u_{n-1}u_n \to_T w_n.$$

It is clear that $(u_0, u_1, \ldots, u_n)$ is a walk in $\Gamma(V,T)$.

Remark 2. Let $u, v, x \in V$ be such that $uv \to_T x$. Combining Axioms A and B we get $u \neq x$.

Lemma 1. *Let $u_0, u_1, v_1, \ldots, v_{i+1} \in V$, where $i \geqslant 1$, let*

$$v_1v_2 \to_T u_0, \ldots, v_iv_{i+1} \to_T u_0$$

and let $u_1u_0 \to_T v_1$. Then

$$v_gv_{g+1} \to_T u_1 \text{ and } u_1u_0 \to_T v_{g+1}$$

for each g, $1 \leqslant g \leqslant i$.

Proof. We proceed by induction on g. First, let $g = 1$. Since $v_1v_2 \to_T u_0$ and $u_1u_0 \to_T v_1$, Axioms C and D imply that $v_1v_2 \to_T u_1$ and $u_1u_0 \to_T v_2$. If $i = 1$, then the proof is complete. Assume that $2 \leqslant g \leqslant i$. According to the induction hypothesis, $u_1u_0 \to_T v_g$. Since $v_gv_{g+1} \to_T u_0$, Axioms C and D imply that $v_gv_{g+1} \to_T u_1$ and $u_1u_0 \to_T v_{g+1}$, which completes the proof. □

Lemma 2. *Let $x_0, \dots, x_j, y_1, \dots, y_{j+1} \in V$, where $j \geqslant 1$, let*

$$y_1 y_2 \to_T x_0, \dots, y_j y_{j+1} \to_T x_0$$

and

$$x_1 x_0 \to_T y_1, \dots, x_j x_{j-1} \to_T y_j.$$

Then

$$y_h y_{h+1} \to_T x_h, \dots, y_j y_{j+1} \to_T x_h$$

and

$$x_h x_{h-1} \to_T y_h, \dots, x_h x_{h-1} \to_T y_{j+1}$$

for each h, $1 \leqslant h \leqslant j$.

Proof. We proceed by induction on h. Since $x_1 x_0 \to_T y_1$, the case when $h = 1$ is covered by Lemma 1. If $j = 1$, then the proof is complete. Assume that $2 \leqslant h \leqslant j$. The induction hypothesis implies that

$$y_h y_{h+1} \to_T x_{h-1}, \dots, y_j y_{j+1} \to_T x_{h-1}.$$

Recall that $x_h x_{h-1} \to_T y_h$. Applying Lemma 1, we get the result. □

Lemma 3. *Let $\Gamma(V,T)$ be connected, let $x_0, \dots, x_n$, $y_1 \in V$, where $n \geqslant 2$, let $(x_0, \dots, x_n)$ be a geodesic in $\Gamma(V,T)$, and let $x_n y_1 \to_T x_0$. Let d denote the distance function of $\Gamma(V,T)$. Then there exist $k \geqslant 0$ and $x_{n+1}, \dots, x_{n+k+1} \in V$ such that $x_{n+1} = y_1$,*

(3) $$x_{n+g} x_{n+g+1} \to_T x_0 \text{ for each } g,\ 0 \leqslant g \leqslant k,$$

(4) $$x_h x_{h-1} \to_T x_{n+h} \text{ for each } h,\ 1 \leqslant h \leqslant k$$

and

(5) *either* (a) $x_n x_{n-1} \to_T x_0$ *and* $d(y_1, x_0) = n - 1$,
or (b) $x_{k+1} x_k$ *non* $\to_T x_{n+k+1}$.

Proof. We distinguish two cases.

Case 1. Assume that there exists an infinite sequence

$$(x_{n+1}, x_{n+2}, x_{n+3}, \dots)$$

of vertices in $\Gamma(V,T)$ such that $x_{n+1} = y_1$ and

$$x_{n+i}x_{n+i+1} \rightarrow_T x_0 \text{ for each } i = 0,1,2,\ldots.$$

Let

$$x_g x_{g-1} \rightarrow_T x_{n+g} \text{ for each } g = 1,2,3,\ldots.$$

Lemma 2 implies that

$$x_h x_{h-1} \rightarrow_T x_{n+h}, x_h x_{h-1} \rightarrow_T x_{n+h+1}, x_h x_{h-1} \rightarrow_T x_{n+h+2}, \ldots$$
$$\text{for each } h = 1,2,3,\ldots.$$

As follows from Remark 2,

$$x_h \neq x_{n+h}, x_{n+h+1}, x_{n+h+2}, \ldots \qquad \text{for each } h = 1,2,3,\ldots.$$

This implies that

$$x_1, x_{n+1}, x_{2n=1}, \ldots$$

are mutually distinct, which is a contradiction to the fact that V is finite. Therefore, there exists $k \geqslant 0$ such that $x_{k+1}x_k$ *non* $\rightarrow_T x_{n+k+1}$. We see that (3), (4) and (5) hold.

Case 2. Let the assumption of Case 1 be not fulfilled. Since $(x_0,\ldots,x_n)$ is a geodesic in $\Gamma(V,T)$ and $n \geqslant 2$, we have $y_1 \neq x_0$. It follows from Axiom H that there exist $x_{n+1},\ldots,x_{n+j+1} \in V$, where $j \geqslant 1$, such that $x_{n+1} = y_1, x_{n+j+1} = x_0$ and

$$x_n x_{n+1} \rightarrow_T x_0, \ldots, x_{n+j}x_{n+j+1} \rightarrow_T x_0.$$

As follows from Remark 1,

$$(x_{n+1},\ldots,x_{n+j+1})$$

is a walk in $\Gamma(V,T)$. Thus $d(x_{n+1},x_0) \leqslant j$. Since $d(x_n,x_0) = n$, we have $d(x_{n+1},x_0) \geqslant n-1$.

First, let

$$x_{i+1}x_i \rightarrow_T x_{n+i+1} \text{ for each } i,\ 0 \leqslant i \leqslant j.$$

Then $x_{j+1}x_j \rightarrow_T x_0$. If $j \geqslant n$, we also have $x_j x_{j+1} \rightarrow_T x_0$, which is a contradiction to Axiom B. Hence $d(x_{n+1},x_0) = n-1$ and $x_n x_{n-1} \rightarrow_T x_0$. Put $k = j$. Then (3), (4) and (5) hold.

Next, let there exist k, $0 \leqslant k \leqslant j$, such that

$$x_{k+1}x_k \text{ non } \rightarrow_T x_{n+k+1}$$

and (4) holds. We see that (3) and (5) hold, too.

Thus the lemma is proved. □

Remark 3. Let $\Gamma(V,T)$ be connected, let $x_0, \ldots, x_n, y_1 \in V$, where $n \geqslant 2$, let $(x_0, \ldots, x_n)$ be a geodesic in $\Gamma(V,T)$, and let $x_n y_1 \to_T x_0$. Let d denote the distance function of $\Gamma(V,T)$. Lemma 3 implies that there exist $k \geqslant 0$ and $x_{n+1}, \ldots, x_{n+k+1} \in V$ such that $x_{n+1} = y_1$ and (3)–(5) hold.

It follows from Remark 1 that

$$(x_0, x_1, \ldots, x_n, \ldots, x_{n+k+1})$$

is a walk in $\Gamma(V,T)$. Axiom A implies that

(6) $$x_g x_{g+1} \to_T x_{g+1} \text{ for each } g,\ 0 \leqslant g \leqslant n+k.$$

Combining (3) and (4) with Lemma 2, we see that if $k \geqslant 1$, then

$$x_{n+h} x_{n+h+1} \to_T x_h, \ldots, x_{n+k} x_{n+k+1} \to_T x_h$$

and

$$x_h x_{h-1} \to_T x_{n+h}, \ldots, x_h x_{h-1} \to_T x_{n+k+1}$$

for each h, $1 \leqslant h \leqslant k$.

Since $x_n x_{n+1} \to_T x_0$, we have

(7) $$x_{n+i} x_{n+i+1} \to_T x_i \text{ for each } i,\ 0 \leqslant i \leqslant k.$$

Proof of Theorem 1. Denote $G = \Gamma(V,T)$. Recall that G is connected. We denote by d, D and S the distance function of G, the diameter of G and the set of all steps in G, respectively. Obviously, $S \subseteq V^3$.

PART ONE (I $\Rightarrow$ II). Let $T = S$. Consider arbitrary $u, v, x, y \in V$. It is easy to see that T fulfils Axioms A, B, E and H. We will prove that T fulfils Axioms C, D, F and G.

(Verification of Axioms C and D). Let $uv \to_T x$ and $xy \to_T v$. Then

$$d(u,v) = 1 = d(x,y), d(u,x) = d(v,x) + 1 \text{ and } d(x,v) = d(y,v) + 1.$$

We get

$$d(u,y) \leqslant d(v,y) + 1 = d(x,v) = d(u,x) - 1 \leqslant d(u,y).$$

Therefore, $d(u,y) = d(v,y) + 1 = d(u,x) - 1$. We see that $xy \to_T u$ and $uv \to_T y$.

(Verification of Axiom F.) Let $uv \to_T x$, $vu \to_T y$ and $xy \to_T y$. Then

$$d(u,x) = d(v,x) + 1, d(v,y) = d(u,y) + 1 \text{ and } d(x,y) = 1.$$

We get

$$d(y,u)+1 \geqslant d(x,u) = d(v,x)+1 \geqslant d(v,y) = d(u,y)+1.$$

We see that $xy \to_T u$.

(Verification of Axiom G.) Let $uv \to_T x$ and $xy \to_T y$. Assume that uv *non* $\to_T y$ and yx *non* $\to_T v$. Then

$$d(u,x) = d(v,x)+1, d(x,y) = 1, d(v,y) \geqslant d(u,y) \text{ and } d(x,v) \geqslant d(y,v).$$

We get

$$d(y,u)+1 \geqslant d(x,u) = d(v,x)+1 \geqslant d(y,v)+1 \geqslant d(u,y)+1.$$

We see that $xy \to_T u$.

Thus T fulfils Axioms A–H.

PART TWO (II $\Rightarrow$ I). Let T fulfil Axioms A - H. We will prove that

$$(8_n) \qquad \text{if } rs \to_S t, \text{ then } rs \to_T t \text{ for every } r,s,t \in V \text{ such that } d(r,t) \leqslant n$$

and

$$(9_n) \qquad \text{if } rs \to_T t, \text{ then } rs \to_S t \text{ for every } r,s,t \in V \text{ such that } d(r,t) \leqslant n$$

for each n, $0 \leqslant n \leqslant D$.

We proceed by induction on n. It is obvious that both (8_0) and (9_0) hold. If $D = 0$, then the theorem is proved. Assume that $D \geqslant 1$.

Consider arbitrary $r_1, r_2, r_3 \in V$ such that $r_1r_2 \to_S r_3$ and $d(r_1,r_3) = 1$. Then $\{r_1, r_2\} \in E(G)$ and $r_2 = r_3$. Since $G = \Gamma(V,T)$, there exists $z \in V$ such that $r_1r_2 \to_T z$ or $r_2r_1 \to_T z$. It follows from Axiom A that $r_1r_2 \to_T r_2$. Since $r_2 = r_3$, we get $r_1r_2 \to_T r_3$. Thus (8_1) holds.

Consider arbitrary $s_1, s_2, s_3 \in V$ such that $s_1s_2 \to_T s_3$ and $d(s_1,s_3) = 1$. Then $s_1s_3 \to_S s_3$. According to (8_1), $s_1s_3 \to_T s_3$. Since $s_1s_2 \to_T s_3$, Axiom E implies that $s_3 = s_2$ and therefore, $s_1s_2 \to_S s_3$. Thus (9_1) holds.

If $D = 1$, then the theorem is proved. Let $2 \leqslant n \leqslant D$. The remainder of the proof will be divided into two sections. In Section 1 we will show that (8_{n-1}) and (9_{n-1}) imply (8_n). In Section 2 we will show that (8_n) and (9_{n-1}) imply (9_n).

Section 1. Consider arbitrary $x_0, x, y \in V$ such that $x_0x \to_S y$ and $d(x_0,y) = n$. Clearly, there exist $x_1, \ldots, x_n \in V$ such that $x_1 = x$, $x_n = y$ and $(x_0, x_1, \ldots, x_n)$ is a geodesic in G. We have $x_0x_1 \to_S x_n$. We want to prove that $x_0x_1 \to_T x_n$. Suppose, to the contrary, that x_0x_1 *non* $\to_T x_n$.

First, let $x_nx_{n-1} \to_T x_0$. Clearly, $x_0x_1 \to_S x_{n-1}$. Since $d(x_0, x_{n-1}) = n-1$, it follows from (8_{n-1}) that $x_0x_1 \to_T x_{n-1}$. According to Axiom C, $x_0x_1 \to_T x_n$, which is a contradiction.

We get x_nx_{n-1} *non* $\to_T x_0$. According to Axiom H, there exists $y_1 \in V$ such that $x_ny_1 \to_T x_0$. As follows from Lemma 3, there exist $k \geqslant 0$ and $x_{n+1}, \dots, x_{n+k+1} \in V$ such that $x_{n+1} = y_1$, $x_{k+1}x_k$ *non* $\to_T x_{n+k+1}$, and (3) and (4) hold. Recall that x_0x_1 *non* $\to_T x_n$ and $d(x_0, x_n) = n$. There exists m, $0 \leqslant m \leqslant k$, such that

(10) $$x_mx_{m+1} \textit{ non } \to_T x_{n+m} \text{ and } \\ d(x_m, x_{n+m}) = n$$

and

(11) $$\text{either } x_{m+1}x_m \textit{ non } \to_T x_{n+m+1} \text{ or } x_{m+1}x_{m+2} \to_T x_{n+m+1} \\ \text{or } d(x_{m+1}, x_{n+m+1}) < n.$$

According to (6), $x_mx_{m+1} \to_T x_{m+1}$. As follows from (7), $x_{n+m}x_{n+m+1} \to_T x_m$.

We distinguish Cases 1.1 and 1.2.

Case 1.1. Let $x_{m+1}x_m \to_T x_{n+m+1}$.

Assume that $d(x_{m+1}, x_{n+m+1}) < n$. According to (9_{n-1}) we have $x_{m+1}x_m \to_S x_{n+m+1}$, and thus $d(x_m, x_{n+m+1}) = d(x_{m+1}, x_{n+m+1}) - 1 < n-1$. This implies that $d(x_m, x_{n+m}) < n$, which contradicts (10). Thus $d(x_{m+1}, x_{n+m+1}) = n$. This means that

$$(x_{n+m+1}, \dots, x_{m+2}, x_{m+1})$$

is a geodesic in G. We have $d(x_{n+m+1}, x_{m+2}) = n-1$ and $d(x_{n+m}, x_{m+2}) = n-2$. Therefore, $x_{n+m+1}x_{n+m} \to_S x_{m+2}$. It follows from (8_{n-1}) that $x_{n+m+1}x_{n+m} \to_T x_{m+2}$.

Let $x_{m+1}x_{m+2} \to_T x_{n+m+1}$. Axiom C implies that $x_{n+m+1}x_{n+m} \to_T x_{m+1}$. We have seen that $x_{n+m}x_{n+m+1} \to_T x_m$. Since $x_mx_{m+1} \to_T x_{m+1}$, it follows from Axiom F that $x_mx_{m+1} \to_T x_{n+m}$, which is a contradiction to (10). Thus $x_{m+1}x_{m+2}$ *non* $\to_T x_{n+m+1}$. Since $x_{m+1}x_m \to_T x_{n+m+1}$ and $d(x_{m+1}, x_{n+m+1}) = n$, we get a contradiction to (11).

Case 1.2. Let $x_{m+1}x_m$ *non* $\to_T x_{n+m+1}$. Recall that $x_{n+m}x_{n+m+1} \to_T x_m$. According to (10), x_mx_{m+1} *non* $\to_T x_{n+m}$. Since $x_mx_{m+1} \to_T x_{m+1}$, Axiom G implies that

$$x_{n+m}x_{n+m+1} \to_T x_{m+1}.$$

Since $d(x_m, x_{n+m}) = n$, $d(x_{m+1}, x_{n+m}) = n-1$. According to (9_{n-1}), $x_{n+m}x_{n+m+1} \to_S x_{m+1}$. Hence $d(x_{m+1}, x_{n+m+1}) = n-2$. Since $x_m x_{m+1} \to_S x_{n+m}$, we get $x_m x_{m+1} \to_S x_{n+m+1}$. Clearly, $d(x_m, x_{n+m+1}) = n-1$. According to (8_{n-1}), $x_m x_{m+1} \to_T x_{n+m+1}$. Recall that $x_{n+m}x_{n+m+1} \to_T x_m$. Axiom C implies that $x_m x_{m+1} \to_T x_{n+m}$, which contradicts (10).

We proved that $x_0 x_1 \to_T x_n$. Hence (8_n) holds.

Section 2. Consider arbitrary $y, y_1, x_0 \in V$ such that $yy_1 \to_T x_0$ and $d(y, x_0) = n$. Clearly, there exist $x_1, \ldots, x_n \in V$ such that $x_n = y$, and $(x_0, x_1, \ldots, x_n)$ is a geodesic in G. We have $x_n y_1 \to_T x_0$. Obviously, $d(y_1, x_0) \geqslant n-1$. We want to prove that $x_n y_1 \to_S x_0$. We see that $x_n y_1 \to_S x_0$ if and only if $d(y_1, x_0) = n-1$. Suppose, to the contrary, that $d(y_1, x_0) \geqslant n$.

As follows from Lemma 3, there exist $k \geqslant 0$ and $x_{n+1}, \ldots, x_{n+k+1} \in V$ such that $x_{n+1} = y_1$, $x_{k+1}x_k$ *non* $\to_T x_{n+k+1}$, and (3) and (4) hold. Recall that $d(x_0, x_n) = n$. There exists m, $0 \leqslant m \leqslant k$, such that

$$d(x_m, x_{n+m}) = n \tag{12}$$

and

$$\text{either } x_{m+1}x_m \text{ non } \to_T x_{n+m+1} \text{ or } d(x_{m+1}, x_{n+m+1}) < n. \tag{13}$$

According to (6), $x_m x_{m+1} \to_T x_{m+1}$. Axiom A implies that $x_{m+1}x_m \to_T x_m$. As follows from (7), $x_{n+m}x_{n+m+1} \to_T x_m$.

We distinguish Cases 2.1 and 2.2.

Case 2.1. Let $d(x_{m+1}, x_{n+m+1}) = n$. Then

$$(x_{n+m+1}, x_{n+m}, \ldots, x_{m+1})$$

is a geodesic in G. Hence $x_{n+m+1}x_{n+m} \to_S x_{m+1}$. It follows from (8_n) that $x_{n+m+1}x_{n+m} \to_T x_{m+1}$. Recall that $x_{n+m}x_{n+m+1} \to_T x_m$. Since $x_{m+1}x_m \to_T x_m$, Axiom F implies that $x_{m+1}x_m \to_T x_{n+m+1}$, which contradicts (13).

Case 2.2. Let $d(x_{m+1}, x_{n+m+1}) < n$.

Assume that $d(x_m, x_{n+m+1}) = n$. Then $d(x_{m+1}, x_{n+m+1}) = n-1$. Therefore, $x_m x_{m+1} \to_S x_{n+m+1}$. According to (8_n), $x_m x_{m+1} \to_T x_{n+m+1}$. Since $x_{n+m}x_{n+m+1} \to_T x_m$, Axiom D implies that $x_{n+m}x_{n+m+1} \to_T x_{m+1}$. Since $d(x_{m+1}, x_{n+m}) = n-1$, it follows from (9_{n-1}) that $x_{n+m}x_{n+m+1} \to_S x_{m+1}$. Therefore, $d(x_{m+1}, x_{n+m+1}) = n-2$, which is a contradiction.

Thus $d(x_m, x_{n+m+1}) < n$. It follows from (12) that

$$d(x_m, x_{n+m+1}) = n-1.$$

Recall that $x_{n+1} = y_1$. If $m = 0$, then $d(x_0, x_{n+1}) = n-1$, which is a contradiction. Let $m \geqslant 1$. Since $m \leqslant k$, Remark 3 implies that

$$x_1 x_0 \rightarrow_T x_{n+m+1}, \ldots, x_m x_{m-1} \rightarrow_T x_{n+m+1}.$$

Consider an arbitrary i, $1 \leqslant i \leqslant m$. If $d(x_i, x_{n+i+1}) < n$, then (9_{n-1}) implies that $x_i x_{i-1} \rightarrow_S x_{n+m+1}$, and therefore, $d(x_{i-1}, x_{n+m+1}) = d(x_i, x_{n+m+1}) - 1$. Since $d(x_m, x_{n+m+1}) = n-1$, we get

$$d(x_0, x_{n+m+1}) = n - m - 1.$$

This means that $m \leqslant n-1$. As follows from Remark 3,

$$(x_{n+1}, \ldots, x_{n+m+1})$$

is a walk in G. Thus $d(x_{n+1}, x_{n+m+1}) \leqslant m$. This means that

$$d(x_0, x_{n+1}) \leqslant d(x_0, x_{n+m+1}) + d(x_{n+m+1}, x_{n+1}) \leqslant n - 1,$$

which is a contradiction.

We have proved that $x_n y_1 \rightarrow_S x_0$. Hence (9_n) holds.

Thus $T = S$, which completes the proof of Theorem 1. □

Remark 4. Let V be a finite nonempty set, and let $T \subseteq V^3$. As we will show, the fact that T fulfils Axioms A–H does not imply that $\Gamma(V, T)$ is connected.

Assume that $V = \{r_1, \ldots, r_n, s_1, \ldots, s_n\}$, where $n \geqslant 3$ and $|V| = 2n$. Put $r_{n+1} = r_1$ and $s_{n+1} = s_1$. Assume that T is the subset of V^3 with the property that $uv \rightarrow_T x$ if and only if one of the following cases a) and b) holds:

a) there exist distinct g and h, $1 \leqslant g \leqslant n$ and $1 \leqslant h \leqslant n$, such that

$$\text{either } (u = r_g,\ v = r_h \text{ and } x = r_h)$$
$$\text{or } (u = s_g,\ v = s_h,\ x = s_h);$$

b) there exist i and j, $1 \leqslant i \leqslant n$ and $1 \leqslant i \leqslant n$, such that

$$\text{either } (u = r_i,\ v = r_{i+1} \text{ and } x = s_j)$$
$$\text{or } (u = s_i,\ v = s_{i+1} \text{ and } x = r_j).$$

It is not difficult to see that T fulfils Axioms A–H and that $\Gamma(V, T)$ has exactly two components.

Let V be a finite nonempty set, and let $R \subseteq \Sigma(V)$. We denote by $[R]$ the subset T of V^3 defined as follows:

$$uv \to_T x \text{ if and only if there exist } n \geqslant 1 \text{ and } u_0, u_1, \dots, u_n \in V \text{ such that } (u_0, u_1, \dots, u_n) \in R,\ u = u_0,\ v = u_1 \text{ and } x = u_n$$

for any $u, v, x \in V$.

Proposition 2. *Let V be a finite nonempty set, and let $R \subseteq \Sigma(V)$. Put $T = [R]$. Assume that there exists a connected graph G with the properties that $V(G) = V$ and R is the set of all geodesics in G. Then $G = \Gamma(V, T)$.*

Proof. Since $V(G) = V(\Gamma(V,T))$, we see that $G = \Gamma(V,T)$ if and only if $E(G) = E(\Gamma(V,T))$.

Consider arbitrary u, $v \in V$.

Let $\{u,v\} \in E(G)$. Then (u,v) is a geodesic in G. Thus $(u,v) \in R$. Clearly, $uv \to_T v$. Since $u \neq v$, we see that $\{u,v\} \in E(\Gamma(V,T))$.

Conversely, let $\{u,v\} \in E(\Gamma(V,T))$. Then $u \neq v$ and there exists $x \in V$ such that $uv \to_T x$ or $vu \to_T x$. Since $T = [R]$, there exist $n \geqslant 1$ and $u_0, u_1, \dots, u_n \in V$ such that $(u_0, u_1, \dots, u_n) \in R, x = u_n$ and either (i) $u = u_0$ and $v = u_1$ or (ii) $u = u_1$ and $v = u_0$. The fact that $(u_0, u_1, \dots, u_n)$ is a geodesic in G implies that $\{u,v\} \in E(G)$.

We have $G = \Gamma(V,T)$, which completes the proof. □

Theorem 2. *Let V be a finite nonempty set, and let $R \subseteq \Sigma(V)$. Put $T = [R]$. Assume that $\Gamma(V,T)$ is connected. Then the following statements* (III) *and* (IV) *are equivalent:*

(III) *R is the set of all geodesics in $\Gamma(V,T)$;*

(IV) *T fulfils Axioms* A–H *(for arbitrary $(u, v, x, y \in V)$ and moreover, R fulfils the following Axioms* X, Y *and* Z *(for arbitrary $m, n \geqslant 1$ and $u, u_0, \dots, u_m, w_0, \dots, w_n \in V$):*

X *$(u) \in R$;*

Y *if $(u, u_m, \dots, u_0) \in R$, then $(u_m, \dots, u_0) \in R$;*

Z *if $(u, u_m, \dots, u_0)$, $(w_n, \dots, w_0) \in R$, $w_0 = u_0$ and $w_n = u_m$, then $(u, w_n, \dots, w_0) \in R$.*

Proof. Denote $G = \Gamma(V,T)$. Recall that G is connected. We denote by d the distance function of G.

PART ONE (III $\Rightarrow$ IV). Let III hold. It is easy to see that R fulfils Axioms X, Y and Z. Since $T = [R]$, we see that T is the set of all steps in G. According to Theorem 1, T fulfils Axioms A–H. Hence IV holds.

PART TWO (IV $\Rightarrow$ III). Let IV hold. Consider arbitrary $v_0, \dots, v_n \in V$, where $n \geqslant 0$. We will prove that

$$(14_n) \qquad (v_n, \dots, v_0) \in R \text{ if and only if } (v_n, \dots, v_0) \text{ is a geodesic in } G.$$

We proceed by induction on n. Let first $n = 0$. It is obvious that (v_0) is a geodesic. According to Axiom X, $(v_0) \in R$. Thus (14_0) holds. We now assume that $n \geqslant 1$.

Let $(v_n, v_{n-1}, \dots, v_0) \in R$. Since $T = [R]$, $v_n v_{n-1} \to_T v_0$. Theorem 1 implies that (v_n, v_{n-1}, v_0) is a step in G. Hence

$$(15) \qquad d(v_n, v_{n-1}) = 1 \text{ and } d(v_n, v_0) = d(v_{n-1}, v_0) + 1.$$

As follows from Axiom Y, $(v_{n-1}, \dots, v_0) \in R$. According to (14_{n-1}), $(v_{n-1}, \dots, v_0)$ is a geodesic in G. It follows from (15) that $(v_n, v_{n-1}, \dots, v_0)$ is a geodesic in G.

Conversely, let $(v_n, v_{n-1}, \dots, v_0)$ be a geodesic in G. Then (15) holds. Hence (v_n, v_{n-1}, v_0) is a step in G. According to Theorem 1, $v_n v_{n-1} \to_T v_0$. Recall that $T = [R]$. It follows from the definition of $[R]$ that there exist $m \geqslant 0$, $u_0, \dots, u_m \in V$ such that $(v_n, u_m, \dots, u_0) \in R$, $u_0 = v_0$ and $u_m = v_{n-1}$. Since $(v_n, v_{n-1}, \dots, v_0)$ is a geodesic, $(v_{n-1}, \dots, v_0)$ is also a geodesic. According to (14_{n-1}), $(v_{n-1}, \dots, v_0) \in R$. Axiom Z implies that $(v_n, v_{n-1}, \dots, v_0) \in R$.

Thus (14_n) holds. The proof of the theorem is complete. □

Combining Theorem 2 with Proposition 2 we get the following characterization of the set of all geodesics in a connected graph.

Corollary 2. *Let V be a finite nonempty set, and let $R \subseteq \Sigma(V)$. Put $T = [R]$. Then there exists a connected graph G with the properties that $V(G) = V$ and R is the set of all geodesics in G if and only if $\Gamma(V, T)$ is connected, T fulfils Axioms* A–H *(for arbitrary $u, v, x, y \in V$) and moreover, R fulfils Axioms* X, Y *and* Z *(for arbitrary $m, n \geqslant 1$ and $u, u_0, \dots, u_m, w_0, \dots, w_n \in V$).*

Another characterization of the set of all geodesics in a connected graph can be found in [5] (cf. also [7] or [8]).

Remark 5. The concept of the set of all geodesics in a connected graph is closely connected to that of the interval function (in the sense of [4]) of a connected graph. A characterization of the interval function of a connected graph was given in [6].

References

[1] *M. Behzad, G. Chartrand and L. Lesniak-Foster*: Graphs & Digraphs. Prindle, Weber & Schmidt, Boston 1979.

[2] *F. Harary*: Graph Theory. Addison-Wesley, Reading (Mass.) 1969.

[3] *D. C. Kay and G. Chartrand*: A characterization of certain ptolemaic graphs. Canad. J. Math. *17* (1965), 342–346.

[4] *H. M. Mulder*: The Interval Function of a Graph. Mathematisch Centrum. Amsterdam 1980.

[5] *L. Nebeský*: A characterization of the set of all shortest paths in a connected graph. Mathematica Bohemica *119* (1994), 15–20.

[6] *L. Nebeský*: A characterization of the interval function of a connected graph. Czechoslovak Math. J. *44 (119)* (1994), 173–178.

[7] *L. Nebeský*: Visibilities and sets of shortest paths in a connected graph. Czechoslovak Math. J. *45(120)* (1995), 563–570.

[8] *L. Nebeský*: On the set of all shortest paths of a given length in a connected graph. Czechoslovak Math. J. *46(121)* (1996), 155–160.

Author's address: Filozofická fakulta Univerzity Karlovy, nám. J. Palacha 2, 116 38 Praha 1, Czech Republic.