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IDEALS AND CONGRUENCE KERNELS OF ALGEBRAS

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§ 1. INTRODUCTION

The concepts of a normal subgroup, a ring ideal and a lattice ideal were extended by A. Ursini in 1972 to the notion of an ideal in universal algebras with 0 [12]. In their 1984 paper [5] H.-P. Gumm and A. Ursini studied and characterized universal algebras $\mathcal{A}$ such that every ideal I of $\mathcal{A}$ ist he kernel (i.e. $I = [0]\theta$) for a unique congruence θ of $\mathcal{A}$. Such an algebra is called ideal determined. As it is well-known ideal determined algebras include groups and rings but not all lattices. In this paper we study algebras $\mathcal{A}$ with a weaker property: every ideal of $\mathcal{A}$ is the kernel of some congruence of $\mathcal{A}$. In Theorem 10 we list 8 equivalent conditions for this property. Here three conditions refer to the kernels of congruences generated by certain sets of the form $\{0\} \times S$, one condition to a certain congruence permutability around 0 and three conditions relate ideals and unary polynomials or translations of fundamental operations.

In Corollary 12 we characterize all varieties $\mathcal{V}$ (with a nullary term 0) such that for every $\mathcal{A} \in \mathcal{V}$ each ideal is a congruence kernel. This condition requires that to each at least ternary term $q(x_1, \dots, x_n)$ of $\mathcal{V}$ in which x_1 appears exactly once there exists a term $p(x_1, \dots, x_n)$ of $\mathcal{V}$ satisfying the identities

$$\begin{aligned} (1) \qquad & p(0,0,0,x_4,\dots,x_n) = 0 \\ & q(x,x,y,x_4,\dots,x_n) = p(q(y,x,y,x_4,\dots,x_n),x,y,x_4,\dots,x_n). \end{aligned}$$

Finally, in Proposition 13 we give a Malt'sev type condition for varieties with a nullary term 0 such that each $\mathcal{A} \in \mathcal{V}$ is permutable at 0 (i.e. $[0](\theta \vee \varphi) = \{a \in A;\ \langle 0,x\rangle \in \theta$ and $\langle x,a\rangle \in \varphi$ for some $x \in A\}$).

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Definitions 1. Let $\mathscr{A} = (A; F)$ be an algebra and let 0 be a fixed element of A. Let f be an n-ary term operation of $\mathscr{A}$ and let $N \subseteq \{1, \ldots, n\}$. Following [5] call f an *N-ideal term operation* (or briefly an *ideal term operation*) of $\mathscr{A}$ if $f(a_1, \ldots, a_n) = 0$ holds whenever $a_1, \ldots, a_n \in A$ satisfy $a_i = 0$ for all $i \in N$.

For example, let $\mathscr{A} = (A; +, -, \cdot, 0)$ be a ring. Then both $x_1 + x_2$ and $x_1 - x_2$ are $\{1, 2\}$-ideal term operations of $\mathscr{A}$. Similary $x_1 \cdot x_2$ is an N-ideal term operation of $\mathscr{A}$ for both $N = \{1\}$ and $N = \{2\}$. Next for a lattice $\mathscr{L} = (L; \vee, \wedge, 0)$ with the least element 0 clearly $x_1 \vee x_2$ is an $\{1, 2\}$-ideal term operation of $\mathscr{L}$ and $x_1 \wedge x_2$ is an N-ideal term operation of $\mathscr{L}$ for both $N = \{1\}$ and $N = \{2\}$.

Denote by $J_{\mathscr{A}}$ the set of all ideal term operations of $\mathscr{A}$. The following fact was noted in [5]:

Proposition 2. *The set $J_{\mathscr{A}}$ is a subclone of the clone of term operations of $\mathscr{A}$.*

Proof. Let $1 \leqslant i \leqslant n$. Clearly the i-th n-ary projection is an $\{i\}$-ideal term operation. Let $f, g \in J_{\mathscr{A}}$ be m-ary and n-ary. Then f is an M-ideal and g is an N-ideal term operation of A for some $M \subseteq \{1, \ldots, m\}$ and $N \subseteq \{1, \ldots, n\}$. It is easy to see that the operation f' obtained from f by exchanging its variables also belongs to $J_{\mathscr{A}}$. Similary $J_{\mathscr{A}}$ is closed under any fusion of variables. Finally set $p := m + n - 1$ and define $h := f * g$ as the p-ary operation on A satisfying $h(a_1, \ldots, a_p) = f(g(a_1, \ldots, a_n), a_{n+1}, \ldots, a_p)$ for all $a_1, \ldots, a_p \in A$. Let $M = \{i_1, \ldots, i_k\}$ and $N = \{j_1, \ldots, j_1\}$ where $1 \leqslant i_1 < \ldots < i_k \leqslant m$ and $1 \leqslant j_1 < \ldots < j_1 \leqslant n$. We have two cases:

1) If $i_1 = 1$ then h is a $\{j_1, \ldots, j_1, i_2 + n - 1, \ldots, i_k + n - 1\}$-ideal term operation of $\mathscr{A}$.

2) If $i_1 > 1$ the h is an $\{i_1 + n - 1, \ldots, i_k + n - 1\}$-ideal term operation.

From Mal'cev's formalism it follows that $J_{\mathscr{A}}$ is a clone. $\square$

Example 3. Let $\mathscr{A} = (A; +, -, \cdot, 0, \{a \colon a \in A\})$ be an associative and commutative ring (with all possible nullary operations). Let $\{F_1, \ldots, F_m\}$ be a family of not necessarily distinct subsets of $\{1, \ldots, n\}$, let $a_1, \ldots, a_m \in A$ and let r_{ij} ($i \in \{1, \ldots, m\}$, $j \in F_i$) be positive integers. Further let $N \subseteq \{1, \ldots, r\}$. The polynomial

$$f(x_1, \ldots, x_n) :\approx \sum_{i=1}^{m} a_i \prod_{j \in F_i} x_j^{r_{ij}}$$

is an N-ideal term operation of $\mathscr{A}$ if and only if N meets each F_i ($i = 1, \ldots, m$).

Definition 4. A nonempty subset I of A is an *ideal* of $\mathscr{A}$ if for every n-ary N-ideal term operation f of $\mathscr{A}$

$$(2) \qquad a_i \in I \quad \text{for all} \quad i \in N \Rightarrow f(a_1, \ldots, a_n) \in I$$

holds for all $a_1, \ldots, a_n \in A$.

Notice that for rings and lattices this definition agrees with the standard one. Consider a group $\mathscr{A} = (A; \cdot, ^{-1}, 0)$. The operations $f(x_1, x_2) \approx x_1 x_2^{-1}$, $g(x_1, x_2) \approx x_1^{-1} x_2$, $h(x_1, x_2) \approx x_2^{-1} x_1 x_2$ are N-ideal term operations for N equal $\{1, 2\}$, $\{1, 2\}$ and $\{1\}$ respectively. It follows that every ideal of $\mathscr{A}$ is a normal subgroup of $\mathscr{A}$. Conversely, it is not difficult to verify that every normal subgroup of $\mathscr{A}$ is an ideal of $\mathscr{A}$.

Denote by $J(\mathscr{A})$ the set of all ideals of $\mathscr{A}$. The poset $J(\mathscr{A}) = (J(\mathscr{A}), \subseteq)$ is a complete lattice in which

$$\bigwedge\{J_i;\ i \in I\} = \bigcap\{J_i;\ i \in I\}$$

for every subset $\{J_i\}_{i \in I}$ of $J(A)$. Thus for every $S \subseteq A$ the ideal generated by S is the least ideal $I(S)$ of $\mathscr{A}$ containing S. We have the following description of $I(S)$ [5]:

Lemma 5. *Let $S \subseteq A$. Then $I(S)$ is the set of all $f(a_1, \ldots, a_n)$ where f is an N-ideal term operation of $\mathscr{A}$ and $a_1, \ldots, a_n \in A$ satisfy $a_i \in S$ for all $i \in N$.*

Proof. Denote by K the set defined in Lemma 5. Clearly $K \subseteq I(S)$. Moreover, $S \subseteq K$ because id_A is a $\{1\}$-ideal term operation. Thus it suffices to show that $K \in J(\mathscr{A})$. Let g be an m-ary M-ideal term operation and let $a_1, \ldots, a_m \in A$ satisfy $a_k \in K$ for all $k \in M$.

1) First consider the case $M = \emptyset$. Then g is constant with value 0 and $0 = g(a_1, \ldots, a_m) \in K$.

2) Thus let $M \neq \emptyset$. Without loss of generality we may assume that $M = \{1, \ldots, p\}$ for some $p \leqslant m$. By the definition of K, for each $1 \leqslant i \leqslant p$ we have $a_i = f_i(b_{i1}, \ldots, b_{il_1})$ for some L_i-ideal term operation f_i and $b_{i1}, \ldots, b_{il_i} \in A$ such that $b_{ij} \in S$ for all $j \in L_i$ $(i = 1, \ldots, p)$. Set $l \colon l_1 + \ldots + l_p$ and

$$L := \bigcup_{j=1}^{p} (L_i + l_1 + \ldots + l_{j-1}) \quad (i = 1, \ldots, p),$$

where for every set X of positive integers and a nonegative integer a, the symbol $X + a$ stands for $\{x + a \colon x \in X\}$. Further define an $(1 + m - p)$-ary operation h on A by setting

$$\begin{aligned} &h(c_{11}, \ldots, c_{1l_1}, \ldots, c_{p1}, \ldots, c_{pl_p}, c_{l+1}, \ldots, c_{l+m-p}) := \\ &\quad := g(f_1(c_{11}, \ldots, c_{1l_1}), \ldots, f_p(c_{p1}, \ldots, c_{pl_p}), c_{l+1}, \ldots, c_{l+m-p}) \end{aligned}$$

for all $c_{11}, \ldots, c_{pl_p}, c_{l+1}, \ldots, c_{l+m-p} \in A$. It is easy to check that h is an L-ideal term operation of A. Finally

$$g(a_1, \ldots, a_m) = h(b_{11}, \ldots, b_{pl_p}, a_{l+1}, \ldots, a_m) \in K.$$

$\square$

Notice that for $J_i \in J(\mathscr{A})$ $(i \in I)$ clearly

$$\bigvee_{i \in I} J_i = I\Big(\bigcup_{i \in I} J_i\Big)$$

and that $\{0\}$ and A are the least and greatest elements of $J(\mathscr{A})$. We abbreviate $I(\{s_1, \ldots, s_n\})$ by $I(s_1, \ldots, s_n)$.

Definition 6. For $S \subseteq A$ and $\varrho \subseteq A^2$ the set $[S]\varrho := \{a \in A \colon (s, a) \in \varrho$ for some $s \in S\}$ is the *hull* of S in ϱ. In particular, the set $[0]\varrho := [\{0\}]\varrho$ is the *kernel* of ϱ. A subset B of A is a *congruence kernel* if $B = [0]\theta$ for some congruence θ of $\mathscr{A}$. The following lemma extends a result from [5].

Lemma 7. *If ϱ is a reflexive subuniverse of $\mathscr{A}^2$ then the kernel of ϱ is an ideal of $\mathscr{A}$.*

Proof. Let f be an n-ary N-ideal term of $\mathscr{A}$ and $a_1, \ldots, a_n \in A$ satisfy $a_i \in I := [0]\varrho$ for all $i \in N$. Set $b_i := 0$ for all $i \in N$ and $b_i := a_i$ otherwise. Then $\beta := f(b_1, \ldots, b_n) = 0$ and $(b_i, a_i) \in \varrho$ due to $(0, a_i) \in \varrho$ for $i \in N$ and $(a_i, a_i) \in \varrho$ otherwise. Thus for $\alpha := f(a_1, \ldots, a_n)$ we have $(0, \alpha) = (\beta, \alpha) = (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \varrho$ proving $\alpha \in [0]\varrho$. □

For the proof of the next theorem we need the following minute sharpening of a well-known result.

Definiton 8. Let f be an n-ary operation on A, let $1 \leqslant i \leqslant n$ and let $a_1, \ldots, a_n \in A$. The selfmap r of A defined by

$$r(x) \approx f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

is an *i-translation* (or shortly a *translation*) of f. For $\mathscr{A} = (A; F)$ denote by $P(\mathscr{A})$ and $T(\mathscr{A})$ the sets of all unary polynomials of $\mathscr{A}$ and of all translations of operations from F, respectively. Further, let $M(\mathscr{A})$ denote the monoid of selfmaps of A generated by $T(\mathscr{A})$ and set

$$\mathscr{A}_P := (A; P(\mathscr{A})),\ \mathscr{A}_M := (A; M(\mathscr{A})),\ \mathscr{A}_T := (A, T(\mathscr{A})).$$

Clearly $\mathscr{A}_p$, $\mathscr{A}_M$ and $\mathscr{A}_T$ are unary algebras on A and $P(\mathscr{A}) \supseteq M(\mathscr{A}) \supseteq T(\mathscr{A})$. The following simple example shows that $M(\mathscr{A})$ may be a proper submonoid of $P(\mathscr{A})$.

Let $\mathscr{N}_5 = (N_5; \vee, \wedge)$ denote the 5-element nonmodular lattice with $N_5 = \{0, a, b, c, 1\}$ and $0 < a < b < 1 > c > 0$. Set $p(x) \approx (x \vee b) \wedge (x \vee c)$. A direct check shows that

$$p(0) = 0,\ p(a) = p(b) = b,\ p(c) = c,\ p(1) = 1.$$

Clearly $p \in P(\mathcal{N}_5)$. We show that $p \notin M(\mathcal{N}_5)$. The translations of $\mathcal{N}_5$ are the selfmaps $x \mapsto x \vee k$ and $x \mapsto x \wedge k$ with $k \in N_5$. Every map from $M(\mathcal{N}_5)$ can be expressed

$$(3) \qquad (\ldots((x \vee k_1) \wedge h_1) \vee \ldots \vee k_n) \wedge h_n$$

for suitable $n > 0$ and $k_1, \ldots, k_n, h_1, \ldots, h_n \in N_5$. Suppose $p \in M(\mathcal{N}_5)$. Choose a representation (3) of p with the least possible n. From $p(1) = 1$ we obtain

$$(4) \qquad (\ldots(h_1 \vee k_2) \wedge \ldots) \vee k_n = 1 = h_n$$

while $p(0) = 0$ yields $(\ldots(k_1 \wedge h_1) \vee \ldots \vee k_n) \wedge 1 = 0$ i.e.

$$(\ldots(k_1 \wedge h_1) \vee \ldots) \wedge h_{n-1} = 0 = k_n.$$

By the minimality of n we obtain $n = 1$ and $p(x) \approx (x \vee 0) \wedge 1 \approx x$. However, this contradicts $p(a) = b$. Thus $p \notin M(N_5)$.

We have:

Lemma 9. *Let $\mathcal{A} = (A; F)$ be an algebra. Then*
(i) $\operatorname{Con} \mathcal{A} = \operatorname{Con} \mathcal{A}_P = \operatorname{Con} \mathcal{A}_M = \operatorname{Con} \mathcal{A}_T$.
(ii) *The following are equivalent for $S \subseteq A$:*
(a) *S is a block of a congruence of $\mathcal{A}$,*
(b)

$$S \cap g(S) \neq \emptyset \Rightarrow g(S) \subseteq S$$

holds for all $g \in P(\mathcal{A})$,
(c) *(5) holds for all $g \in M(\mathcal{A})$.*

P r o o f. (i) From $P(\mathcal{A}) \supseteq M(\mathcal{A}) \supseteq T(\mathcal{A})$ and the fact that $P(\mathcal{A})$ is the set of unary polynomials of $\mathcal{A}$ we obtain $\operatorname{Con} \mathcal{A} \subseteq \operatorname{Con} \mathcal{A}_P \subseteq \operatorname{Con} \mathcal{A}_M \subseteq \operatorname{Con} \mathcal{A}_T$. To prove $\operatorname{Con} \mathcal{A}_T \subseteq \operatorname{Con} \mathcal{A}$ let $\theta \in \operatorname{Con} \mathcal{A}_T$, let $f \in F$ be n-ary and let

$$(a_1, b_1), \ldots, (a_n, b_n) \in \theta.$$

For $i = 0, \ldots, n$ set

$$c_i = f(b_1, \ldots, b_i, a_{i+1}, \ldots, a_n)$$

and notice that $c_0 = f(a_1, \ldots, a_n)$ while $c_n = f(b_1, \ldots, b_n)$. For $i = 1, \ldots, n$ denote by t_i the translation

$$t_i(x) \approx f(b_1, \ldots, b_{i-1}, x, a_{i+1}, \ldots, a_n).$$

As $t_i \in T$ and $\theta \in \operatorname{Con} \mathscr{A}_T$, we have

$$(c_{i-1}, c_i) = (t_i(a_i), t_i(b_i)) \in \theta.$$

By transitivity,

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) = (c_0, c_n) \in \theta.$$

(Notice that in this standard proof the symmetry of θ has not been used and so (i) holds if we replace Con by Quao where Quao $\mathscr{A}$ denotes the set of all compatible quasiorders (= reflexive and transitive relations). The equality Quao $\mathscr{A}$ = Quao $\mathscr{A}_P$ was observed in [7], p. 10).

(ii) Let $S \subseteq A$. (a)$\Rightarrow$(b): If S is a block of some $\theta \in \operatorname{Con} \mathscr{A}$, then clearly every polynomial g of $\mathscr{A}$ satisfies (5). (b)$\Rightarrow$(c): Trivial. (c)$\Rightarrow$(a): Let (5) hold for every $g \in M(\mathscr{A})$. Denote by θ the reflexive and transitive hull of the binary relation $\bigcup\{g(S^2) : g \in M(\mathscr{A})\}$. It is easy to verify that $\theta \in \operatorname{Con} \mathscr{A}_M$. As $\operatorname{Con} \mathscr{A}_M = \operatorname{Con} \mathscr{A}$, by (i), it remains to show that S is a block of θ. As $\mathrm{id}_A \in M(\mathscr{A})$ clearly $S^2 = \mathrm{id}_A(S^2) \subseteq \theta$. Suppose to the contrary that S is not a block of θ. By the definition of θ there exist $s, s' \in S$ and $g \in M(\mathscr{A})$ such that $g(s) \in S$ while $g(s') \notin S$ in contradiction to (5). $\square$

In this paper we study algebras $\mathscr{A}$ with 0 such that every ideal of $\mathscr{A}$ is a congruence kernel. The next theorem characterizes such algebras. As usual, for a binary relation ϱ on A we denote by $Cg(\varrho)$ the least congruence of $\mathscr{A}$ containing ϱ. For $\varrho = \{\langle a, b\rangle\}$ we abbreviate $Cg(\{a, b\})$ by $Cg(a, b)$.

Theorem 10. *The following are equivalent for an algebra $\mathscr{A} = (A; F)$ with 0:*

(i) *Every ideal of $\mathscr{A}$ is a congruence kernel.*

(ii) *$I(S) = [0]Cg(\{0\} \times S)$ for every subset S of A.*

(iii) *$I(S) = [0]Cg(\{0\} \times S)$ for every finite subset S of A.*

(iv) *$I(S) = [I(S \setminus \{s\})]Cg(0, s)$ for every finite nonempty subset S of A and each $s \in S$.*

(v) *For every finite subset $S = \{s_1, \ldots, s_n\}$ of A and $\theta_i = Cg(0, s_i)$ $(i = 1, \ldots, n)$*

$$I(S) = [0](\theta_1 \circ \ldots \circ \theta_n).$$

(vi) *For every ideal I of $\mathscr{A}$, all $a, b \in I$ and every $p \in P(\mathscr{A})$*

$$p(a) \in I \Rightarrow p(b) \in I.$$

(vii) *For every ideal I of $\mathscr{A}$, all $a, b \in I$ and every $m \in M(\mathscr{A})$*

$$m(a) \in I \Rightarrow m(b) \in I.$$

(viii) $p(a) \in I(a, b, p(b))$ *for all* $a, b \in A$ *and every* $p \in P(\mathscr{A})$.

(ix) $m(a) \in I(a, b, m(b))$ *for all* $a, b \in A$ *and every* $m \in M(\mathscr{A})$.

Proof. (i)$\Rightarrow$(ii): Let (i) hold and let $S \subseteq A$. The $I(S) = [0]\tau$ for some $\tau \in \operatorname{Con} \mathscr{A}$. Set $\theta := Cg(\{0\} \times S)$. From $S \subseteq I(S) = [0]\tau$ we obtain $\{0\} \times S \subseteq \tau$ and so $[0]\theta \subseteq [0]\tau$. Clearly $S \subseteq [0]\theta$. By Lemma 7 the set $[0]\theta$ is an ideal of $\mathscr{A}$ and therefore $I(S) \subseteq [0]\theta$. Together $I(S) \subseteq [0]\theta \subseteq [0]\tau = I(S)$; hence $I(S) = [0]\theta$ proving (ii). Next (ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(iv): Let (iii) hold and let $S = \{s_1, \ldots, s_n\}$ be a finite subset of A. Set $S' := \{s_1, \ldots, s_{n-1}\}$; $K := I(S')$ and $\theta := Cg(0, s_n)$.

1) Let $n = 1$. Then $I(\emptyset) = \{0\}$. Applying (iii) to $S = \{s_1\}$ we obtain the required $I(S) = [0]\theta = [I(\emptyset)]\theta = [I(S')]\theta$.

2) Thus let $n > 1$. To prove $I(S) \subseteq [I(S')]\theta$ let $w \in I(S)$ be arbitrary. By Lemma 5 we have $w = f(a_1, \ldots, a_m)$ for an m-ary M-ideal term operation f of $\mathscr{A}$ and $a_1, \ldots, a_m \in A$ such that $a_i \in S$ for all $i \in M$. If $M = \emptyset$ then f is constant with value 0 and $w = 0 \in [I(S')]\theta$. Thus let $M \neq \emptyset$. For notational simplicity let $M = \{1, \ldots, p\}$ for some $1 \leqslant p \leqslant m$. Without loss of generality we may assume that each s_i appears at most once among $a_1, \ldots, a_p$. (Indeed, if some s_i appears more than once, it suffices to fuse the coresponding variables). We distinguish two cases. (1) Let $s_n \notin \{a_1, \ldots, a_p\}$. Then $w \in I(S') \subseteq [I(S')]\theta$ and we are done. (2) Thus let $s_n \in \{a_1, \ldots, a_p\}$, e.g. let $s_n = a_1$. Set $v := f(0, a_2, \ldots, a_m)$. Again from Lemma 5 and $I(S') = I(S' \cup \{0\})$ we obtain that $v \in I(S')$. Moreover, $(v, w) \in \theta$ because f is a term operation of $\mathscr{A}$. Together we have the required $w \in [I(S')]\theta$ and $\subseteq$. To prove $I(S) \supseteq [I(S')]\theta$ let $w \in [I(S')]\theta$. Then $(v, w) \in \theta$ for some $v \in I(S')$. By (iii) clearly $I(S') = [0]Cg(\{0\} \times S')$. Thus $(0, w) \in Cg(\{0\} \times S') \vee \theta = Cg(\{0\} \times S') \vee Cg(0, s_n) = Cg(0 \times S)$. Thus $w \in [0]Cg(\{0\} \times S)$ and so by (iii) we have $w \in I(S)$. Thus (iv) holds.

(iv)$\Rightarrow$(v) Let (iv) hold and let $S = \{s_1, \ldots, s_n\} \subseteq A$. For $i = 1, \ldots, n$ set $\theta_i := Cg(0, s_i)$ and $S_i := \{s_1, \ldots, s_i\}$. From (iv) we get $I(S_1) = [I(\emptyset)]\theta_1 = [0]\theta_1$. By an easy induction we obtain

$$I(S) = I(S_n) = (\ldots(([0]\theta_1)\theta_2)\ldots) = [0](\theta_1 \circ \theta_2 \circ \ldots \circ \theta_n).$$

(v)$\Rightarrow$(iii): Let (v) hold and let $S = \{s_1, \ldots, s_n\} \subseteq A$. For $i = 1, \ldots, n$ set $\theta_i := Cg(0, s_i)$. Further set $\sigma := Cg(\{0\} \times S)$ and $K := [0]\sigma$. Notice that $\sigma = \theta_1 \vee \ldots \vee \theta_n$ (in the lattice of equivalences on A). By Lemma 7 the set K is an ideal of $\mathscr{A}$. Clearly $S \subseteq K$ and whence $I(S) \subseteq K$. To prove $K \subseteq I(S)$ let $v \in K$, i.e. $(0, v) \in \sigma = \theta_1 \vee \ldots \vee \theta_n$. There exist $m \geqslant 1$, $0 = b_0, b_1, \ldots, b_m = v$ in A and $j_0, j_1, \ldots, j_{m-1} \in \{1, \ldots, n\}$ such that $(b_i, b_{i+1}) \in \theta_{j_i}$ for $i = 0, \ldots, m-1$. We need the following:

Claim. $[0](\theta_1 \circ \ldots \circ \theta_n) = [0](\theta_{\pi(1)} \circ \ldots \circ \theta_{\pi(n)})$ for every permutation π of $\{1, \ldots, n\}$.

Proof of the claim. Apply (v) to $S = \{s_{\pi(1)}, \ldots, s_{\pi(n)}\}$ to obtain $I(S) = [0](\theta_{\pi(1)} \circ \ldots \circ \theta_{\pi(n)})$.

Using repeatedly the claim we obtain $(0, v) \in \theta_1 \circ \ldots \circ \theta_n$, hence $v \in [0](\theta_1 \circ \ldots \circ \theta_n) = I(S)$ by (v). Thus $K \subseteq I(S)$ and (iii) holds.

(iii)⇒(ii): Let (iii) hold and let $S \subseteq A$. Set $\sigma := Cg(\{0\} \times S)$. Again by Lemma 7 and $S \subseteq [0]\sigma$ we have $I(S) \subseteq [0]\sigma$. For the converse let $v \in [0]\sigma$. Then $(0, v) \in \sigma$. The congruence σ is compactly generated and so $(0, v) \in \sigma' := Cg(\{0\} \times S')$ for some finite subset S' of S. From (iii) we obtain $v \in [0]\sigma' = I(S') \subseteq I(S)$. Thus $[0]\sigma \subseteq I(S)$.

(ii)⇒(i): Trivial. (i)⇔(vi)⇔(vii): Lemma 9 (ii) (a)⇔(b)⇔(c).

(vi)⇒(viii): Let (vi) hold and let $a, b \in A$ and $p \in P(\mathscr{A})$. Set $I := I(a, b, p(b))$. As $p(b) \in I$, the condition (vi) yields $p(a) \in I$. (viii)⇒(ix): Trivial.

(xi)⇒(i): Let (ix) hold. Suppose to the contrary that (i) does not hold. Then there exists an ideal S of $\mathscr{A}$ which is the kernel of no congruence of $\mathscr{A}$. By Lemma 9 (ii) (c) ⇒ (a) there exist $m \in M(\mathscr{A})$ and $a, b \in S$ such that $m(a) \notin S$ while $m(b) \in S$. Observe that by (ix) we have $m(a) \in I(a, b, m(b)) \subseteq I(S) = S$ in contradiction to $m(a) \notin S$. □

Corollary 11. *Let $\mathscr{A}$ be such that to every two-element subset T of A there exists a binary term operation p_T of $\mathscr{A}$ satisfying $p_T(0, 0) = 0$ and $Cg(\{0\} \times T) \subseteq Cg(0, t)$ for some $t = p_T(a, b)$ with $a, b \in T$. Then every ideal of $\mathscr{A}$ is a congruence kernel if and only if $I(x)$ is a congruence kernel for every $x \in A$.*

Proof. (⇒) Obvious. (⇐) Let $I(x)$ be a congruence kernel for all $x \in A$. We need the following:

Claim. For every finite subset S of A we have $Cg(\{0\} \times S) = Cg(0, s)$ for some $s \in I(S)$.

Proof of the claim. By induction on $n := |S|$. The claim is evident for $n \leqslant 1$. Thus assume that the claim holds for some $n \geqslant 1$ and let $S = \{s_1, \ldots, s_{n+1}\}$. Set $S' := \{s_1, \ldots, s_n\}$. By the induction hypothesis $Cg(\{0\} \times S') = Cg(0, s')$ for some $s' \in I(S')$. Set $T := \{s', s_{n+1}\}$ and $\theta := Cg(\{0\} \times T)$. By the hypothesis $\theta \subseteq Cg(0, t)$ for some $t := p_T(a, b)$ with $a, b \in T$. Clearly $(0, t) = (p_T(0, 0), p_T(a, b)) \in \theta$; whence $Cg(0, t) \subseteq \theta$ and $\theta = Cg(0, t)$. As $p_T(0, 0) = 0$, clearly p_T is an $\{1, 2\}$-ideal term operation and so $t \in I(T) \subseteq I(S)$. This concludes the induction step.

For the remaining part, we verify (iii) from Theorem 10. Let S be a finite subset of A. By the claim and the hypothesis $I(S) \subseteq [0]Cg(\{0\} \times S) = [0]Cg(0, s) = I(s) \subseteq I(S)$. □

For varieties we obtain:

Corollary 12. *The following conditions are equivalent for a variety $\mathscr{V}$ of algebras of the same type with a nullary term* 0:

(i) *Every ideal of each $\mathscr{A} \in \mathscr{V}$ is a congruence kernel.*

(ii) *To every $n \geqslant 3$ and each term $q(x_1, \ldots, x_n)$ of $\mathscr{V}$ in which x_1 occurs exactly once, there exists an n-ary term p of $\mathscr{V}$ satisfying the following identities:*

$$p(0, 0, 0, x_4, \ldots, x_n) = 0, \tag{6}$$

$$q(x_1, x_1, x_2, \ldots, x_{n-1}) = p(q(x_2, x_1, x_2, x_3, \ldots, x_{n-1}), x_1, \ldots, x_{n-1}). \tag{7}$$

Proof. (i)⇒(ii): Let (i) hold, let $n > 1$ and let $q(x_1, \ldots, x_n)$ be an n-ary term of $\mathscr{V}$ in which x_1 occurs exactly once (e.g. $(x_2 \wedge x_3) \vee (x_4 \wedge (x_3 \vee (x_1 \wedge x_2)))$ is such a term in the variety of lattices). Denote by $\mathscr{Z}$ the free algebra of $\mathscr{V}$ on $n-1$ generators $x_1, \ldots, x_{n-1}$. For every $z \in Z$ set

$$m(z) := q(z, x_1, \ldots, x_{n-1}). \tag{8}$$

It is easy to see that $m \in M(\mathscr{Z})$ (in the above example $m = t_1 \circ t_2 \circ t_3 \circ t_4$ where $t_1(z) \approx (x_1 \wedge x_2) \vee z$, $t_2(z) \approx x_3 \wedge z$, $t_3(z) \approx x_2 \vee z$, $t_4(z) \approx z \wedge x_1$). By assumption $\mathscr{Z} \in \mathscr{V}$ satisfies (i) and therefore by Theorem 10 (i)⇒(iii) the algebra $\mathscr{Z}$ also satisfies (ix). For $a = x_1$ and $b = x_2$ we obtain $m(x_1) \in I(x_1, x_2, m(x_2))$ where by (8)

$$m(x_1) = q(x_1, x_1, \ldots, x_{n-1}), \quad m(x_2) = q(x_2, x_1, \ldots, x_{n-1}).$$

Set $S := \{x_1, x_2, q(x_2, x_1, \ldots, x_{n-1})\}$. From $m(x_1) \in I(S)$ and Lemma 5 we obtain

$$q(x_1, x_1, \ldots, x_{n-1}) = m(x_1) = g(a_1, \ldots, a_k)$$

where g is an N-ideal term operation of $\mathscr{Z}$ and $a_1, \ldots, a_k \in Z$ satisfy $a_i \in S$ for all $i \in N$.

Notice that each $a_i \in Z \setminus S$ is of the form $h_i(x_1, \ldots, x_{n-1})$ for some term h_i of $\mathscr{V}$. It follows that

$$g(a_1, \ldots, a_k) = p(q(x_2, x_1, \ldots, x_{n-1}), x_1, \ldots, x_{n-1}).$$

for some $\{1, 2, 3\}$-ideal term operation p of $\mathscr{V}$. Thus (ii) holds. (ii)⇒(i): Let (ii) hold, let $\mathscr{A} \in \mathscr{V}$, let $a_1, a_2 \in A$ and let $m \in M(\mathscr{A})$. Then there exists $k \geqslant 1$, a k-ary term $r(x_1, \ldots, x_k)$ of $\mathscr{V}$ and $a_3, \ldots, a_{k+1} \in A$ such that (1) x_1 appears at most once in r and (2) $m(x) = r^{\mathscr{A}}(x, a_3, \ldots, a_{k+1})$ for all $x \in A$ (where, as usualy $r^{\mathscr{A}}$ denotes the k-ary term operation of A which to arbitrary $b_1, \dots, b_k \in A$ assigns the value calculated in $\mathcal{A}$ according to r). Set $n := k + 2$ and define the n-ary term q of $\mathcal{V}$ by

$$q(x_1, \dots, x_n) = r(x_1, x_4, \dots, x_n)$$

(i.e. q differs from r only in two dummy variables). By (ii) to q there exists an n-ary term p of $\mathcal{V}$ satisfying (6) and (7). Now

$$(*) \qquad \begin{aligned} m(a_1) &= q^{\mathcal{A}}(a_1, a_1, a_2, \dots, a_{n-1}) \\ &= p^{\mathcal{A}}(q^{\mathcal{A}}(a_2, a_1, a_2, a_3, \dots, a_{n-1}), a_1, \dots, a_{n-1}) \\ &= p^{\mathcal{A}}(r^{\mathcal{A}}(a_2, \dots, a_{n-1}), a_1, \dots, a_{n-1}). \end{aligned}$$

According to (6) the operation $p^{\mathcal{A}}$ is an $\{1, 2, 3\}$-ideal term of $\mathcal{A}$. Now $(*)$ and Lemma 5 show that $m(a_1) \in I(a_1, a_2, m(a_2))$. Thus (ix) of Theorem 10 is satisfied and so (i) holds. $\square$

Example 13. 1) Consider the variety of all groups (with the neutral element 0). For $n \geqslant 3$ each term $q(x_1, \dots, x_n)$ in which x_1 occurs exactly once is of the form $ax_1^j b$ where a and b are terms in $x_2, \dots, x_n$ and $j \in \{-1, 1\}$. Put

$$p(x_0, \dots, x_{n-1}) := x_0 b^{-1}(x_1, \dots, x_{n-1}) x_2^{-j} x_1^{j} b(x_1, \dots, x_{n-1}).$$

Clearly p satisfies (6). We check (7). Abbrevite $(x_1, \dots, x_{n-1})$ by u and set $\alpha := a(u)$ and $\beta := b(u)$. Then $q(x_1, u) = \alpha x_1{}^j \beta$, $q(x_2, u) = \alpha x_2{}^j \beta$ and

$$p(q(x_2, u), u) = q(x_2, u)\beta^{-1} x_2{}^{-j} x_1{}^j \beta = \alpha x_2{}^j \beta\beta^{-1} x_2{}^{-j} x_1{}^j \beta = \alpha x_1{}^j \beta = q(x_1, u)$$

proving (7). From Corollary 12 we obtain that every group ideal is a congruence kernel. As group ideals are exactly the normal subgroups this is just the elementary fact relating normal subgroups and group congruences.

2) Consider the variety $\mathcal{V}$ of distributive lattices with 0. Let $n \geqslant 3$ and let $q(x_1, \dots, x_n)$ be a term of $\mathcal{V}$. Then q can be written as (1) $(x_1 \wedge a) \vee b$ or (2) $x_1 \vee b$ where a and b are terms of $\mathcal{V}$ in variables $x_2, \dots, x_n$. Consider the case (1). Set

$$p(x_1, \dots, x_n) := (x_1 \wedge b) \vee (x_2 \wedge a).$$

Clearly p satisfies (6). We check (7). Again abbreviate $(x_1, \dots, x_{n-1})$ by u and $a(u)$ and $b(u)$ by α and β. Now

$$\begin{aligned} p(q(x_2, u), u) &= (q(x_2, u) \wedge \beta) \vee (x_2 \wedge \alpha) = (((x_2 \wedge \alpha) \vee \beta) \wedge \beta) \vee (x_1 \wedge \alpha) = \\ &= \beta \vee (x_1 \wedge \alpha) = q(x_1, u). \end{aligned}$$

The case (2) is similar but simpler.

From Corollary 12 we obtain that every ideal of a distributive lattice is a congruence kernel. This is a known result [4]; in fact, in [4] it is also shown that among lattices only distributive lattices have this property.

Folowing [2, 3, 5] we say that $\mathscr{A}$ is permutable at 0 if $[0](\theta \circ \psi) = [0](\psi \circ \theta)$ for all $\theta, \psi \in \operatorname{Con} \mathscr{A}$. We have

Proposition 14. *Let $\mathscr{V}$ be a variety of algebras of the same type such that 0 is a nullary term of $\mathscr{V}$. Then*

1) *The following are equivalent:*

(i) *Every $\mathscr{A} \in \mathscr{V}$ is permutable at 0,*

(ii)

$$b(x, x) \approx 0, \;\; b(x, 0) \approx x \tag{9}$$

for a binary term b of $\mathscr{V}$, and

(iii)

$$t(x, x, y) \approx y, \;\; t(0, x, x) \approx 0 \tag{10}$$

for a ternary term t of $\mathscr{V}$.

2) *If $\mathscr{V}$ satisfies one of* (i)–(iii), *then for every $\mathscr{A} \in \mathscr{V}$ each ideal of $\mathscr{A}$ is a congruence kernel.*

P r o o f. 1) The equivalence of (i)–(iii) is shown in [5] pp. 48–49. 2) Let (iii) hold for $\mathscr{V}$ and let t be a term of $\mathscr{V}$ satisfying (10). Let $\mathscr{A} \in \mathscr{V}$ and let I be an ideal of $\mathscr{A}$. We verify the condition (vi) of Theorem 10. Let $p \in P(\mathscr{A})$ satisfy $p(i) \in I$ for some $i \in I$ and let $i' \in I$. There exists an m-ary term operation q of $\mathscr{A}$ and $a_2, \dots, a_m \in A$ such that $p(x) \approx q(x, a_2, \dots, a_m)$. Set

$$s(x_1, \dots, x_{m+2}) :\approx t(x_1, q(x_2, x_4, \dots, x_{m+2}), q(x_3, x_4, \dots, x_{m+2})).$$

By the second half of (10)

$$s(0, 0, 0, x_4, \dots, x_{m+2}) \approx t(0, q(0, x_4, \dots, x_{m+2}), q(0, x_4, \dots, x_{m+2})) \approx 0$$

and so s is an $\{1, 2, 3\}$-ideal term operation of $\mathscr{A}$. By the first half of (10) and the definition of s

$$p(i') = t(p(i), p(i), p(i')) = s(p(i), i, i', a_2, \dots, a_m).$$

Here $p(i), i, i' \in I$ and so $p(i') \in I$ as well. $\square$

Example 15. Consider the variety $\mathscr{V}$ of all pseudocomplemented meet-semilatices $\mathscr{A} = (A; \wedge, *, 0)$ with 0 (i.e. for every $a \in A$ the element a^* is the greatest element y such that $a \wedge y = 0$). The term $b(x, y) :\approx x \wedge y^*$ satisfies (9) and therefore every ideal of a pseudocomplemented meet-semilattice with 0 is a congruence kernel.

References

[1] *Bělohlávek R., Chajda I.*: Congruences and ideals in semiloops. Acta Sci. Math. (Szeged) *59* (1994), 43–47.

[2] *Chajda I.*: A localization of some congruence conditions in varieties with nullary operations. Annales Univ. Sci Budapest, Sectio Math. *30* (1987), 17–23.

[3] *Duda J.*: Arithmeticity at 0. Czech. Math. J *27* (1987), 197–206.

[4] *Grätzer G., Schmidt E.T.*: Ideals and congruence relations in lattices. Acta Math. Acad. Sci. Hungar. *9* (1958), 137–175.

[5] *Gumm H.-P., Ursini A.*: Ideals in universal algebras. Algebra Universalis *19* (1984), 45–54.

[6] *Hashimoto J.*: Ideal theory fo lattices. Mathem. Japon. *2* (1952), 149–186.

[7] *Larose B.*: M. Sc. thesis. Université de Montréal, 1990.

[8] *Mal'tsev A.I.*: On the general theory of algebraic systems (Russian). Matem. Sbornik *35* (1954), 3–20.

[9] *Matthiessen G.*: Ideals, normal sets and congruences. Colloq. Math. Soc. J. Bolyai Szeged (Hungary) *17* (1975), 295–310.

[10] *Raftery J.G.*: Ideal determined varieties need not be congruence 3-permutable. Preprint University of Natal, Pietermaritzburg, 1992.

[11] *Ursini A.*: Sulle varietá di algebra con una buona teoria degli ideali. Boll. U.M.I. (4) *6* (1972), 90–95.

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