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ON HALF LATTICE ORDERED GROUPS

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The notion of half lattice ordered groups was introduced and studied by Giraudet and Lucas [3]; it is a generalization of the notion of a lattice ordered group.

Each half lattice ordered group can be represented as a group of monotone transformations of a linearly ordered set [3].

We apply the same terminology and notation as in [3]. In particular, if G is a half lattice ordered group, then $G\uparrow$ is the connected component of G containing the neutral element e of G. This substructure $G\uparrow$ of G is a lattice ordered group.

The half lattice ordered group G fails to be uniquely determined by the lattice ordered group $G\uparrow$. In [3] it was proved that there exist half lattice ordered groups G_1 and G_2 such that G_1 is not isomorphic to G_2, $G_1\uparrow = G_2\uparrow$ and $G_1\uparrow \neq G_1$, $G_2\uparrow \neq G_2$.

In the present paper we investigate congruence relations on and small direct products of half lattice ordered groups. The motivation of introducing the latter concept is as follows.

Let $\mathcal{H}$ be the class of all half lattice ordered groups and let $\mathcal{H}_1$ be the class of all elements of $\mathcal{H}$ which fail to be lattice ordered groups. If I is a nonempty set and if $G_i \in \mathcal{H}$ for each $i \in I$, then the direct product $\prod\limits_{i\in I} G_i$ need not belong to $\mathcal{H}$.

Let $G_i \in \mathcal{H}_1$ for each $i \in I$. We construct a substructure G^0 of $\prod\limits_{i\in I} G_i$ such that G^0 belongs to $\mathcal{H}_1$ and satisfies a certain maximality condition. G^0 will be said to be a small direct product of the system $(G_i)_{i\in I}$.

The relations between direct product decompositions of the lattice ordered group $G\uparrow$ and small direct product decompositions of G will be dealt with.

Sample results:

Each congruence relation on a half lattice ordered group G is determined by an ℓ-ideal of the lattice ordered group $G\uparrow$ which is normal in G.

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Let $G \in \mathcal{H}_1$. If $G\!\uparrow = \prod\limits_{i\in I} A_i$ is such that, for each $i \in I$, A_i is normal in G and $A_i \neq \{e\}$, then G can be expressed as a small direct product of a system $(G_i)_{i\in I}$ with $G_i\!\uparrow = A_i$ for each $i \in I$.

If C is a normal convex chain in G such that $e \in C$ and C has neither an upper bound nor a lower bound in G, then there exist $G_1, G_2 \in \mathcal{H}_1$ such that (i) G is a small direct product of G_1 and G_2, and (ii) $C = G_1\!\uparrow$.

We define a set $SD_r(G)$ of small direct product decompositions of G which will be called regular. Each small direct product decomposition of G is isomorphic to an element of $SD_r(G)$. It is proved that under a natural partial order the set $SD_r(G)$ is a meet-semilattice.

It is shown that any two small direct product decompositions of G have isomorphic refinements.

Let us recall that an analogous theorem for lexicographic product decompositions of linearly ordered groups was proved by Maltsev [6]; this result was generalized by Fuchs [2] and by the author [5].

1. Preliminaries

We recall the definition of a half lattice ordered group (cf. [3], Section 1).

Let G be a group with the neutral element e. Further, suppose that G is a partially ordered set.

We denote by $G\!\uparrow$ and $G\!\downarrow$ the set of all $x \in G$ such that, whenever $y, z \in G$ and $y \leqslant z$, then $xy \leqslant xz$ or $xy \geqslant xz$, respectively.

G is said to be a *half lattice ordered group* if the following conditions are satisfied:

1) the partial order $\leqslant$ on G is nontrivial (i.e., there are $x_1, x_2 \in G$ with $x_1 < x_2$);
2) if $x, y, z \in G$ and $y \leqslant z$, then $yx \leqslant zx$;
3) $G = G\!\uparrow \cup G\!\downarrow$;
4) $G\!\uparrow$ is a lattice.

In what follows we assume that G is a half lattice ordered group. Let $\mathcal{H}$ be as above. Next let $\mathcal{H}_1$ be the class of all elements G of $\mathcal{H}$ such that $G\!\downarrow \neq \emptyset$.

It is obvious that $\mathcal{H} \setminus \mathcal{H}_1$ is the class of all lattice ordered groups with more than one element.

1.1. Proposition. (Cf. [3]). *Let $G \in \mathcal{H}_1$. Then*

(i) *$G\!\uparrow$ is a subgroup of G having the index 2;*
(ii) *the partially ordered sets $G\!\uparrow$ and $G\!\downarrow$ are isomorphic and, at the same time, dually isomorphic;*
(iii) *if $x \in G\!\uparrow$ and $y \in G\!\downarrow$, then x and y are incomparable.*

Let I be a nonempty set and for each $i \in I$ let G_i be a half lattice ordered group. Hence for each $i \in I$ we consider the structure

$$(G_i; \leqslant, \cdot),$$

where $\leqslant$ is a partial order on G_i and $\cdot$ is a group operation on G_i such that the conditions 1)–4) are satisfied.

We can construct the direct product

$$G^1 = \prod_{i \in I} G_i$$

in the usual way (i.e., the partial order and the group operation in G^1 are defined component-wise).

For $g \in G^1$ and $i \in I$ we denote by g_i the component of g in G_i.

2.1. Lemma. *Let G^1 be as above and let $\operatorname{card} I \geqslant 2$. Then the following conditions are equivalent:*

(i) *G^1 is a lattice ordered group;*
(ii) *G^1 is a half lattice ordered group;*
(iii) *for each $i \in I$, G_i is a lattice ordered group.*

Proof. The relations (i) $\Leftrightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are obviously valid. Suppose that (iii) fails to hold. Hence there exists $i(1) \in I$ with $G_{i(1)} \downarrow \neq \emptyset$. Next there is $i(2) \in I$ such that $i(2) \neq i(1)$.

Choose $y, z \in G^1$ such that

$$y_i < z_i \quad \text{for each} \quad i \in I.$$

Thus $y < z$. There exists $x \in G^1$ with

$$x_{i(1)} \in G_{i(1)} \downarrow, \quad x_i \in G_i\uparrow \quad \text{for each} \quad i \in I \setminus \{i(1)\}.$$

Then

$$x_{i(1)} y_{i(1)} > x_{i(1)} z_{i(1)},$$
$$x_i y_i < x_i z_i \quad \text{for each} \quad i \in I \setminus \{i(1)\}.$$

Hence the elements xy and xz are incomparable. Thus $x \notin G\uparrow \cup G\downarrow$. Therefore G^1 is not a half lattice ordered group. □

Again, let G^1 be as above. We denote by G^0 the set of all $g \in G^1$ such that either

$$g_i \in G_i{\uparrow} \quad \text{for each} \quad i \in I, \tag{1}$$

or

$$g_i \in G_i{\downarrow} \quad \text{for each} \quad i \in I. \tag{2}$$

Then G^0 is a subgroup of the group G^1. The partial order on G^0 is inherited from that in G^1.

2.2. Lemma. *G^0 is a half lattice ordered group.*

Proof. We have to verify that the conditions 1)–4) above are valid. Let $i \in I$. Since $G_i \in \mathcal{H}$ there exists $x^i \in G_i$ with $e < x^i$. Hence $x^i \in G_i{\uparrow}$. Let $g \in G^1$ be such that $g_i = x^i$ for each $i \in I$. Then $g > e$. In view of the definition of G^0 we have $g \in G^0$ and $e \in G^0$. Hence 1) holds.

Since the multiplication in G^0 is performed component-wise we infer that 2) is valid.

The set $G^0{\uparrow}$ consists of those elements g of G^0 which satisfy (1); similarly, $G^0{\downarrow}$ is the set of elements of G^0 satisfying (2). Thus the condition 3) holds. The validity of 4) is obvious. □

2.3. Lemma. *Let G^2 be a subgroup of G^1 and let $\leqslant$ be the partial order on G^2 which is inherited from G^1. Suppose that G^2 is a half lattice ordered group such that $G^0 \subseteq G^2$. Then $G^0 = G^2$.*

Proof. We proceed similarly as in the proof of 2.1. By way of contradiction, suppose that G^2 fails to be a subset of G^0. Thus there are $i(1)$ and $i(2)$ in I and $g \in G^2$ such that

$$g_{i(1)} \in G_{i(1)}{\uparrow}, \quad g_{i(2)} \in G_{i(2)}{\downarrow}.$$

For each $i \in I$ we have $G_i \neq \{e\}$ and hence in view of 1.1, $G_i{\uparrow} \neq \{e\}$; thus there exists $g^i \in G_i{\uparrow}$ with $e < g^i$. According to the definition of G^0 there exists $z \in G^0$ such that $z_i = g^i$ for each $i \in I$. Hence $e, z \in G^2$ and $e < z$. Then

$$g_{i(1)}e_{i(1)} < g_{i(1)}z_{i(1)},$$
$$g_{i(2)}e_{i(2)} > g_{i(2)}z_{i(2)}.$$

Therefore the elements $g = ge$ and gz are incomparable in G^2, which is a contradiction. □

The half lattice ordered group G^0 will be said to be *the small direct product* of half lattice ordered groups G_i $(i \in I)$; we denote it by the symbol

$$(s)\prod_{i\in I} G_i.$$

It is obvious that if G^1 is a lattice ordered group (i.e., if $G^1 \downarrow = \emptyset$) then $G^0 = G^1$.

In our construction, all G_i are half lattice ordered groups, thus $G_i \neq \{e\}$. On the other hand, by considering direct product decompositions of a lattice ordered group, one-element direct factors can be taken into account (this occurs when forming common refinements of two direct decompositions.) In the case of lattice ordered groups the notions of a direct product with all factors distinct from $\{e\}$ and a small direct product coincide.

If φ is an isomorphism of a half lattice ordered group H onto $(s)\prod\limits_{i\in I} G_i$, $h \in H$, $\varphi(h) = (\dots, g^i, \dots)_{i\in I}$ and if no confusion can occur, then we can identify the elements h and $\varphi(h)$, and in this sense we write

$$H = (s)\prod_{i\in I} G_i; \tag{3}$$

the relation (3) is said to be *a small direct product decomposition* of H. In particular, if $i \in I$ and $g^i \in G_i$, then the element g^i is identified with the element g of G such that $g_i = g^i$ and $g_{i(1)} = e$ whenever $i(1) \in I$ and $i(1) \neq i$.

If a more thorough description is needed then instead of (3) we apply the notation where the isomorphism under consideration is explicitly written.

Let (3) be valid. If, moreover, for each $i \in I$ we have

$$G_i = (s)\prod_{j\in J(i)} G_{ij},$$

then

$$H = (s)\prod_{i\in I, j\in J(i)} G_{ij}. \tag{4}$$

The small direct product decomposition (4) will be called *a refinement* of (3).

Throughout this paper we shall apply without further reference the known facts on direct product decompositions of lattice ordered groups (cf. , e.g. [1]). In particular, we apply the notion of internal direct decomposition as in [1], Section 5.3. Namely, if H is a lattice ordered group and if we have an isomorphism φ of H onto a direct product $\prod\limits_{i\in I} H_i$, then for each $i(0) \in I$ we can construct the set $H^0_{i(0)} = \{h \in H:$

$\varphi(h)_i = e$ for each $i \in I \setminus \{i(0)\}\}$. Then $H^0_{i(0)}$ is an ℓ-subgroup of H which is isomorphic to $H_{i(0)}$; we call $H^0_{i(1)}$ an internal direct factor of H. To simplify the notation, we use the following convention:

2.4. Convention. *Under the assumptions as above, $H_{i(0)}$ will be identified with $H^0_{i(0)}$.*

3. Congruence relations

Several results and methods from this section will be applied below for investigating small direct product decompositions.

In what follows we assume that G is a half lattice ordered group which fails to be lattice ordered. Under the notation as above, G can be viewed as a structure with a group operation and two binary partial operations $\vee, \wedge$ (partial lattice operations).

From this point of view the following definition is a natural one.

3.1. Definition. *An equivalence ϱ on G is said to be a congruence relation if it satisfies the following conditions:*

(i) *ϱ is a congruence relation with respect to the group operation;*

(ii) *if $\circ \in \{\wedge, \vee\}$, $x, y, z \in G$, $y\varrho z$ and if $x \circ y$ exists in G, then $x \circ z$ exists in G and $(x \circ y)\varrho(x \circ z)$.*

For $u, v \in G\uparrow$ (or $u, v \in G\downarrow$, respectively) we put $u\varrho^{(1)}v$ (or $u\varrho^{(2)}v$) iff $u\varrho v$. Then from 3.1 we obtain

3.2. Lemma. (i) *$\varrho^{(1)}$ is a congruence relation on the lattice ordered group $G\uparrow$.*
(ii) *$\varrho^{(2)}$ is a congruence relation of the lattice $G\downarrow$.*

We apply the symbols G/ϱ, $G\uparrow/\varrho^{(1)}$ and $G\downarrow/\varrho^{(2)}$ in the usual sense.

Let $x \in G$. We denote $\overline{x}(\varrho) = \{y \in G : x\varrho y\}$. Next we put $\overline{G}(\varrho) = \{\overline{x}(\varrho) : x \in G\}$. If no misunderstanding can occur, then we write $\overline{x}$ and $\overline{G}$ instead of $\overline{x}(\varrho)$ and $\overline{G}(\varrho)$.

For $\overline{x}, \overline{y} \in \overline{G}$ we put $\overline{x} \leqslant \overline{y}$ if there are $x_1 \in \overline{x}$ and $y_1 \in \overline{y}$ with $x_1 \leqslant y_1$. Next we put $\overline{x} \cdot \overline{y} = \overline{xy}$. Then

(i) $\overline{G}$ turns out to be a partially ordered set;

(ii) $\overline{G}$ is a group with respect to the operation $\cdot$ and $\overline{x} \cdot \overline{y} = \overline{xy}$.

In view of (i) and (ii) we can construct the sets $\overline{G}\uparrow$ and $\overline{G}\downarrow$. Clearly $\overline{G} = G/\varrho$.

3.3. Remark. Let $\varrho_{\max}$ be the largest equivalence relation on G. Next let $\varrho_{(2)}$ be the equivalence on G such that for $x, y \in G$ we have $x\varrho_{(2)}y$ iff either $x, y \in G\uparrow$ or $x, y \in G\downarrow$. Then both $\varrho_{\max}$ and $\varrho_{(2)}$ are congruence relations on G. Next, $\operatorname{card}\overline{G}(\varrho_{\max}) = 1$, $\operatorname{card}\overline{G}(\varrho_{2)} \leqslant 2$ and the partial orders on both $\overline{G}(\varrho_{\max})$, $\overline{G}(\varrho_{(2)})$ are trivial. Hence neither $\overline{G}(\varrho_{\max})$ nor $\overline{G}(\varrho_{(2)})$ is a half lattice ordered group.

3.4. Lemma. *Let ϱ be a congruence relation on G such that $\varrho_{\max} \neq \varrho \neq \varrho_{(2)}$. Then the partial order $\leqslant$ on $\overline{G}$ is non-trivial.*

Proof. In view of the assumption there exist $x, y \in G$ such that (i) $\overline{x} \neq \overline{y}$, and (ii) either $x, y \in G\uparrow$ or $x, y \in G\downarrow$. Hence there exist

$$u = x \wedge y, \quad v = x \vee y.$$

Thus $\overline{u} \leqslant \overline{v}$. If $\overline{u} = \overline{v}$, then 3.2 yields that $\overline{x} = \overline{y}$, which is a contradiction. □

3.5. Lemma. *Let ϱ be a congruence relation on G and let $\overline{x}, \overline{y}, \overline{z} \in \overline{G}$, $\overline{y} \leqslant \overline{z}$. Then $\overline{y} \cdot \overline{x} \leqslant \overline{z} \cdot \overline{x}$.*

Proof. There are $y_1 \in \overline{y}$ and $z_1 \in \overline{z}$ such that $y_1 \leqslant z_1$. Then $y_1x \leqslant z_1x$. Hence $\overline{y_1x} \leqslant \overline{z_1x}$ and $\overline{y_1x} = \overline{y}_1 \cdot \overline{x} = \overline{y} \cdot \overline{x}$, $\overline{z_1x} = \overline{z} \cdot \overline{x}$. □

3.6. Lemma. *Let ϱ be a congruence relation on G. Then $\overline{G} = \overline{G}\uparrow \cup \overline{G}\downarrow$.*

Proof. It is obvious that

$$x \in G\uparrow \Longrightarrow \overline{x} \in \overline{G}\uparrow, \quad x \in G\downarrow \Longrightarrow \overline{x} \in \overline{G}\downarrow.$$

Now it suffices to apply the relation $G = G\uparrow \cup G\downarrow$. □

3.7. Lemma. *Let ϱ be a congruence relation on G, $\varrho_{\max} \neq \varrho \neq \varrho_{(2)}$. Then $\overline{G}\uparrow \cap \overline{G}\downarrow = \emptyset$.*

Proof. By way of contradiction, suppose that $\overline{x} \in \overline{G}\uparrow \cap \overline{G}\downarrow$. Let $\overline{y}, \overline{z} \in \overline{G}$, $\overline{y} \leqslant \overline{z}$. In view of the assumption we have $\overline{x} \cdot \overline{y} \leqslant \overline{x} \cdot \overline{z}$ and, at the same time, $\overline{x} \cdot \overline{y} \geqslant \overline{x} \cdot \overline{z}$, whence $\overline{x} \cdot \overline{y} = \overline{x} \cdot \overline{z}$. Then $\overline{y} = \overline{z}$. Hence the partial order on $\overline{G}$ is trivial, which contradicts 3.4. □

3.8. Lemma. *Let ϱ be a congruence relation on G, $\varrho_{\max} \neq \varrho \neq \varrho_{(2)}$. Then $\overline{G}\uparrow$ is a lattice.*

Proof. Let $\varrho^{(1)}$ be as above. In view of 3.7, the partially ordered set $\overline{G}\uparrow$ coincides with $G\uparrow/\varrho^{(1)}$, whence it is a lattice. □

3.9. Proposition. *Let ϱ be a congruence relation on G such that $\varrho_{\max} \neq \varrho \neq \varrho_{(2)}$. Then $\overline{G}$ is a half lattice ordered group.*

Proof. This is a consequence of 3.4, 3.5, 3.6 and 3.8. □

The maximal equivalence relation on $G\uparrow$ will be denoted by $\tau_{\max}$. Let τ be a congruence relation of the lattice ordered group $G\uparrow$, $\tau \neq \tau_{\max}$. For $u, v \in G$ we put $u\varrho v$ if and only if $u^{-1}v \in G\uparrow$ and $e\tau u^{-1}v$.

The definition of G implies that the relation $u^{-1}v \in G\uparrow$ is valid iff either $u, v \in G\uparrow$ or $u, v \in G\downarrow$. Next, for $u, v \in G\uparrow$ we have

$$u\varrho v \Longleftrightarrow u\tau v.$$

3.10. Lemma. *ϱ is an equivalence relation on G.*

Proof. It is obvious that the relation ϱ is reflexive. Let $u\varrho v$, thus $u^{-1}v\tau e$. Then $(u^{-1}v)^{-1}\tau e$, whence $v^{-1}u\tau e$ and $v\varrho u$. Thus ϱ is symmetric.

Let $x, y, z \in G$, $x\varrho y$, $y\varrho z$. Hence $x^{-1}y\tau e$ and $y^{-1}z\tau e$. We have either $x, y, z \in G\uparrow$ or $x, y, z \in G\downarrow$. This yields that $x^{-1}z \in G\uparrow$. Next, $x^{-1}z = (x^{-1}y)(y^{-1}z)\tau e$, whence $x\varrho z$. Therefore ϱ is transitive. □

3.11. Lemma. *Let $x, y, z \in G$, $y\varrho z$. Then $xy\varrho xz$.*

Proof. We have $e\tau y^{-1}z$. From $y^{-1}z = (y^{-1}x^{-1})(xz) = (xy)^{-1}(xz)$ we obtain that $xy\varrho xz$. □

3.12. Lemma. *The following conditions are equivalent:*

(i) *If $x, y, z \in G, y\varrho z$, then $yx\varrho zx$.*

(ii) *If $x \in G\downarrow, t \in G\uparrow$ and $t\tau e$, then $x^{-1}tx\tau e$.*

(iii) *If x and t are as in (ii), then $tx\varrho x$.*

Proof. ((i)$\Longrightarrow$(ii)) Let (i) be valid. Let x and t be as in (ii). Then $t\varrho e$, hence according to 3.11 we have $x^{-1}t\varrho x^{-1}$ and thus (i) yields that $x^{-1}tx\varrho e$. Thus $x^{-1}tx\tau e$.

((ii)$\Longrightarrow$(iii)) Let (ii) be valid and let x, t be as in (ii). Then $t^{-1} \in G\uparrow$ and $t^{-1}\tau e$. Thus in view of (ii), $x^{-1}t^{-1}x\tau e$. Hence $(tx)^{-1}x\tau e$. This yields that $tx\varrho x$.

((iii)$\Longrightarrow$(i)) Let (iii) be valid and let x, y, z be as in (i). Then $e\varrho y^{-1}z$. Put $y^{-1}z = t$. Hence $t \in G\uparrow$ and $e\tau t$.

First suppose that x belongs to $G\uparrow$. Since τ is a congruence relation on $G\uparrow$ we obtain that $x\tau tx$, thus $e\tau x^{-1}y^{-1}zx$ yielding that $yx\varrho zx$.

Now assume that x belongs to $G\downarrow$. From $t\tau e$ we get, applying (iii), the relation $tx\varrho x$. Thus in view of 3.11 we obtain $x^{-1}tx\varrho e$. Therefore $x^{-1}y^{-1}zx\varrho e$ and hence $yx\varrho zx$. □

3.13. Lemma. *Let $\circ \in \{\wedge, \vee\}$, $x, y, z \in G$, $y\varrho z$ and suppose that $x \circ y$ exists in G. Then $x \circ z$ exists in G and $(x \circ y)\varrho(x \circ z)$.*

Proof. Let $\circ$ be the partial operation $\wedge$ (for the partial operation $\vee$ we proceed analogously).

From the relation $y\varrho z$ and from the fact that $x \wedge y$ exists we obtain that either

(i) $$x, y, z \in G\uparrow$$

or

(ii) $$x, y, z \in G\downarrow$$

holds. Hence $x \circ z$ exists in G.

Assume that (i) is valid. Then, since ϱ coincides with τ on $G\uparrow$ and τ is a congruence relation on $G\uparrow$, we infer that $x \wedge y \varrho x \wedge z$ holds.

Next let us suppose that (ii) is valid. Choose a fixed element u in $G\downarrow$ and consider the mappings

$$\varphi_1(t_1) = ut_1 \quad (t_1 \in G\downarrow),$$
$$\varphi_2(t_2) = u^{-1}t_2 \quad (t_2 \in G\uparrow).$$

Then φ_1 is a dual isomorphism of the lattice $G\downarrow$ onto the lattice $G\uparrow$ and $\varphi_2 = \varphi_1^{-1}$. Thus

$$\varphi_1(x \wedge y) = \varphi_1(x) \vee \varphi_1(y),$$

$$\varphi_1(x \vee z) = \varphi_1(x) \wedge \varphi_1(z).$$

According to 3.11,

$$\varphi_1(y)\varrho\varphi_1(z)$$

and hence (cf. the case (i) where $\wedge$ is replaced by $\vee$)

$$\varphi_1(x) \vee \varphi_1(y)\varrho\varphi_1(x) \vee \varphi_1(z),$$
$$\varphi_1(x \wedge y)\varrho\varphi_1(x \wedge z).$$

If we apply the mapping φ_2 then from the last relation we get (in view of 3.11)

$$x \wedge y\varrho x \wedge z.$$

□

3.14. Proposition. *Let ϱ be as above. Then the following conditions are equivalent:*

(i) *ϱ is a congruence relation on G.*

(ii) *Some of the conditions from 3.12 is satisfied.*

Proof. The implication (i)$\Longrightarrow$(ii) is obvious. The inverse implication is a consequence of 3.10–3.13. □

If τ and ϱ are as above, then ϱ will be said to be *a G-extension* of τ. It is obvious that if τ has a G-extension, then this G-extension is uniquely determined.

By using this term, Proposition 3.14 can be expressed as follows:

3.14.1. Proposition. *Let τ be a congruence relation on the lattice ordered group $G\uparrow$. Then the following conditions are equivalent:*

(i) *The G-extension of τ is a congruence relation on G.*

(ii) *The set $\{x \in G\uparrow : x\tau e\}$ is normal in G.*

It is easy to verify that if ϱ is a congruence relation on G, then ϱ is a G-extension of $\varrho^{(1)}$.

Let $\operatorname{Con} G\uparrow$ and $\operatorname{Con} G$ be the systems of all congruence relations on $G\uparrow$ and on G, respectively; these systems are partially ordered in the usual way. Then $\operatorname{Con} G\uparrow$ and $\operatorname{Con} G$ are complete lattices. Let $\operatorname{Con}_1 G\uparrow$ be the system of all $\tau \in \operatorname{Con} G\uparrow$ satisfying the condition (i) from 3.14.1.

As an immediate consequence of 3.14.1 we obtain

3.14.2. Proposition. *$\operatorname{Con}_1 G\uparrow$ is a closed sublattice of the lattice $\operatorname{Con} G\uparrow$.*

Let φ be a mapping of $\operatorname{Con}_1 G$ into $\operatorname{Con} G$ such that, for each $\tau \in \operatorname{Con}_1 G$, $\varphi(\tau)$ is the G-extension of τ.

3.15. Proposition. *φ is an isomorphism of $\operatorname{Con}_1 G$ onto $\operatorname{Con} G$.*

Proof. If $\varrho \in \operatorname{Con} G$, then $\varphi(\varrho^{(1)}) = \varrho$; hence φ is an epimorphism. Let $\tau_i \in \operatorname{Con}_1 G\uparrow$, $\varrho_i = \varphi(\tau_i)$ $(i = 1, 2)$.

Let $\tau_1 \leqslant \tau_2$, $y, z \in G$, $y\varrho_1 z$. Then $y^{-1}z\tau_1 e$, whence $y^{-1}z\tau_2 e$ and thus $y\varrho_2 z$. Therefore $\varrho_1 \leqslant \varrho_2$.

Conversely, assume that $\varrho_1 \leqslant \varrho_2$. We have $\tau_1 = \varrho_1^1, \tau_2 = \varrho_2^1$, thus $\tau_1 \leqslant \tau_2$, which completes the proof. □

3.16. Proposition. *Let $\tau_i \in \operatorname{Con}_1 G\uparrow$, $\varrho_i = \varphi(\tau_i)$ $(i = 1, 2)$. Then τ_1, τ_2 are permutable if and only if ϱ_1, ϱ_2 are permutable.*

Proof. Assume that τ_1 and τ_2 are permutable. Let $x, y, z \in G$, $x\varrho_1 y$, $y\varrho_2 z$. Then we have either (i) $x, y, z \in G\uparrow$, or (ii) $x, y, z \in G\downarrow$. If (i) is valid, then $x\tau_1 y$, $y\tau_2 z$, hence there is $u \in G\uparrow$ such that $x\tau_2 u$, $u\tau_2 z$. This yields that $x\varrho_2 u$, $u\varrho_2 z$. If (ii) holds, then we take any $t \in G\downarrow$ and obtain $tx\varrho_1 ty$, $ty\varrho_2 tz$ and $tx, ty, tz \in G\uparrow$. Hence $tx\tau_1 ty$, $ty\tau_2 tz$. Thus there is $v \in G\uparrow$ such that $tx\tau_2 v$, $v\tau_1 tz$. Then $tx\varrho_2 v$ and $v\varrho_1 tz$. There exists $w \in G\downarrow$ such that $v = tw$. We get $x\varrho_2 w$, $w\varrho_1 z$. Hence ϱ_1 and ϱ_2 are permutable.

Conversely, suppose that ϱ_1 and ϱ_2 are permutable. Let $x, y, z \in G\uparrow$, $x\tau_1 y$, $y\tau_2 z$. Then $x\varrho_1 y$, $y\varrho_2 z$. There exists $u \in G$ such that $x\varrho_2 u$, $u\varrho_1 z$. We have $u \in G\uparrow$ and hence $x\tau_2 u$, $u\tau_1 z$. □

4. Two-factor small direct products

For a two-factor small direct product decomposition of a half lattice ordered group G we apply the notation

$$G = (s)G_1 \times G_2; \tag{1}$$

G_1 and G_2 are said to be s-factors of G. Let $\mathcal{S}(G)$ be the system of all s-factors of G.

If $g \in G$ and $i \in \{1, 2\}$, then the component of g in G_i will be denoted by g_i.

4.1. Lemma. *Let* (1) *be valid. Then*

(i) *for the lattice ordered group* $G\uparrow$ *we have a direct product decomposition*

$$G\uparrow = G_1\uparrow \times G_2\uparrow;$$

(ii) *for the lattice* $G\downarrow$ *we have a direct product decomposition*

$$G\downarrow = G_1\downarrow \times G_2\downarrow.$$

Proof. This is an immediate consequence of the definition of the small direct product. □

Let (1) be valid. For $x, y \in G$ we put $x\varrho_1 y$ if the following conditions are satisfied:

(i) either $x, y \in G\uparrow$ or $x, y \in G\downarrow$;

(ii) $x_1 = y_1$.

Similarly we define the binary relation ϱ_2 on G (the condition (ii) is replaced by $x_2 = y_2$).

The definitions of ϱ_1 and ϱ_2 imply

4.2. Lemma. *Let* (1) *be valid. Then*

(i) ϱ_1 *and* ϱ_2 *are congruence relations on* G;

(ii) ϱ_1 *and* ϱ_2 *are permutable;*

(iii) $\varrho_1 \wedge \varrho_2 = \varrho_{\min}$;

(iv) *if either* $x, y \in G\uparrow$ *or* $x, y \in G\downarrow$, *then there is* $z \in G$ *such that* $x\varrho_1 z$ *and* $z\varrho_2 y$.

4.3. Lemma. *Suppose that* ϱ_1 *and* ϱ_2 *are congruence relations on* G *such that the conditions* (i)–(iv) *from* 4.2 *are satisfied and* $\varrho_{\max} \neq \varrho_i \neq \varrho_{(2)}$ $(i = 1, 2)$. *Put* $G_i = G/\varrho_i$ $(i = 1, 2)$. *Then the mapping* $\psi\colon G \longrightarrow G_1 \times G_2$ *defined by* $\psi(x) = (\overline{x}(\varrho_1), \overline{x}(\varrho_2))$ *gives a small direct product decomposition of* G.

Proof. According to 3.9, G_1 and G_2 are half lattice ordered groups. In view of (iii), ψ is a monomorphism. If $x \in G\uparrow$, then $\overline{x}(\varrho_1) \in G_1\uparrow$ and $\overline{x}(\varrho_2) \in G_2\uparrow$, hence $\psi(x) \in G_1\uparrow \times G_2\uparrow$. Similarly, if $x \in G\downarrow$, then $\psi(x) \in G_1\downarrow \times G_2\downarrow$. Thus ψ is a mapping of G into $(G_1\uparrow \times G_2\uparrow) \cup (G_1\downarrow \times G_2\downarrow)$.

Let $(\overline{x}(\varrho_1), \overline{y}(\varrho_2)) \in G_1\uparrow \times G_2\uparrow$. According to (iv) there exists $z \in G\uparrow$ such that $x\varrho_1 z$ and $z\varrho_2 y$. Then $\psi(z) = (\overline{x}(\varrho_1), \overline{y}(\varrho_2))$. An analogous consideration can be performed for $G_1\downarrow \times G_2\downarrow$. Thus ψ is an epimorphism of G onto $(G_1\uparrow \times G_2\uparrow) \cup (G_1\downarrow \times G_2\downarrow)$.

Let $x, y \in G$, $x \leqslant y$. Since ϱ_1 and ϱ_2 are congruence relations on G we have $\overline{x}(\varrho_1) \leqslant \overline{y}(\varrho_1)$ and $\overline{x}(\varrho_2) \leqslant \overline{y}(\varrho_2)$, thus $\psi(x) \leqslant \psi(y)$. Conversely, assume that $\psi(x) \leqslant \psi(y)$. This means that $\overline{x}(\varrho_1) \leqslant \overline{y}(\varrho_1)$ and $\overline{x}(\varrho_2) \leqslant \overline{y}(\varrho_2)$. Hence either $x, y \in G\uparrow$ or $x, y \in G\downarrow$. We first suppose that $x, y \in G\uparrow$. Let us denote by $\varrho_i^{\uparrow}$ the relation ϱ_i reduced to $G\uparrow$ $(i = 1, 2)$. From (i)–(iv) and from 3.16 we obtain that the mapping φ reduced to $G\uparrow$ is an isomorphism of the lattice $G\uparrow$ onto $G_1\uparrow \times G_2\uparrow$. A similar result holds for the lattice $G\downarrow$. Hence ψ is an isomorphism with respect to the partial order.

From the fact that ψ is an injective mapping of G onto $(G_1\uparrow \times G_2\uparrow) \cup (G_1\downarrow \times G_2\downarrow)$ and from the condition (i) in 4.2 we obtain that ψ is an isomorphism with respect to the group operation. □

Combining 4.2 and 4.3 we obtain

4.4. Theorem. *Let* ϱ_1 *and* ϱ_2 *be congruence relations on* G *with* $\varrho_{\max} \neq \varrho_i \neq \varrho_{(2)}$ $(i = 1, 2)$. *Then the following conditions are equivalent:*

(i) *The conditions* (i)–(iv) *from* 4.2 *are satisfied.*

(ii) *The mapping* $\psi(x) = (\overline{x}(\varrho_1), \overline{x}(\varrho_2))$ *is an isomorphism of* G *onto* $(s)(G/\varrho_1) \times (G/\varrho_2)$.

Now let us investigate the relations between two-factor direct product decompositions of the lattice ordered group $G\uparrow$ and two-factor small direct product decompositions of G.

Let us have a direct product decomposition

$$G\uparrow = A \times B, \quad A \neq \{e\} \neq B \tag{2}$$

of the lattice ordered group $G\uparrow$.

For $x \in G\uparrow$ we denote by $x(A)$ and $x(B)$ the components of x on A and in B, respectively.

Let $x, y \in G\uparrow$. We put $x\tau_1 y$ ($x\tau_2 y$) if $x(A) = y(A)$ (or $x(B) = y(B)$, respectively).

4.5. Lemma. τ_1 *and* τ_2 *are congruence relations on* $G\uparrow$ *satisfying the conditions* (i), (ii), (iii) *of* 4.2, *and also the condition*

(iv$_1$) *if* $x, y \in G\uparrow$, *then there is* $z \in G\uparrow$ *with* $x\tau_1 z$, $z\tau_2 y$.

Proof. The validity of these conditions is a consequence of (2). □

Let us construct binary relations ϱ_1^0 and ϱ_2^0 by means of τ_1 and τ_2 by the same method as we did in Section 3 for τ and ϱ.

4.6. Lemma. *Assume that* A *is a normal subset of* G. *Then* ϱ_1^0 *is a congruence relation on* G.

Proof. This is a consequence of 3.14.1. □

4.7. Lemma. *If* A *is a normal subset of* G, *then* B *is a normal subset of* G *as well.*

Proof. Assume that A is a normal subset of G. The relation (2) yields that

$$B = A^\delta = \{x \in G\uparrow \colon |x| \wedge |a| = e \text{ for each } a \in A\}.$$

Let $z \in G$. If $z \in G\uparrow$, then from (2) we obtain that $z^{-1}Bz = B$. Let $z \in G\downarrow$. Then the mapping $\varphi\colon G\uparrow \longrightarrow G\uparrow$ defined by $\varphi(t) = z^{-1}tz$ for each $t \in G\uparrow$ is a dual automorphism of the lattice $G\uparrow$ with $\varphi(e) = e$. Thus $\varphi(A^\delta) = A^\delta$, which completes the proof. □

4.8. Lemma. *Let* (2) *be valid and suppose that A is a normal subset of G. Then ϱ_1^0 and ϱ_2^0 are congruence relations on G satisfying the conditions* (i)–(iv) *from* 4.2.

Proof. This is a consequence of 4.7, 4.6 and 3.14.1. □

4.9. Theorem. *Let* (2) *be valid and let ϱ_1^0, ϱ_2^0 be as above. Then $G = (s)G/\varrho_1^0 \times G/\varrho_2^0$.*

Proof. This result is valid in view of 4.4 and 4.8. □

4.10. Proposition. *Under the assumptions and notation as in* 4.9, *the lattice ordered groups $(G/\varrho_1^0)\uparrow$ and A are isomorphic; moreover, under the convention as in* 2.4, $(G/\varrho_1^0)\uparrow = A$.

Proof. We have

$$(G/\varrho_1^0)\uparrow = \{\overline{g}(\varrho_1^0) \colon g \in G\uparrow\},$$

whence $(G/\varrho_1^0)\uparrow = (G\uparrow)/\tau_1$, where τ_1 is as above. Next, $(G\uparrow)/\tau_1$ is isomorphic to A. Under the convention as in 2.4 we clearly have $(G/\varrho_1^0)\uparrow = A$. □

5. The general case

Consider the relation

$$G = (s) \prod_{i \in I} G_i. \tag{1}$$

Let $i(0)$ be a fixed element of I. We put

$$G'_{i(0)} = \{g \in G \colon g_{i(0)} = e\}.$$

From the definition of the small direct product we immediately obtain

5.1. Lemma. *Let* (1) *be valid and let $i(0) \in I$. Then $G = (s)G_{i(0)} \times G'_{i(0)}$.*

5.2. Lemma. *Let I be a nonempty set and for each $i \in I$ let G_i be an s-factor of G. For $g \in G$ and $i \in I$ let g_i be the component of g in G_i. Put $\varphi(g) = (g_i)_{i \in I}$. Then φ is a mapping of G into $(s) \prod_{i \in I} G_i$.*

Proof. Let $g \in G\uparrow$. Then for each $i \in I$ we have $g_i \in G_i\uparrow$. Similarly, if $g \in G\downarrow$, then $g_i \in G_i\downarrow$ for each $i \in I$. Hence $\varphi(g) \in (s) \prod_{i \in I} G_i$. □

5.3. Proposition. *Let $I, (G_i)_{i\in I}$ and φ be as in 5.2. Then the following conditions are equivalent:*

(i) *φ is an isomorphism of G onto $(s)\prod_{i\in I} G_i$.*

(ii) *φ is a bijection.*

Proof. The relation (i)$\Longrightarrow$(ii) obviously holds. Let (ii) be valid. From the definition of φ we infer that φ is a homomorphism with respect to the group operation. Thus, in view of (ii), φ is an isomorphism with respect to the group operation. Put

$$\varphi_1 = \varphi|G\uparrow, \quad \varphi_2 = \varphi|G\downarrow.$$

In view of 5.2, φ_1 is a bijection of $G\uparrow$ onto $\prod_{i\in I}(G_i\uparrow) = ((s)\prod_{i\in I} G_i)\uparrow$ and, similarly, φ_2 is a bijection of $G\downarrow$ onto $((s)\prod_{i\in I} G_i)\downarrow$. We have to verify that φ_1 is an isomorphism of the lattice $G\uparrow$ onto the lattice $\prod_{i\in I} G_i\uparrow$, and that an analogous result is valid for φ_2.

Let $g, g' \in G\uparrow, g < g'$. Then we have $g_i \leqslant g_i'$ for each $i\in I$, thus $\varphi_1(g) \leqslant \varphi_1(g')$. Since φ_1 is a bijection we obtain that $g_1(g) < g_1(g')$.

Conversely, suppose that $\varphi(g) < \varphi(g')$. Then $g' < g$ cannot hold. By way of contradiction, assume that g and g' are incomparable. Put $u = g\wedge g'$. Then $u\neq g$. In view of the definition of φ_1 we conclude that φ_1 is a homomorphism with respect to the operation $\wedge$, whence

$$\varphi_1(u) = \varphi_1(g\wedge g') = (g_i\wedge g_i')_{i\in I} = (g_i)_{i\in I} = \varphi_1(g),$$

which is a contradiction. Therefore $g < g'$.

For φ_2 we can apply analogous arguments. □

5.4. Lemma. *Let φ_1 and φ_2 be as in the proof of 5.3. Then the following conditions are equivalent:*

(i) *φ is a bijection.*

(ii) *φ_1 is a bijection.*

Proof. The implication (i)$\Longrightarrow$(ii) is obvious. Let (ii) be valid. We have to prove that φ_2 is a bijection.

Let $g, g' \in G\downarrow$, $g\neq g'$. Choose any $x\in G\downarrow$. Then $xg, xg' \in G\uparrow$ and $xg\neq xg'$. Thus $\varphi(xg)\neq\varphi(xg')$. Since

$$\varphi(xg) = \varphi(x)\varphi(g) = \varphi(x)\varphi_2(g), \ \varphi(xg') = \varphi(x)\varphi_2(g')$$

we obtain that $\varphi_2(g) \neq \varphi_2(g')$.

For each $i \in I$ let $g^i \in G_i\downarrow$. Choose $x \in G\downarrow$. Hence $x_i \in G_i\downarrow$ for each $i \in I$. Next, $x_i g^i \in G_i\uparrow$ for each $i \in I$. Hence there exists $g_1 \in G\uparrow$ such that

$$(g_1)_i = x_i g^i \quad \text{for each} \quad i \in I.$$

Put $g_2 = x^{-1} g_1$. Then $g_2 \in G$ and

$$(g_2)_i = (x^{-1})_i (x_i g^i) = g_i$$

for each $i \in I$. Thus φ_2 is a bijection. □

5.5. Theorem. *Assume that $G\uparrow = \prod_{i\in I} A_i$ and that all A_i are normal in G, $A_i \neq \{e\}$. Then there are half ordered groups G_i such that $G_i\uparrow = A_i$ for each $i \in I$ and $G = (s)\prod_{i\in I} G_i$.*

Proof. Let $i(0) \in I$. There exists a direct factor $A'_{i(0)}$ of $G\uparrow$ such that $G\uparrow = A_{i(0)} \times A'_{i(0)}$. Since $A_{i(0)}$ is normal in G, in view of 4.7 the set $A'_{i(0)}$ is also normal in G. Hence according to 4.9 and 4.10 there exists a small direct decomposition

$$G = (s) G_{i(0)} \times G'_{i(0)}$$

such that $G_{i(0)}\uparrow = A_{i(0)}$.

Let φ, φ_1 and φ_2 be as above. In view of $G\uparrow = \prod_{i\in I} A_i$ we obtain that φ_1 is a bijection. Thus according to 5.4, φ is a bijection as well. Therefore 5.3 yields that $G = (s)\prod_{i\in I} G_i$. □

The following example shows that a direct factor of $G\uparrow$ need not be, in general, a normal subset of the group G.

Let H_1 be the additive group of all integers with the natural linear order and $H_2 = H_1$. Put $H = H_1 \times H_2$. Next, let F and F' be as in [3], p. 87. By applying [3], Lemma III.3 we construct the half ordered groups $G_{H,F}$ and $G_{H,F'}$. Then

$$G_{H,F}\uparrow = G_{H,F'}\uparrow = H.$$

It can be easily verified that neither H_1 nor H_2 are normal subgroups of $G_{H,F'}$. On the other hand, both H_1 and H_2 are normal in $G_{H,F}$.

5.6. Theorem. *Let G be a half lattice ordered group and let $C \subseteq G$, $c \in C$. Suppose that*

(i) *C is a convex chain in G which has no upper bound and no lower bound;*
(ii) *the set $c^{-1}C$ is normal in G.*

Then there exists an s-factor G_1 of G such that $G_1\uparrow = c^{-1}C$.

Proof. The set $c^{-1}C$ is a convex chain in $G\uparrow$ which has no upper bound and no lower bound in $G\uparrow$. Thus in view of [2], $c^{-1}C$ is a direct factor of the lattice ordered group $G\uparrow$. Hence according to 5.5, there is an s-factor G_1 of G such that $G_1\uparrow = c^{-1}C$. □

6. Regular decompositions

Consider a small direct product decomposition

$$G = (s)\prod_{i\in I} G_i. \tag{α}$$

Let $i \in I$. For $x, y \in G$ we put $x\varrho^i y$ if $x_i = y_i$. Then ϱ^i is a congruence relation on G.

Let $g^i \in G_i$ and let $\varphi_i(g^i)$ be the set of all $x \in G$ such that $x_i = g^i$. Then φ_i is an isomorphism of G_i onto G/ϱ^i.

For each $x \in G$ we put

$$\varphi(x) = (\overline{x}(g^i))_{i\in I}.$$

The mapping φ determines a small direct product decomposition

$$G = (s)\prod_{i\in I} \overline{G}_i, \tag{$\overline{\alpha}$}$$

where $\overline{G}_i = G/\varrho^i$ for each $i \in I$. We will say that $\overline{\alpha}$ is *a regular decomposition corresponding to the small direct decomposition* α.

A small direct product decomposition β of G will be called *regular* if there exists a small direct product decompositions β_1 of G such that $\beta = \overline{\beta_1}$.

Let us have another small direct decomposition

$$G = (s)\prod_{j\in J} G_j. \tag{β}$$

The small direct product decompositions α and β are called *isomorphic* if there exists a bijection $\psi\colon I \longrightarrow J$ such that for each $i \in I$ the half lattice ordered groups G_i and $G_{\psi(i)}$ are isomorphic.

Next, α and β are said to be *equivalent* (notation: $\alpha \approx \beta$) if $\overline{\alpha} = \overline{\beta}$; in other words, if there exists a bijective mapping $\psi\colon I \longrightarrow J$ such that $\varrho^i = \varrho^{\psi(i)}$ for each $i \in I$. It is obvious that $\alpha \approx \overline{\alpha}$. The relation $\approx$ is an equivalence on the class $SD(G)$ of all small direct product decompositions of G. Put $SD_r(G) = \{\overline{\alpha}\colon \alpha \in SD(G)\}$.

It is clear that if α, β are regular and if $\alpha \approx \beta$, then $\alpha = \beta$.

If $\alpha \in SD(G)$, then α and $\overline{\alpha}$ are isomorphic (in view of the isomorphisms φ_i above). This yields that if α and β are equivalent, then they are isomorphic.

On the other hand, if α and β are isomorphic, then they need not be equivalent.

Let H be a lattice ordered group, $H \neq \{e\}$. We denote by $D(H)$ the class of all direct product decompositions of H. Next, let $D_1(H)$ be the subclass of H containing those direct product decompositions all factors in which are distinct from $\{e\}$. We can introduce an analogous equivalence on $D_1(H)$ as we did for $SD(G)$ above; this equivalence on $D_1(H)$ will be denoted by the same symbol $\approx$.

Assume that G, G_i and A_i $(i \in I)$ are as in 5.5. We apply the notation α as above and denote

$$(\alpha_1) \qquad G\uparrow = \prod_{i \in I} A_i.$$

Let us put $f(\alpha_1) = \alpha$.

6.1. Proposition. *Let $\alpha_1, \alpha_2 \in D_1(G\uparrow)$. Then*

$$\alpha_1 \approx \alpha_2 \Longleftrightarrow f(\alpha_1) \approx f(\alpha_2).$$

P r o o f. This is a consequence of the construction performed in Section 5. □

The definition of $\overline{\alpha}$ implies that $SD(G)/\approx$ is a set, and so is $D_1(G\uparrow)$. For $\alpha \in SD(G)$ we denote by $\alpha(\approx)$ the class of all $\beta \in SD(G)$ with $\alpha \approx \beta$. For $\alpha_1 \in D_1(G\uparrow)$ the symbol $\alpha_1(\approx)$ has an analogous meaning.

Let $\alpha_1(\approx) \in D_1(G\uparrow)/\approx$. We put $\overline{f}(\alpha_1(\approx)) = f(\alpha_1)(\approx)$. Then $\overline{f}$ is a correctly defined mapping of $D_1(G\uparrow)/\approx$ into $SD(G)/\approx$.

From 5.5 and 6.1 we obtain

6.2. Corollary. *$\overline{f}$ is a bijection of the set $D_1(G\uparrow)/\approx$ onto $SD(G)/\approx$.*

Let α and β be as above. We put $\alpha \leqslant \beta$ if for each $i \in I$ there exists $j \in J$ such that

$$\overline{e}(\varrho^i) \supseteq \overline{e}(\varrho^j).$$

Analogously we define the relation $\leqslant$ on the class $D_1(G\uparrow)$. From these definitions we obtain

6.3. Lemma. *The relation $\leqslant$ is a quasiorder on the class $SD(G)$. If $\alpha_1, \alpha_2 \in D_1(G)$, then*

$$\alpha_1 \leqslant \alpha_2 \Longleftrightarrow f(\alpha_1) \leqslant f(\alpha_2).$$

Next, if $\alpha, \beta \in SD(G)$, then

$$\alpha \leqslant \beta \Longleftrightarrow \overline{\alpha} \leqslant \overline{\beta}.$$

6.4. Lemma. *Let α and β be as above, $i \in I$, $j \in J$. Then the following conditions are equivalent:*

(i) $\overline{e}(\varrho^i) \supseteq \overline{e}(\varrho^j)$.

(ii) $G_i{\uparrow} \subseteq G_j{\uparrow}$.

Proof. Let $\overline{e}(\varrho^i) \supseteq \overline{e}(\varrho^j)$. In view of 5.1,

$$G = (s)G_i \times G'_i.$$

Analogously we have

$$G = (s)G_j \times G'_j.$$

Hence

$$G{\uparrow} = G_i{\uparrow} \times G'_i{\uparrow},$$

$$G{\uparrow} = G_j{\uparrow} \times G'_j{\uparrow}.$$

Next, $\overline{e}(\varrho^i) \cap G{\uparrow} = G'_i{\uparrow}$ and $\overline{e}(\varrho^j) \cap G{\uparrow} = G'_j{\uparrow}$. From (i) we obtain $G'_i{\uparrow} \supseteq G'_j{\uparrow}$ and this yields that $G_i{\uparrow} \subseteq G_j{\uparrow}$.

The proof of the implication (ii)$\Rightarrow$(i) is similar. □

6.4.1. Corollary. *Let $\alpha, \beta \in SD(G)$. Then the following conditions are equivalent:*

(i) $\alpha \leqslant \beta$;

(ii) *for each $i \in I$ there exists $j \in J$ such that $G_i{\uparrow} \subseteq G_j{\uparrow}$.*

6.5. Lemma. *Let α, β be as above. Then the following conditions are equivalent:*

(i) $\alpha \leqslant \beta$ *and* $\beta \leqslant \alpha$;

(ii) $\alpha \approx \beta$.

Proof. Let (i) be valid. Choose $i \in I$. In view of the relation $\alpha \leqslant \beta$ and of 6.4 there exists $j \in J$ such that $G_i{\uparrow} \subseteq G_j{\uparrow}$. Since $G_i \neq \{e\}$ we have $G_i{\uparrow} \neq \{e\}$. Let $j(1) \in J$, $j(1) \neq j$. If $G_i{\uparrow} \subseteq G_{j(1)}{\uparrow}$, then $G_j{\uparrow} \cap G_{j(1)}{\uparrow} \neq \{e\}$, which is impossible. Hence we obtain a mapping $\psi\colon I \to J$ defined by $\psi(i) = j$ (where i, j are as above).

Similarly, $\beta \leqslant \alpha$ yields that there is $i(1) \in I$ with $G_j\uparrow \subseteq G_{i(1)}\uparrow$. Then $G_i\uparrow \cap G_{i(1)}\uparrow \neq \{e\}$ and thus $i = i(1)$. From this we obviously infer that ψ is a bijection; moreover, $G_i\uparrow = G_{\psi(i)}\uparrow$ for each $i \in I$. Thus $G_i'\uparrow = G'_{\psi(i)}\uparrow$ for each $i \in I$. Hence $\alpha_1 \approx \alpha_2$. According to 6.1 we obtain that $\alpha \approx \beta$.

Conversely, suppose that (ii) holds. Hence according to 6.1, $\alpha_1 \approx \beta_1$. Let ψ be as in the definition of $\approx$. Then

$$G_i' = \overline{e}(\varrho^i) = \overline{e}(\varrho^{\psi(i)}) = G'_{\psi(i)}$$

for each $i \in I$. Thus

$$G_i'\uparrow = G'_{\psi(i)}\uparrow,$$
$$G_i\uparrow = G_{\psi(i)}\uparrow$$

for each $i \in I$. Hence in view of 6.4 we obtain that (i) holds. □

6.6. Theorem. *Let $\alpha, \beta \in SD(G)$. There exists $\gamma \in SD(G)$ such that*

(i) $\gamma \leqslant \alpha$ *and* $\gamma \leqslant \beta$;

(ii) *if* $\gamma' \in SD(G)$ *and* $\gamma' \leqslant \alpha$, $\gamma' \leqslant \beta$, *then* $\gamma' \leqslant \gamma$.

Proof. Let $\alpha_1 \in D_1(G\uparrow)$, $\alpha = f(\alpha_1)$. Suppose that β_1 has an analogous meaning. Without loss of generality we can suppose that α_1 and β_1 are internal direct decompositions of $G\uparrow$. There exists a common refinement of α_1 and β_1, namely (cf., e.g., [1])

$$G\uparrow = \prod_{i\in I, j\in J} (G_i\uparrow \cap G_j\uparrow).$$

Let $K = \{(i,j) : i \in I, j \in J \text{ and } G_i\uparrow \cap G_j\uparrow \neq \{e\}\}$. Then $K \neq \emptyset$ and

$$(\gamma_1) \qquad G\uparrow = \prod_{(i,j)\in K} (G_i\uparrow \cap G_j\uparrow).$$

All $G_i\uparrow \cap G_j\uparrow$ are normal in G. Hence there exists $\gamma \in SD(G)$ with $\gamma = f(\gamma_1)$.

Clearly $\gamma_1 \leqslant \alpha_1$ and $\gamma_1 \leqslant \beta_1$. Thus in view of 6.3 $\gamma \leqslant \alpha$ and $\gamma \leqslant \beta$.

Let $\gamma' \in SD(G)$, $\gamma' \leqslant \alpha$, $\gamma' \leqslant \beta$. There is $\gamma_1 \in D_1(G)$ with $f(\gamma_1') = \gamma'$. Then $\gamma_1' \leqslant \alpha_1$ and $\gamma_1' \leqslant \beta_1$. Again, without loss of generality we can suppose that γ_1' is an internal direct product decomposition of $G\uparrow$. Hence γ_1' is a refinement of both α_1 and γ_1. Thus γ_1' is a refinement of γ_1. This yields that $\gamma_1' \leqslant \alpha_1$ and $\gamma_1' \leqslant \beta_1$. Therefore $\gamma' \leqslant \alpha$ and $\gamma' \leqslant \beta$. □

On the set $SD_r(G)$ we consider the relation $\leqslant$ which is inherited from $SD(G)$.

6.7. Lemma. *The relation $\leqslant$ is a partial order on $SD_r(G)$.*

Proof. This is a consequence of 6.1 and of the fact that for $\alpha, \beta \in SD_r(G)$ we have $\alpha \approx \beta \Rightarrow \alpha = \beta$. □

6.8. Corollary. *With respect to the relation $\leqslant$, $SD_r(G)$ is a meet-semilattice.*

Proof. This follows from 6.6 and 6.7. $\square$

7. Common refinements

In the present section we prove that any two small direct product decompositions of a half lattice ordered group G have isomorphic refinements.

Let α and β be as in Section 6.

7.1. Lemma. *Suppose that α and β are regular and that $\alpha \leqslant \beta$. For $j \in J$ let $I(j) = \{i \in I \colon \overline{e}(\varrho^i) \supseteq \overline{e}(\varrho^j)\}$. Then $I(j) \neq \emptyset$ for each $j \in J$.*

Proof. Let $j \in J$. By way of contradiction, suppose that $I(j) = \emptyset$. Let $i \in I$. In view of 6.4, $G_i\uparrow \subseteq G'_j\uparrow$ for each $i \in I$. This yields that $G\uparrow \subseteq G'_j\uparrow$ and thus $G_j\uparrow = \{e\}$, which is impossible. $\square$

7.2. Lemma. *Let α, β be as in 7.1 and let $j \in J$, $g \in G$. We put*

$$\chi(\overline{g}(\varrho^j)) = (\ldots, \overline{g}(\varrho^i), \ldots)_{i \in I(j)}.$$

Then χ is a mapping of G_j into $(s) \prod\limits_{i \in I(j)} G_i$.

Proof. If g, $g' \in G$ such that $\overline{g}(\varrho^j) = \overline{g'}(\varrho^j)$, then for each $i \in I(g)$ we have $\overline{\varrho}(\varrho^i) = \overline{\varrho'}(\varrho^i)$, whence χ is a correctly defined mapping on G_j.

For $\overline{\varrho}(g^j) \in G_j\uparrow$ the relation $g \in G\uparrow$ is valid and hence $\overline{g}(\varrho^i) \in G_i\uparrow$ for each $i \in I(j)$. Analogously, if $\overline{g}(\varrho^j) \in G_j\downarrow$, then $\overline{g}(\varrho^i) \in G_i\downarrow$ for each $i \in I(j)$. Thus $\chi(G_j) \subseteq (s) \prod\limits_{i \in I)j)} G_i$. $\square$

7.3. Lemma. *χ is a homomorphism with respect to the group operation and also with respect to the partial lattice operations $\wedge$ and $\vee$.*

Proof. This is an immediate consequence of the definition of the mapping χ. $\square$

7.4. Lemma. *$G_j\uparrow = \prod\limits_{i \in I(j)} G_i\uparrow$ for each $j \in J$.*

Proof. Let $j \in J$ and $i \in I(j)$. In view of 6.4.1 we have $G_i\uparrow \subseteq G_j\uparrow$, whence $G_i\uparrow \cap G_j\uparrow = G_i\uparrow$. Since $G\uparrow$ is a lattice ordered group, the relation

$$G_j\uparrow = \prod_{i \in I} (G_i\uparrow \cap G_j\uparrow)$$

is valid. If $i(1) \in I \setminus I(j)$, then there exists $j(1) \in J$ with $j(1) \neq j$ such that $G_{i(1)} \subseteq G_{j(1)}$, whence

$$G_{i(1)} \cap G_j \subseteqq G_{j(1)} \cap G_j = \{e\}.$$

Therefore

$$G_j\uparrow = \prod_{i \in I(j)} G_i\uparrow.$$

□

7.5. Lemma. *The mapping χ is a monomorphism.*

P r o o f. Let g, $g' \in G$ and suppose that $\chi(\overline{g}(\varrho^j)) = \chi(\overline{g'}(\varrho^j))$. Hence we have either (i) g, $g' \in G\uparrow$, or (ii) g, $g' \in G\downarrow$. If (i) holds, then $\overline{g}(\varrho^j)$ and $\overline{g'}(\varrho^j)$ belong to $G_j\uparrow$ and hence in view of 7.4 we obtain that $\overline{g}(\varrho^j) = \overline{g'}(\varrho^j)$. Let (ii) be valid. Then e, $g^{-1}g' \in G\uparrow$ and

$$\chi(\overline{e}(\varrho^j)) = \chi(\overline{g^{-1}g'}(\varrho^j)).$$

This yields that $\overline{e}(\varrho^j) = \overline{g^{-1}g'}(\varrho^j)$, whence $\overline{g}(\varrho^j) = \overline{g'}(\varrho^j)$. □

7.6. Lemma. *χ is an epimorphism.*

P r o o f. Let

$$(\overline{g^i}(\varrho^i))_{i \in I(j)} \in (s) \prod_{i \in I(j)} G_i.$$

Then either

(i) $g^i \in G\uparrow$ for each $i \in I(j)$,

or

(ii) $g^i \in G\downarrow$ for each $i \in I(j)$.

First assume that (i) is satisfied. Then in view of 7.4 there is $g \in G\uparrow$ such that $\chi(\overline{g}(\varrho^j)) = (\overline{g^i}(\varrho^i))_{i \in I(j)}$.

Next suppose that (ii) is valid. Choose $g \in G\downarrow$. Hence $\overline{g}(\varrho^j) \in G_j\downarrow$ and $g_i g^i \in G_i\uparrow$ for each $i \in I(j)$. Therefore there exists $g' \in G$ such that

$$\chi(\overline{g'}(\varrho^j)) = (g_i g^i)_{i \in I(j)}.$$

Then

$$\chi(\overline{g^{-1}}(\varrho^j)\overline{g'}(\varrho^j)) = (g^i)_{i \in I(j)}$$

which completes the proof. □

7.7. Proposition. *Let $\alpha \leqslant \beta$. Then the mapping χ determines a small direct product decomposition*

$$G_j = (s) \prod_{i \in I(j)} G_i.$$

Proof. This is a consequence of 7.1–7.6. □

7.8. Corollary. *Let α and β be regular and $\alpha \leqslant \beta$. Then α is a refinement of β.*

The definition of an isomorphism of small direct product decompositions implies

7.9. Lemma. *Let α, β be small direct decompositions of G and suppose that α is isomorphic to β. Let γ be a refinement of α. Then there exists a refinement γ' of β such that γ is isomorphic to γ'.*

7.10. Theorem. *Let α and β be small direct product decompositions of a half lattice ordered group G. Then α and β have isomorphic refinements.*

Proof. Let γ be as in 6.6. Then $\gamma \leqslant \alpha$ and $\gamma \leqslant \beta$. In view of 6.3 we have $\overline{\gamma} \leqslant \overline{\alpha}$ and $\overline{\gamma} \leqslant \overline{\beta}$. Since $\overline{\alpha}, \overline{\beta}$ and $\overline{\gamma}$ are regular, from 7.8 we obtain that $\overline{\gamma}$ is a refinement of both $\overline{\alpha}$ and $\overline{\beta}$. Next, $\alpha \approx \overline{\alpha}$ and $\beta \approx \overline{\beta}$, thus by applying 7.9 we get that there exist $\gamma', \gamma'' \in SD(G)$ such that

γ' is a refinement of α and γ' is isomorphic to $\overline{\gamma}$;
γ'' is a refinement of β and γ'' is isomorphic to $\overline{\gamma}$.

Hence γ' and γ'' are isomorphic. □

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