Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 47 (1997), No. 1, 113–126

Persistent URL: http://dml.cz/dmlcz/127343

Czechoslovak Mathematical Journal, 47 (122) 1997, Praha

RETRACT IRREDUCIBILITY OF CONNECTED MONOUNARY ALGEBRAS II

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(Received October 10, 1994)

This paper is a continuation of [1]; for references, definitions and notation cf. [1]. The following result was proved in [1]:

(R) *Let $\mathscr{A} = (A, f)$ be a connected monounary algebra possessing a one element cycle $\{c\}$. Then the following conditions are equivalent:*

(i) *$\mathscr{A}$ is retract irreducible;*
(ii) *if a and b are elements of A such that $f(a) = f(b)$, then either $a = b$ or $c \in \{a, b\}$.*

Let $(\mathbb{N}, f)$ be a monounary algebra such that $f(n) = n + 1$ for each $n \in \mathbb{N}$. In the present paper we prove

(R1) *Let $\mathscr{A} = (A, f)$ be a connected monounary algebra which has no one-element cycle. Then the following conditions are equivalent:*

(i) *$\mathscr{A}$ is retract irreducible;*
(ii) *either $\mathscr{A} \cong (\mathbb{N}, f)$ or there are $n \in \mathbb{N}$ and a prime p such that $\mathscr{A}$ is a cycle with p^n elements.*

In the whole paper suppose that (A, f) is a connected monounary algebra and that $f(x) \neq x$ for each $x \in A$ (i.e., (A, f) has no one-element cycle).

1.1. Proposition. *Let p be a prime, $n \in \mathbb{N}$. Suppose that (A, f) is a monounary algebra consisting of one cycle, $\operatorname{card} A = p^n$. Then (A, f) is retract irreducible.*

Proof. Assume that $(A, f) \in R(\prod_{i \in I}(A_i, f))$ and that (A_i, f) is a connected monounary algebra for each $i \in I$. Then (A, f) is isomorphic to a subalgebra of $\prod_{i \in I}(A_i, f)$ and there exists $x \in \prod_{i \in I} A_i$ such that

$$f^{p^n}(x) = x, \text{ i.e., } f^{p^n}(x(i)) = x(i) \text{ for each } i \in I,$$
$$f^k(x) \neq x \text{ for each } k \in N,\ k < p^n.$$

This implies that if $i \in I$, then $x(i)$ belongs to a cycle having m_i elements, where m_i/p^n. Suppose that $m_i < p^n$ for each $i \in I$. Then

$$m = \text{l.c.m.}\{m_i : i \in I\} < p^n$$

and we have

$$f^m(x(i)) = x(i) \text{ for each } i \in I, \text{ i.e., } f^m(x) = x,$$

which is a contradiction. Therefore there is $i \in I$ such that $m_i = p^n$. Let M be the cycle of (A_i, f). Then

(1) $$(M, f) \cong (A, f);$$

(2) $$\text{if } y \in f^{-1}(M), \text{ then there is } z \in M \text{ with } f(y) = f(z)$$
$$\text{and } s_f(y) \leqslant \infty = s_f(z).$$

By [1], 1.3, $(A, f) \in R(A_i, f)$, thus (A, f) is retract irreducible. □

In 1.2–1.9 suppose that (A, f) is a connected monounary algebra containing a cycle C with $1 < \operatorname{card} C < \operatorname{card} A$.

1.2. Construction. Let $\operatorname{card} C = n$, $C = \{a_1, \dots, a_n\}$, $f(a_1) = a_2$, ..., $f(a_n) = a_1$. Denote $I = \{1, 2, \dots, n\}$ and put

$$A_i = \{x \in A : (\exists k \in \mathbb{N})(f^k(x) = a_i, f^{k-1}(x) \notin C)\} \text{ for } i \in I.$$

If $i \in I$, consider a new element $a_i' \notin A$ such that $a_i' \neq a_j'$ for each $i, j \in I$, $i \neq j$. For $i \in I$ define

$$B_i = \{a_i'\} \cup A_i,$$
$$g(x) = \begin{cases} f(x) & \text{if } \{x, f(x)\} \subseteq A_i, \\ a_i' & \text{if } x = a_i' \text{ or } x \in A_i,\ f(x) = a_i, \end{cases}$$
$$B_0 = C,$$
$$g(x) = f(x) \text{ for each } x \in C.$$

Further let

$$(B, g) = \prod_{i \in I \cup \{0\}} (B_i, g).$$

1.3. Lemma. *If $i \in I \cup \{0\}$, then $(A, f) \notin R(B_i, g)$.*

Proof. Since (A, f) is not a cycle and (B_0, g) is a cycle, (A, f) cannot be isomorphic to a subalgebra of (B_0, g), thus we obtain that $(A, f) \notin R(B_0, g)$. Let $i \in I$. The algebra (B_i, g) is connected and it contains a one-element cycle, the algebra (A, f) contains a cycle with more than one element, hence $(A, f) \notin R(B_i, g)$. □

1.4. Notation. If $i \in I$ and $t \in A_i$, then there is a unique $k(t) \in \mathbb{N}$ such that $f^{k(t)}(t) = a_i$, $f^{k(t)-1}(t) \notin C$. Further let $m(t) \in I$ be such that

$$m(t) \equiv i - k(t) \pmod{n}.$$

Let us define the following sets for $i \in I$:

$$\begin{gathered} T_0 = \{x \in B \colon x(j) = a'_j \text{ for each } j \in I\}, \\ T_i = \{x \in B \colon x(j) = a'_j \text{ for each } j \in I - \{i\}, x(i) \in A_i, \\ x(0) = a_{m(x(i))}\}, \\ T = \bigcup_{i \in I \cup \{0\}} T_i. \end{gathered}$$

Notice that $T_i \cap T_j = \emptyset$ for each $i, j \in I \cup \{0\}$, $i \neq j$. If $x \in T_0$, then $x(0) \in B_0 = C \subseteq A$. If $i \in I$ and $x \in T_i$, then $x(i) \in A_i \subseteq A$.

Define a mapping $\nu \colon T \to A$ as follows: if $x \in T_i$ for some $i \in I \cup \{0\}$, then $\nu(x) = x(i)$.

1.5. Lemma. *The mapping ν is bijective.*

Proof. Let $x, y \in T$, $\nu(x) = \nu(y) = d \in A$. Then either $d \in C$ or $d \in A_i$ for some $i \in I$. If $d \in C$, then $d \notin A_i$ for each $i \in I$, thus $\{x, y\} \subseteq T_0$ and we get

$$\begin{gathered} x(0) = \nu(x) = \nu(y) = y(0), \\ x(i) = a'_i = y(i) \text{ for each } i \in I. \end{gathered}$$

Let $d \in A_i$ for some $i \in I$. Then $\{x, y\} \subseteq T_i$, which implies

$$\begin{gathered} x(i) = \nu(x) = \nu(y) = y(i), \\ x(j) = a'_j = y(j) \text{ for each } j \in I - \{i\}, \\ x(0) = a_{m(x(i))} = a_{m(y(i))} = y(0). \end{gathered}$$

Therefore the mapping ν is injective.

Let $z \in C$. There is $x \in T_0$ with $x(0) = z$ and $x(i) = a'_i$ for each $i \in I$. Then $\nu(x) = z$. Next let $i \in I$ and $z \in A_i$. There exists $x \in T_i$ with $x(j) = a'_j$ for each $j \in I - \{i\}$, $x(i) = z$, $x(0) = a_{m(z)}$. Then $\nu(x) = z$, thus ν is a surjective mapping. □

1.6. Lemma. *(T, g) is a monounary algebra and ν is an isomorphism of (T, g) onto (A, f).*

Proof. According to 1.5 it suffices to show that ν is a homomorphism. Let $x \in T_0$. Put $y = g(x)$. Then

$$y(i) = (g(x))(i) = g(x(i)) = g(a'_i) = a'_i \text{ for each } i \in I,$$
$$\nu(x) = x(0), \;\; \nu(y) = y(0),$$

thus we obtain

$$\nu(g(x)) = \nu(y) = y(0) = (g(x))(0) = g(x(0)) = f(x(0)) = f(\nu(x)). \tag{1}$$

Now let $i \in I$, $x \in T_i$, $y = g(x)$. If $k(x(i)) = 1$, then $y(i) = g(x(i)) = a'_i$, $y \in T_0$, which implies

$$\begin{aligned} \nu(g(x)) = \nu(y) = y(0) = (g(x))(0) = \\ = g(x(0)) = f(x(0)) = f(a_{m(x(i))}). \end{aligned} \tag{2}$$

Since $m(x(i)) + 1 \equiv i \pmod n$, we have $f(a_{m(x(i))}) = a_i$, therefore (2) yields

$$\nu(g(x)) = a_i = f(x(i)) = f(\nu(x)). \tag{3}$$

Let $k(x(i)) > 1$. Then $k(y(i)) = k(x(i)) - 1$ and

$$y(0) = (g(x))(0) = g(x(0)) = f(x(0)) = a_j,$$

where

$$j \equiv m(x(i)) + 1 \equiv i - k(x(i)) + 1 \equiv i - k(y(i)) \pmod n,$$

thus $y \in T_i$ and we get

$$\nu(g(x)) = \nu(y) = y(i) = (g(x))(i) = g(x(i)) = g(\nu(x)) = f(\nu(x)). \tag{4}$$

The relations (1), (3) and (4) yield that ν is a homomorphism, therefore ν is an isomorphism of (T, g) onto (A, f). □

1.7. Lemma. *Let K be a connected component of (B, g) with $K \cap T = \emptyset$.*

(a) *If K contains a cycle with m elements, then n/m.*

(b) *If K contains no cycle and x is a fixed element of K, then there is $y \in T$ such that $s_g(g^k(x)) \leqslant s_g(g^k(y))$ for each $k \in \mathbb{N} \cup \{0\}$.*

Proof. (a) If K contains a cycle with m elements and z is an element of this cycle, then we have

$$g^m(z(0)) = (g^m(z))(0) = z(0).$$

Since $z(0) \in C$, this yields that n divides m.

(b) Let K contain no cycle and let $x \in K$. Take $y \in T_0$ with $y(0) = a_1$. Then y belongs to a cycle, $s_g(y) = \infty$ and

$$s_g(g^k(x)) \leqslant \infty = s_g(g^k(y)) \text{ for each } k \in \mathbb{N} \cup \{0\}.$$

□

1.8. Lemma. *If $y \in g^{-1}(T)$, then there exists $z \in T$ with $g(y) = g(z)$ and $s_g(y) \leqslant s_g(z)$.*

Proof. By 1.6, $\nu\colon (T, g) \to (A, f)$ is an isomorphism, thus (T, g) is a connected monounary algebra with a cycle T_0. Let $y \in g^{-1}(T)$, $g(y) = t$. If $t \in T_0$, then there is $z \in T_0$ with $f(z) = t$ and we have

$$g(y) = g(z), \ s_g(y) \leqslant \infty = s_g(z). \tag{1}$$

Suppose that $t \in T_i$, $i \in I$. Consider $z \in T_i$ such that $z(j) = a'_j$ for $j \in I - \{i\}$, $z(i) = y(i)$, $z(0) = a_{m(y(i))}$. We have

$$(g(y))(j) = t(j) = a'_j = g(a'_j) = g(z(j)) = (g(z))(j) \text{ for } j \in I - \{i\},$$
$$(g(y))(i) = g(y(i)) = g(z(i)) = (g(z))(i),$$
$$(g(y))(0) = t(0) = a_{m(t(i))}.$$

By 1.4, $m(t(i)) \equiv i - k(t(i)) = i - k((g(y))(i)) = i - k(f(y(i))) = i - [k(y(i)) - 1] \equiv$ $\equiv m(y(i)) + 1 \pmod n$, thus $a_{m(t(i))} = g(a_{m(y(i))})$ and we get

$$(g(y))(0) = g(a_{m(y(i))}) = g(z(0)) = (g(z))(0).$$

Hence

$$g(y) = g(z). \tag{2}$$

By the definition of z we have

$$s_g(z(j)) = \infty \text{ for each } j \in I \cup \{0\} - \{i\},$$
$$s_g(z(i)) = s_g(y(i)),$$

and these relations imply

$$s_g(z) \geqslant s_g(y). \tag{3}$$

According to (1)–(3), the assertion is valid. □

1.9. Corollary. *T is a retract of (B, g).*

Proof. This a consequence of 1.7, 1.8 and of 1.3, [1]. □

1.10. Proposition. *If (A, f) is a connected monounary algebra containing a cycle C with $1 < \operatorname{card} C < \operatorname{card} A$, then (A, f) is retract reducible.*

Proof. The assertion follows from 1.3, 1.6 and 1.9. □

1.11. Proposition. *Suppose that (A, f) is a connected monounary algebra containing a cycle of cardinality which is not a power of a prime. Then (A, f) is retract reducible.*

Proof. Let C be a cycle of (A, f). Then $n = \operatorname{card} C > 1$ and there are $p, q \in \mathbb{N} - \{1\}$ with $n = p \cdot q$, g.c.d.$(p, q) = 1$. If $C \neq A$, then (A, f) is retract reducible by 1.10. Assume that $C = A$. Take monounary algebras (B, f), (D, f), which form a p-element cycle or a q-element cycle, respectively. It is obvious that $(A, f) \notin R(B, f) \cup R(D, f)$. Further, $(B, f) \times (D, f)$ is a cycle with n elements, i.e., it is isomorphic to (A, f). Thus

$$(A, f) \in R((B, f) \times (D, f))$$

and (A, f) is retract reducible. □

2. (A, f) POSSESSING NO CYCLE AND WITH $a \in A$, $s_f(a) = \infty$

In this section suppose that (A, f) is a connected monounary algebra possessing no cycle and such that there is $a \in A$ with $s_f(a) = \infty$. Then there are distinct elements $a_i \in A$ for $i \in \mathbb{Z}$ such that $f(a_i) = a_{i+1}$ for each $i \in \mathbb{Z}$.

2.0. Notation. Let $n \in \mathbb{N}$. If $i \in \mathbb{Z}$, then there exists a unique $j \in \{0, \ldots, n-1\}$ such that $i \equiv j \pmod{n}$. Denote

$$i_n = \{k \in \mathbb{Z} \colon k \equiv i \pmod{n}\}$$

and let

$$\mathbb{Z}_n = \{0_n, 1_n, \ldots, (n-1)_n\}.$$

Then $i_n \in \mathbb{Z}_n$ for each $i \in \mathbb{Z}$. For elements of $\mathbb{Z}_n$ the operation $+$ is taken modulo n.

2.1. Construction. Let $C = \{a_i \colon i \in \mathbb{Z}\}$. If $i \in \mathbb{Z}$, then we put

$$A_i = \{x \in A - C \colon f(x) = a_i\},$$
$$D_i = \{y \in A \colon (\exists x \in A_i)(\exists n \in \mathbb{N} \cup \{0\})(f^n(y) = x)\}.$$

Consider a set $P = \{p_i \colon i \in \mathbb{Z}\}$ of distinct primes. For $i \in \mathbb{Z}$ let $(\mathbb{Z}_{p_i}, g)$ be a cyclic monounary algebra, where

$$g(z_{p_i}) = (z+1)_{p_i} \text{ for each } z_{p_i} \in \mathbb{Z}_{p_i}.$$

Further let

$$B_i = \mathbb{Z}_{p_i} \cup D_i,$$
$$g(x) = \begin{cases} f(x) & \text{if } x \in D_i - A_i, \\ i_{p_i} & \text{if } x \in A_i, \end{cases}$$
$$(B, g) = \prod_{i \in \mathbb{Z}} (B_i, g).$$

The definition of (B, g) implies that no connected component of (B, g) contains a cycle.

2.2. Lemma. *If $i \in \mathbb{Z}$, then $(A, f) \notin R(B_i, g)$.*

Proof. Since (B_i, g) is connected and contains a cycle $\mathbb{Z}_{p_i}$ and (A, f) contains no cycle, the assertion is obvious. □

2.3. Notation. Let

$$T_0 = \{b \in B\colon (\exists k \in \mathbb{Z})(b(i) = k_{p_i} \text{ for each } i \in \mathbb{Z})\},$$
$$T_1 = \{b \in B\colon (\exists i \in \mathbb{Z})(\exists n \in \mathbb{N})(b(j) = (i-n)_{p_j}$$
$$\text{for each } j \in \mathbb{Z} - \{i\}, f^{n-1}(b(i)) \in A_i)\},$$
$$T = T_0 \cup T_1.$$

Define a mapping $\nu\colon T \to A$ as follows. If $b \in T_0$, $k \in \mathbb{Z}$, $b(i) = k_{p_i}$ for each $i \in \mathbb{Z}$, then we put $\nu(b) = a_k$. If $b \in T_1$, $i \in \mathbb{Z}$, $n \in \mathbb{N}$, $b(j) = (i-n)_{p_j}$ for each $j \in \mathbb{Z} - \{i\}$, $f^{n-1}(b(i)) \in A_i$, then we put $\nu(b) = b(i)$.

2.4. Lemma. *The mapping ν is bijective.*

Proof. If $b, t \in T$, $\nu(b) = \nu(t)$, then either $\{b,t\} \subseteq T_0$ or $\{b,t\} \subseteq T_1$, since $\nu(T_0) \subseteq C$, $\nu(T_1) \subseteq A - C$. In the first case obviously $b = t$. If $\{b,t\} \subseteq T_1$, then there are $i, i' \in \mathbb{Z}$, $n, n' \in \mathbb{N}$ with

$$b(j) = (i-n)_{p_j} \text{ for each } j \in \mathbb{Z} - \{i\},$$
$$t(j) = (i'-n')_{p_j} \text{ for each } j \in \mathbb{Z} - \{i'\},$$
$$f^{n-1}(b(i)) \in A_i,\ f^{n'-1}(t(i')) \in A_{i'}.$$

We have $b(i) = \nu(b) = \nu(t) = t(i)$, thus $i = i'$, $n = n'$. Then $b(j) = t(j)$ for each $j \in \mathbb{Z}$ and $b = t$.

Now let $x \in A$. If $x \in C$, then $x = a_k$ for some $k \in \mathbb{Z}$. Put $b(i) = k_{p_i}$ for each $i \in \mathbb{Z}$. Then $b \in T_0$ and $x = \nu(b)$. If $x \in A - C$, then there is $i \in \mathbb{Z}$ with $x \in D_i$, thus there is $n \in \mathbb{N}$ such that $f^{n-1}(x) \in A_i$. We can take an element $b \in T_1$ satisfying the relation $b(j) = (i-n)_{p_j}$ for each $j \in \mathbb{Z} - \{i\}$, $b(i) = x$, and we obtain $\nu(b) = x$. Hence the mapping ν is a bijection. □

2.5. Lemma. *(T,g) is a monounary algebra and ν is an isomorphism of (T,g) onto (A,f).*

Proof. By 2.4 we have to prove that ν is a homomorphism. Let $b \in T_0$, $b(i) = k_{p_i}$ for each $i \in \mathbb{Z}$, where $k \in \mathbb{Z}$. Take $g(b) = t$. Then $t(i) = (k+1)_{p_i}$ for each $i \in \mathbb{Z}$ and

(1) $$\nu(g(b)) = \nu(t) = a_{k+1} = f(a_k) = f(\nu(b)).$$

Let $b \in T_1$, i.e., there are $i \in \mathbb{Z}$, $n \in \mathbb{N}$ such that $b(j) = (i-n)_{p_j}$ for each $j \in \mathbb{Z} - \{i\}$, $f^{n-1}(b(i)) \in A_i$. If $g(b) = t$, then

(2) $$t(j) = g(b(j)) = (i-n+1)_{p_j} = (i-(n-1))_{p_j}.$$

Consider two cases: a) $n > 1$, b) $n = 1$. If a) is valid, then $t(i) = g(b(i)) = f(b(i))$ and

$$f^{n-2}(t(i)) = f^{n-2}(f(b(i))) = f^{n-1}(b(i)) \in A_i, \tag{3}$$

thus (2) and (3) yield

$$\nu(g(b)) = \nu(t) = t(i) = f(b(i)) = f(\nu(b)). \tag{4}$$

Let b) hold. Then $b(i) \in A_i$,

$$t(i) = g(b(i)) = i_{p_i}, \tag{5}$$

and, by (2),

$$t(j) = i_{p_j} \text{ for each } j \in \mathbb{Z} - \{i\}. \tag{6}$$

In view of (5) and (6), $\nu(t) = a_i$ and thus

$$\nu(g(b)) = \nu(t) = a_i = f(b(i)) = f(\nu(b)). \tag{7}$$

Therefore ν is a homomorphism. □

2.6. Lemma. *If $y \in g^{-1}(T)$, then there exists $z \in T$ with $g(y) = g(z)$ and $s_g(y) \leqslant s_g(z)$.*

Proof. According to 2.5, ν is an isomorphism of (T, g) onto (A, f). Notice that $\nu^{-1}(C) = T_0$, thus

$$s_g(t) = \infty \text{ for each } t \in T_0. \tag{1}$$

Let $y \in g^{-1}(T)$, $g(y) = t$. If $t \in T_0$, then there is $z \in T_0$ with $g(z) = t$ and then, by (1),

$$s_g(y) \leqslant \infty = s_g(z). \tag{2}$$

Now suppose that $t \in T_1$. There are $i \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $t(j) = (i-n)_{p_j}$ for each $j \in \mathbb{Z} - \{i\}$, $f^{n-1}(t(i)) \in A_i$. Put

$$z(j) = (i-n-1)_{p_j} \text{ for each } j \in \mathbb{Z} - \{i\},$$
$$z(i) = y(i).$$

Then $z \in T_1$ and

$$(g(z))(j) = g(z(j)) = (i-n)_{p_j} = t(j) = (g(y))(j) \text{ for each } j \in \mathbb{Z} - \{i\},$$
$$(g(z))(i) = g(z(i)) = g(y(i)) = (g(y))(i),$$

i.e.,

$$g(z) = g(y). \tag{3}$$

Further,

$$s_g(z(j)) = \infty \text{ for each } j \in \mathbb{Z} - \{i\},$$
$$s_g(z(i)) = s_g(y(i)),$$

which implies that $s_g(z) \geqslant s_g(y)$. □

2.7. Lemma. *T is a retract of (B, g).*

Proof. (T, g) is a connected subalgebra of (B, g) according to 2.5. In 2.6 it was shown that the condition (a) of 1.3, [1] is valid. Let K be a connected component of (B, g). By the definition, K contains no cycle, thus (b2) of 1.3, [1] is trivially satisfied. Let $x_0 \in K$. There is $t \in T_0$ with $s_g(t) = \infty$, thus

$$s_g(g^k(x_0)) \leqslant \infty = s_g(g^k(t)) \text{ for each } k \in \mathbb{N} \cup \{0\},$$

hence (b1) of 1.3, [1] holds, too. Therefore T is a retract of (B, g). □

2.8. Proposition. *Let (A, f) be a connected monounary algebra possessing no cycle and such that there is $a \in A$ with $s_f(a) = \infty$. Then (A, f) is retract reducible.*

Proof. The required assertion is obtained in view of 2.1, 2.2, 2.5 and 2.7. □

3. (A, f) WITH $s_f(x) \neq \infty$ FOR EACH $x \in A$

Suppose that $s_f(x) \neq \infty$ for each $x \in A$. Then (A, f) contains no cycle.

3.1. Lemma. *Let (E, f) be a connected monounary algebra and let (M, f) be a subalgebra of (E, f) such that $M = \{e_n : n \in \mathbb{N}\}$, $e_n \neq e_m$ for each $n, m \in \mathbb{N}$, $n \neq m$, $f(e_n) = e_{n+1}$ for each $n \in \mathbb{N}$. Then M is a retract of (E, f) if and only if*

$$f^{-k}(e_k) = \emptyset \text{ for each } k \in \mathbb{N}. \tag{1}$$

Proof. Suppose that M is a retract of (E, f) and suppose that there are $k \in \mathbb{N}$ and $x \in f^{-k}(e_k)$. Let h be a corresponding retraction endomorphism of (E, f) onto (M, f). Then there is $j \in \mathbb{N}$ with $h(x) = e_j$. We obtain

$$e_k = h(e_k) = h(f^k(x)) = f^k(h(x)) = f^k(e_j) = e_{j+k},$$

which is a contradiction.

Conversely, suppose that (1) is valid. If $x \in E$, then there exists $n(x) \in \mathbb{N} \cup \{0\}$ such that $f^{n(x)}(x) \in M$, $f^m(x) \notin M$ for each $m \in \mathbb{N} \cup \{0\}, m < n(x)$. Then $n(x)$ is uniquely determined. Next we have $f^{n(x)}(x) = e_{k(x)}$ for some (uniquely determined, too) $k(x) \in \mathbb{N}$. According to (1), $n(x) < k(x)$ and we can define

$$h(x) = e_{k(x)-n(x)}.$$

For $x \in M$ we have $n(x) = 0$ and $h(x) = x$. Further, $f(x) \in M$, thus

$$h(f(x)) = f(x) = f(h(x)).$$

Let $x \in E - M$. Put $y = f(x)$. If $n(x) > 1$, then $n(y) = n(x) - 1$, $k(y) = k(x)$ and

$$h(f(x)) = h(y) = e_{k(y)-n(y)} = e_{k(x)-n(x)+1} = f(e_{k(x)-n(x)}) = f(h(x)).$$

If $n(x) = 1$, then $y = e_{k(x)}$, $k(x) > 1$ (by (1)) and we get

$$h(f(x)) = h(e_{k(x)}) = e_{k(x)} = f(e_{k(x)-1}) = f(h(x)).$$

Thus M is a retract of (E, f). □

3.2. Corollary. *Assume that $A = \{a_n : n \in \mathbb{N}\}$, $a_m \neq a_n$ for each $m, n \in \mathbb{N}, m \neq n$ and that $f(a_n) = a_{n+1}$ for each $n \in \mathbb{N}$. Suppose that (E, f) is a connected monounary algebra. The following conditions are equivalent:*

(i) *$(A, f) \in R(E, f)$;*

(ii) *there exist distinct elements $e_n \in E$ $(n \in \mathbb{N})$ such that $f(e_n) = e_{n+1}$ and $f^{-n}(e_n) = \emptyset$ for each $n \in \mathbb{N}$.*

3.3. Proposition. *Assume that $A = \{a_n : n \in \mathbb{N}\}$, $a_m \neq a_n$ for each $m, n \in \mathbb{N}$, $m \neq n$ and that $f(a_n) = a_{n+1}$ for each $n \in \mathbb{N}$. Then (A, f) is retract irreducible.*

Proof. Suppose that there are connected monounary algebras (B_i, f), $i \in I$, such that

$$(A, f) \in R\Big(\prod_{i \in I}(B_i, f)\Big), \tag{1}$$

$$(A, f) \notin R(B_i, f) \text{ for each } i \in I. \tag{2}$$

Put

$$(B, f) = \prod_{i \in I} (B_i, f).$$

There exists $(M, f) \cong (A, f)$ such that M is a retract of (B, f). By 3.2, there are distinct elements $b_n \in B$ $(n \in \mathbb{N})$ such that

(3) $$f(e_n) = e_{n+1} \text{ for each } n \in \mathbb{N},$$

(4) $$f^{-n}(e_n) = \emptyset \text{ for each } n \in \mathbb{N}.$$

If $i \in I$, then

(3′) $$f(e_n(i)) = e_{n+1}(i) \text{ for each } n \in \mathbb{N}.$$

If the elements $e_n(i)$, $n \in \mathbb{N}$ are not mutually distinct, then

(5.1) $$(B_i, f) \text{ contains a cycle.}$$

If the elements $e_n(i)$, $n \in \mathbb{N}$ are distinct, then 3.2, (2) and (3′) imply

(5.2) $$f^{-n}(e_n(i)) \neq \emptyset \text{ for some } n \in \mathbb{N}.$$

Therefore, in both the cases, for each $i \in I$, there exist $n \in \mathbb{N}$ and $t_i \in B_i$ such that

(6) $$f^n(t_i) = e_n(i).$$

Take $t \in B$ with $t(i) = t_i$ for each $i \in I$. Then (6) implies

$$f^n(t) = e_n,$$

which contradicts (4). □

In 3.4–3.8 suppose that the assumption of 3.3 is not satisfied.

3.4. Construction. Since $s_f(x) \neq \infty$ for each $x \in A$, the set $L = \{a \in A\colon f^{-1}(a) = \emptyset\}$ is nonempty. The assumption of 3.3 is not satisfied, thus

$$\{k \in \mathbb{N}\colon \operatorname{card} f^{-1}(f^k(a)) > 1\} \neq \emptyset \text{ for each } a \in L;$$

put

$$k(a) = \min\{k \in \mathbb{N}\colon \operatorname{card} f^{-1}(f^k(a)) > 1\}.$$

Further let

$$m = \min\{k(a)\colon a \in L\},$$
$$J = \{a \in L\colon k(a) = m\},$$
$$V = \{f^m(a)\colon a \in J\}.$$

For each $v \in V$ such that $f^{-m}(v) \subseteq J$ we choose a fixed element of the set $f^{-m}(v)$ and we denote this fixed element by $\overline{v}$. Then define

$$I = \{a \in J\colon f^{-m}(f^m(a)) \not\subseteq J\} \cup \{a \in J\colon f^{-m}(f^m(a)) \subseteq J, a \neq \overline{f^m(a)}\}.$$

Let $c \notin A$ be a new element and if $a \in I$, then put

$$A_a = \{a, f(a), \dots, f^{m-1}(a)\},$$
$$B_a = A_a \cup \{c\},$$
$$g(x) = \begin{cases} f(x) & \text{if } x \in A_a - \{f^{m-1}(a)\}, \\ c & \text{if } x \in \{c, f^{m-1}(a)\}. \end{cases}$$

Denote

$$B_0 = A - \bigcup_{a \in I} A_a.$$

Then (B_0, f) is a subalgebra of (A, f) and we put

$$(B_0, g) = (B_0, f),$$
$$(B, g) = \prod_{a \in I \cup \{0\}} (B_a, g).$$

3.5. Lemma. *If $a \in I \cup \{0\}$, then $(A, f) \notin R(B_a, g)$.*

Proof. If $a \in I$, then (B_a, g) contains a cycle, thus obviously $(A, f) \notin R(B_a, g)$. The relation $(A, f) \notin R(B_0, g)$ can be proved in the same way as in the proof of 5.2, [1]. □

3.6. Lemma. *If $a \in I$, then there exists an endomorphism φ_a of (A, f) such that $\varphi_a(x) \neq x$ iff $x \in A_a$ and $\varphi_a(A_a) \subseteq B_0$.*

Proof. Analogously as 5.3, [1]. □

3.7. Notation. Denote

$$T_0 = \{b \in B\colon b(0) \in B_0, b(a) = c \text{ for each } a \in I\}$$

and if $a \in I$, then

$$T_a = \{b \in B\colon b(a) \in A_a, b(i) = c \text{ for each } i \in I - \{a\},\\ b(0) = \varphi_a(b(a))\}.$$

Let

$$T = \bigcup_{a \in I \cup \{0\}} T_a.$$

The proofs of the assertions of the following lemma are the same as proofs of 5.5, [1] or 5.6, [1], respectively.

3.8. Lemma. $(T, g) \cong (A, f)$ *and* T *is a retract of* (B, g).

3.9. Proposition. *Suppose that* (A, f) *is a connected monounary algebra not fulfilling the assumption of* 3.3 *and such that* $s_f(x) \neq \infty$ *for each* $x \in A$. *Then* (A, f) *is retract reducible.*

Proof. It is a consequence of 3.4, 3.5 and 3.8. □

4. Proof of Theorem (R1)

Let (A, f) be a connected monounary algebra not possessing a one-element cycle. Then some of the following possibilities occurs:

(1) (A, f) consists of one cycle, $\operatorname{card} A = p^n$, p is a prime, $n \in \mathbb{N}$.
(2) (A, f) consists of one cycle and (1) is not valid.
(3) (A, f) contains a cycle C with $1 < \operatorname{card} C < \operatorname{card} A$.
(4) (A, f) contains no cycle and there is $a \in A$ with $s_f(a) = \infty$.
(5) (A, f) is isomorphic to $(\mathbb{N}, f)$, where $f(n) = n + 1$ for each $n \in \mathbb{N}$.
(6) (A, f) does not satisfy (5) and $s_f(x) \neq \infty$ for each $x \in A$.

If (1) or (5) holds, then (A, f) is retract irreducible by 1.1 and 3.3. If (2), (3), (4) or (6) is valid, then 1.11, 1.10, 2.8 and 3.9 imply that (A, f) is retract reducible.

References

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