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PRINCIPAL CONVERGENCES ON LATTICE ORDERED GROUPS

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All lattice ordered groups dealt with in the present paper are assumed to be abelian.

Let G be a lattice ordered group. The set of all sequential convergences on G will be denoted by $\operatorname{Conv} G$; this set is partially ordered in a natural way. It was investigated in the papers [1]–[4], [6]–[9]. For the sake of brevity, we shall say "convergence" instead of "sequential convergence".

The partially ordered set $\operatorname{Conv} G$ need not be, in general, a lattice. It possesses the least element (the discrete convergence) which will be denoted by $d(G)$. For each $\alpha \in \operatorname{Conv} G$, the interval $[d(G), \alpha]$ of $\operatorname{Conv} G$ is a complete Brouwerian lattice (cf. [2]).

A convergence $\alpha \in \operatorname{Conv} G$ will be said to be principal if there exists a sequence (x_n) in α such that, whenever $\alpha_1 \in \operatorname{Conv} G$ and $(x_n) \in \alpha_1$, then $\alpha \leqslant \alpha_1$.

If $\alpha \in \operatorname{Conv} G$ and if the interval $[d(G), \alpha]$ is finite, then α is principal. In the present paper the following results will be established:

(A) Let $\alpha \in \operatorname{Conv} G$, $\alpha \neq d(G)$. Assume that the interval $[d(G), \alpha]$ is finite. Then $[d(G), \alpha]$ is a Boolean algebra.

(B) Let α be as in (A). Then $[d(G), \alpha]$ is a direct factor of the partially ordered set $\operatorname{Conv} G$.

(C) Let $\alpha \in \operatorname{Conv} G$. Then the following conditions are equivalent:

 (i) If $\alpha_1 \in \operatorname{Conv} G$, $\alpha_1 \leqslant \alpha$, then α_1 is principal.

 (ii) The interval $[d(G), \alpha]$ of $\operatorname{Conv} G$ is finite.

(B) generalizes a result of [4]. Some further results on $\operatorname{Conv} G$ will also be proved. For instance, it will be shown that if $d(G) < \alpha \in \operatorname{Conv} G$ and if the interval $[d(G), \alpha]$ of $\operatorname{Conv} G$ is a chain, then α is an atom of $\operatorname{Conv} G$.

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For convergences in a lattice ordered group we will apply the same definitions and notation as in [8], Section 1.

Let G be a lattice ordered group. If α is a principal convergence on G and if (x_n) is as above, then α is said to be *generated* by (x_n).

1.1. Lemma. *Let $\alpha \in \operatorname{Conv} G$. Assume that the interval $[d(G), \alpha]$ of $\operatorname{Conv} G$ is finite. Then α is principal.*

Proof. Put $\operatorname{card}[d(G), \alpha] = n$. We apply the induction on n. First let $n = 1$: set $x_n = 0$ for each $n \in \mathbb{N}$. Then α is generated by (x_n). Next assume that $n > 1$. There exists $\alpha_1 \in \operatorname{Conv} G$ such that α covers α_1. Hence by the induction hypothesis there exists (y_n) in α_1 such that α_1 is generated by (y_n). Further, there exists $(z_n) \in \alpha \setminus \alpha_1$. Put $x_n = y_n \vee z_n$ for each $n \in \mathbb{N}$. Then $(x_n) \in \alpha$, thus there exists $\delta \in \operatorname{Conv} G$ such that δ is generated by (x_n). We have $\alpha_1 < \delta \leqslant \alpha$; because α_1 is covered by α we infer that $\delta = \alpha$. Hence α is principal. □

The fact that $\operatorname{Conv} G$ possesses the least element $d(G)$ implies that when dealing with direct product decompositions of $\operatorname{Conv} G$ it suffices (without loss of generality) to consider only such direct factors which are convex subsets of $\operatorname{Conv} G$ and contain the element $d(G)$; if X is such a direct factor and $\alpha \in \operatorname{Conv} G$, then the component of α in X is the element $\sup\{\beta \in X : \beta \leqslant \alpha\}$. (Cf. also [4].)

Proof of (A). Again, assume that the interval $[d(G), \alpha]$ of $\operatorname{Conv} G$ is finite. Since $d(G) < \alpha$, we have $\operatorname{card}[d(G), \alpha] > 1$. Hence the set of atoms of the lattice $[d(G), \alpha]$ is nonempty; let this set be $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

In view of [4], the interval $[d(G), \alpha_1]$ of $\operatorname{Conv} G$ is a direct factor of $\operatorname{Conv} G$. Thus there is a convex subset Z of $\operatorname{Conv} G$ such that

$$\operatorname{Conv} G = [d(G), \alpha_1] \times Z. \tag{1}$$

In view of (1) and of the fact that $\alpha_1 \leqslant \alpha$ there exists $\beta \in Z$ such that

$$\alpha = \alpha_1 \vee \beta, \quad \alpha_1 \wedge \beta = d(G). \tag{2}$$

First assume that $k = 1$. If $\beta > d(G)$, then the interval $[d(G), \beta]$ of $\operatorname{Conv} G$ is finite and has a cardinality greater or equal to 2, thus there is an atom γ of $\operatorname{Conv} G$ with $\gamma \leqslant \beta$. Hence $\gamma \leqslant \alpha$ and $\gamma \neq \alpha_1$, which is a contradiction. Therefore $\beta = d(G)$ and thus according to (2) we obtain $\alpha = \alpha_1$, whence $[d(G), \alpha] = \{d(G), \alpha_1\}$. We have verified that the assertion holds for $k = 1$.

Let $k > 1$ and suppose that the assertion is valid for $k-1$. If γ is an atom of the interval $[d(G), \beta]$, then $\gamma \in \{\alpha_2, \alpha_3, \dots, \alpha_k\}$. Conversely, let $i \in \{2, 3, \dots, k\}$. Then $\alpha_1 = \alpha_i \wedge \alpha = \alpha_i \wedge (\alpha_1 \vee \beta)$. Since the lattice $[d(G), \beta]$ is Brouwerian and $\alpha_i \wedge \alpha_1 = d(G)$ we infer that $\alpha_i = \alpha_i \wedge \beta$, hence $\alpha_i \leqslant \beta$. Thus the set of all atoms of the interval $[d(G), \beta]$ is $\{\alpha_2, \alpha_3, \dots, \alpha_k\}$. Hence $[d(G), \beta]$ is a Boolean algebra with 2^{k-1} elements. Next, the relations (2) yield that the interval $[d(G), \alpha]$ is a direct product $[d(G), \alpha_1] \times [d(G), \beta]$. Hence $[d(G), \alpha]$ is a Boolean algebra with 2^k elements. □

Let X be a nonempty subset of G and let $(a_n) \in (G^{\mathbb{N}})^+$. If there exists $m \in \mathbb{N}$ such that $a_n \in X$ whenever $n \geqslant m$, then we say that (a_n) *ultimately deals* on X. In such a case let $m(0)$ be the least m with the property; we put $a_n[X] = a_{m(0)+n-1}$ for each $n \in \mathbb{N}$.

Let H be a convex ℓ-subgroup of G and $\alpha \in \operatorname{Conv} G$. The set of all sequences $(a_n) \in \alpha$ such that (a_n) ultimately deals on H will be denoted by α_H. Then $\alpha_H \neq \emptyset$. Next, let $\alpha_{[H]}$ be the set of all sequences $(a_n[H])$, where (a_n) runs over the set α_H.

For $X \subseteq G$ we put $X^\delta = \{g \in G \colon |g| \wedge |x| = 0 \text{ for each } x \in X\}$.

1.2. Lemma. *Let α_1 be an atom of $\operatorname{Conv} G$. Then there exists a uniquely determined ℓ-subgroup $H = C(\alpha_1) \neq \{0\}$ of G such that*

(i) *H is linearly ordered and it is a convex ℓ-subgroup of G;*

(ii) *$\alpha_1 \subseteq \alpha_H$;*

(iii) *if Z is as in (1) and $\gamma \in Z$, then each sequence belonging to Z ultimately deals on H.*

Proof. This is a consequence of [6], Theorem 4.7 (including the facts mentioned in the proof of the quoted theorem). □

Proof of (B). Under the assumptions as in (B), let $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be the set of all atoms of the interval $d[(G), \alpha]$. If $k = 1$, then according to (A) we have $\alpha = \alpha_1$, and thus the assertion under consideration holds in view of [6], Theorem 4.7.

Suppose that $k > 1$ and that the assertion is valid for $k-1$. Let β be as in the proof of (A). Then, in view of the induction hypothesis, $[d(G), \beta]$ is a direct factor of $\operatorname{Conv} G$. Hence there is a convex subset Z_1 of $\operatorname{Conv} G$ with $d(G) \in Z_1$ such that

$$\operatorname{Conv} G = [d(G), \beta] \times Z_1. \tag{3}$$

From (2) we obtain

$$[d(G), \alpha_1] \cap [d(G), \beta] = \{d(G)\}. \tag{4}$$

The relations (1), (2) and (4) yield

$$(5) \qquad [d(G),\alpha_1]\cap Z_1=[d(G),\alpha_1], \quad [d(G),\beta]\cap Z=[d(G),\beta].$$

Since $\operatorname{Conv} G$ is connected, according to [5] the direct decompositions (1) and (3) have a common refinement

$$\begin{aligned}\operatorname{Conv} G = {} & ([d(G),\alpha_1]\cap[d(G),\beta])\times([d(G),\alpha_1]\cap Z_1)\\ & \times(Z\cap[d(G),\beta])\times(Z\cap Z_1).\end{aligned}$$

Hence in view of (4) and (5) we get

$$\begin{aligned}\operatorname{Conv} G &= [d(G),\alpha_1]\times[d(G),\beta]\times(Z\cap Z_1)\\ &= [d(G),\alpha]\times(Z\cap Z_1),\end{aligned}$$

completing the proof. □

1.3. Corollary. (Cf. [4].) *Let α be an atom of $\operatorname{Conv} G$. Then the interval $[d(G),\alpha]$ is a direct factor of $\operatorname{Conv} G$.*

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In this section the assertion (C) above will be dealt with.

Again, let G be a lattice ordered group. A sequence (a_n) in G will be said to be *strictly disjoint* if $a_n>0$ for each $n\in\mathbb{N}$ and $a_n\wedge a_m=0$ whenever n and m are distinct positive integers.

The following two lemmas are consequences of [2], Theorem 7.3.

2.1. Lemma. *Let (a_n) be a strictly disjoint sequence in G. Then there exists $\alpha\in\operatorname{Conv} G$ such that α is generated by (a_n).*

2.2. Lemma. *Let I be a nonempty set and for each $i\in I$ let α_i be a principal convergence on G generated by a sequence (a_n^i). Assume that for each $n,m\in\mathbb{N}$ and for each pair of distinct elements $i(1)$ and $i(2)$ of I the relation $a_n^{i(1)}\wedge a_m^{i(2)}=0$ is valid. Then $\alpha=\bigvee_{i\in I}\alpha_i$ does exist in $\operatorname{Conv} G$.*

2.3. Lemma. *Let I, α_i and (a_n^i) be as in 2.2. Assume that the set I is infinite. Then α is not principal.*

Proof. By way of contradiction, suppose that α is generated by a sequence (a_n). Then Theorem 2.2 of [3] yields that for each subsequence (a'_n) of (a_n) there is a subsequence (a''_n) of (a'_n) having the property that there exists a finite subset $I(1)$ of I and a positive integer k such that

$$a''_n \leqslant k \sum b^i_n \quad (i \in I(1)) \tag{6}$$

is valid for each $n \in \mathbb{N}$, where (b^i_n) is an appropriately chosen subsequence of (a^i_n) for each $i \in I(1)$.

Let I' be the union of all sets $I(1)$ which have the above mentioned property. Hence

$$\alpha = \bigvee \alpha_i \quad (i \in I').$$

We distinguish two cases. First suppose that $I' \neq I$. Thus there exists $j \in I \setminus I'$. Since $(a^j_n) \in \alpha$ and α is generated by (a_n), then (again in view of [3], Thm. 2.2) there exist subsequences (c^t_n) $(t = 1, 2, \ldots, m)$ of (a_n), a subsequence (a^{*j}_m) of (a^j_n) and a positive integer k' such that

$$a^{*j}_m \leqslant k' \sum c^t_n \quad (t = 1, 2, \ldots, n) \tag{7}$$

is valid for each $n \in \mathbb{N}$.

There exists a subsequence $(n(1))$ of the sequence $1, 2, 3, \ldots$ such that for each $t \in \{1, 2, \ldots, m\}$ the subsequence $(c^t_{n(1)})$ of (c^t_n) satisfies analogous conditions as (a''_n) above. In view of (7),

$$a^{*j}_{n(1)} \leqslant k' \sum c^t_{n(1)} \quad (t = 1, 2, \ldots, m)$$

holds for each member $n(1)$ of the sequence $(n(1))$ under consideration. But according to (6) we have $c^t_n \wedge a^{*j}_n = 0$ for $t = 1, 2, \ldots, m$, whence $a^{*j}_n = 0$ for each $n \in \mathbb{N}$, which is a contradiction.

Further, suppose that $I' = I$. Thus I' is infinite.

Let us denote by S the system of all subsequences (a''_n) of (a_n) which have the properties as above (cf. (6)). We construct a system (a''_{nk}) $(k \in \mathbb{N})$ as follows.

Let (a''_{n1}) be an arbitrary sequence belonging to S. Thus there exists a least finite subset $I(1)$ of I such that

$$a''_{n1} \leqslant k_1(b^{i(11)}_n + \ldots + b^{i(k(1),1)}_n)$$

is valid for each $n \in \mathbb{N}$, where $k_1 \in \mathbb{N}$, $k(1) \in \mathbb{N}$, $I(1) = \{i(1,1), i(2,1), \ldots, i(k(1),1)\}$, $(b^{i(11)}_n)$ is a subsequence of $(a^{i(1,1)}_n), \ldots, (b^{i(k(1),1)}_n)$ is a subsequence of $(a^{i(k(1),1)}_n)$. In view of the minimality of $I(1)$ there exists $n(1) \in \mathbb{N}$ such that

$$a''_{n(1),1} \wedge b^{i(1,1)}_{n(1)} > 0.$$

Now, since I' is infinite, there exists (a''_{n2}) in S such that (under analogous assumptions and notation as in the case of (a''_{n1})) we have

$$a''_{n2} \leqslant k_2(b_n^{i(1,2)} + \ldots + b_n^{i(k(2),2)}) \quad \text{for each } n \in \mathbb{N}$$

and $i(1,m) \notin I(1)$. By induction, for each $m \in \mathbb{N}$ there is $(a''_{nm}) \in S$ such that

$$a''_{nm} \leqslant k_m(b_n^{i(1,m)} + \ldots + b_n^{i(k(m),m)}) \quad \text{for each } n \in \mathbb{N}$$

and $i(1,m) \notin I(1) \cup I(2) \cup \ldots \cup I(m-1)$; moreover, the minimality of $I(m)$ (analogous to the minimality of $I(1)$) is satisfied.

For each $m \in \mathbb{N}$ there exists $n(m) \in \mathbb{N}$ such that

$$a''_{n(m),m} \wedge b_n^{i(1,m)} > 0 \tag{8}$$

is valid.

Let us consider the subsequence

$$a''_{n(1),1},\ a''_{n(2),2},\ a''_{n(3),3},\ \ldots \tag{9}$$

of (a_n). Let (c_n) be a subsequence of the sequence (9). Since the set $\{i(1,1), i(1,2), i(1,3), \ldots\}$ is infinite, the relation (8) yields that (c_n) does not belong to S, which is a contradiction. □

2.4. Lemma. *Let $\alpha \in \operatorname{Conv} G$, $(a_n) \in \alpha$ and suppose that (a_n) is strictly disjoint. Then there exists $\alpha_1 \in \operatorname{Conv} G$ with $\alpha_1 \leqslant \alpha$ such that α_1 fails to be principal.*

Proof. There exist infinite subsets I_m of $\mathbb{N}$ ($m = 1, 2, \ldots$) such that $I_m \cap I_{m(1)} = \emptyset$ whenever m and $m(1)$ are distinct positive integers. For each $m \in \mathbb{N}$ let (a_n^m) be the subsequence (a_n) consisting of those a_n for which the relation $n \in I_m$ holds. Now, by applying 2.2 and 2.3 we infer that there exists α_1 with the desired properties. □

Let us consider the following condition for an element $\alpha \in \operatorname{Conv} G$:

(c) There exist principal convergences α_1 and α_2 in $[d(G), \alpha]$ such that $\alpha_1 \neq d(G) \neq \alpha_2$ and $\alpha_1 \wedge \alpha_2 = d(G)$.

2.5. Lemma. *Let $\alpha \in \operatorname{Conv} G$. Assume that α contains a strictly disjoint sequence. Then α satisfies the condition (c).*

Proof. Let (a_n) be a strictly disjoint sequence belonging to α. For each $n \in \mathbb{N}$ put $b_n = a_{2n-1}$, $c_n = a_{2n}$. Then (b_n) and (c_n) belong to α as well. There exist α_1 and α_2 in $\operatorname{Conv} G$ such that α_1 is generated by (b_n) and α_2 is generated by (c_n). We have $d(G) < \alpha_i < \alpha$ for $i = 1, 2$. Next, from [3], Theorem 2.2 we obtain that $\alpha_1 \wedge \alpha_2 = d(G)$. □

2.6. Lemma. *Let $0 < v \in G$ be such that the interval $[0, v]$ of G is a chain. Let $\alpha \in \operatorname{Conv} G$, $(q_n) \notin d(G)$, $q_n \in [0, v]$ for each $n \in \mathbb{N}$. Let δ be the principal convergence on G generated by (q_n). Then δ is an atom of* $\operatorname{Conv} G$.

Proof. Since $(q_n) \notin d(G)$ we have $\delta > d(G)$. Let $\delta' \in \operatorname{Conv} G$, $d(G) < \delta' \leqslant \delta$. There exists $(q'_n) \in \delta' \setminus d(G)$. From [3], Theorem 2.2 it follows that (q'_n) ultimately deals on $[0, v]$. Put $C = \bigcup[-nv, nv]$ $(n = 1, 2, \ldots)$. Then C is a linearly ordered subgroup of G; next, both (q_n) and (q'_n) ultimately deal on C. Now from [2], Theorem 3.9 we obtain that $\delta' = \delta$. Hence δ is an atom of $\operatorname{Conv} G$. □

2.7. Lemma. *Let $\alpha \in \operatorname{Conv} G$, $\alpha > d(G)$. Assume that the interval $[d(G), \alpha]$ of $\operatorname{Conv} G$ does not contain any atom. Then α contains a strictly disjoint sequence.*

Proof. Since $\alpha \neq d(G)$, there exists $(a_n) \in \alpha \setminus d(G)$. Without loss of generality we can assume that $a_n > 0$ for each $n \in \mathbb{N}$.

We distinguish two cases.

(a) First suppose that there exists (a_n) as above such that, whenever $n \in \mathbb{N}$ and $v_n \in G$, $0 < v_n \leqslant a_n$, then the interval $[0, v_n]$ fails to be a chain. Hence there are $v'_n, v''_n \in [0, v_n]$ such that $0 < v'_n < v_n$, $0 < v''_n < v_n$ and $v'_n \wedge v''_n = 0$. Now by the same method as in [9], Lemma 2.1 we can verify that there exists a strictly disjoint sequence (b_n) belonging to α.

(b) Next assume that there is $n(1) \in \mathbb{N}$ having the property that there exists $v_1 \in G$ with $0 < v_1 \leqslant a_{n(1)}$ such that the interval $[0, v_1]$ of G is a chain; let $n(1)$ be the least positive integer which has this property. For each $n \in \mathbb{N}$ put $q_n = v_1 \wedge a_{n(1)+n}$. Then $(q_n) \in \alpha$; let δ be the principal convergence on G generated by (q_n). Since $[0, v_1]$ is a chain, in view of 2.6 either $\delta = d(G)$ or δ is an atom of $\operatorname{Conv} G$. The latter case is impossible, since $\delta \leqslant \alpha$. Thus there is $m \in \mathbb{N}$ such that $q_n = 0$ for each $n \geqslant m$. If there exists a positive integer $n(2) = n(1) + m$ such that $[0, v_2]$ is a chain for an element v_2 with $0 < v_2 \leqslant a_{n(2)}$, then we take the least element $n(2)$ with this property. Now either

(i) we can construct in this way a sequence $(v_n) \in \alpha$,

or

(ii) there exists $k \in \mathbb{N}$ such that the sequence (a_{k+n}) has the property investigated in the case (a).

If (i) holds, then (v_m) is strictly disjoint. If (ii) is valid, then it suffices to apply (a). □

2.8. Lemma. *Let $\alpha \in \operatorname{Conv} G$. Assume that the interval $[d(G), \alpha]$ of $\operatorname{Conv} G$ is infinite and that α_1 is an atom of $[d(G), \alpha]$. Then α satisfies the condition (c) and there is $\alpha' \in [d(G), \alpha]$ such that α' is not principal.*

Proof. Since $[d(G), \alpha]$ is infinite we have $\alpha_1 < \alpha$; next, α_1 is an atom of $\operatorname{Conv} G$. Hence according to [4], Theorem 4.7, the interval $[d(G), \alpha_1]$ is a direct factor of $\operatorname{Conv} G$. Thus we have $\operatorname{Conv} G = [d(G), \alpha_1] \times Z$ for a convex subset Z of $\operatorname{Conv} G$ with $d(G) \in Z$. Let β and γ be the components of α in $[d(G), \alpha_1]$ and in Z respectively. Then $\alpha = \beta \vee \gamma$, $\beta \wedge \gamma = d(G)$, $\beta \in \{d(G), \alpha_1\}$, $\gamma \in Z$. If $\beta = d(G)$, then $\alpha_1 = \alpha_1 \wedge \alpha = \alpha_1 \wedge \gamma = d(G)$, which is a contradiction. Hence $\beta = \alpha_1$. If $\gamma = d(G)$, then we would have $\alpha = \alpha_1$, which is impossible. Thus $\gamma > d(G)$ and therefore α satisfies the condition (c).

Next, the interval $[d(G), \gamma]$ of $\operatorname{Conv} G$ is infinite. If this interval does not contain any atom, then it suffices to apply 2.4 and 2.7. If $[d(G), \gamma]$ contains an atom α_2, then we proceed by applying the same steps for γ as we did for α above. In this way either (i) we obtain a sequence (α_n) of distinct atoms of $\operatorname{Conv} G$ which are elements of $[d(G), \alpha]$, or (ii) we arrive at an element γ' of $\operatorname{Conv} G$ such that $d(G) < \gamma' < \alpha$ and $[d(G), \gamma']$ does not contain any atom. In the case (i) for each α_m ($m \in \mathbb{N}$) there exists a sequence (a_n^m) which generates α_m. Next, according to 1.2 there exists a convex ℓ-subgroup C_m of G such that C_m is linearly ordered and (a_n^m) ultimately deals on C_m. If $m(1)$ and $m(2)$ are distinct positive integers, then $\alpha_{m(1)} \neq \alpha_{m(2)}$ and this yields that $C_{m(1)} \cap C_{m(2)} = \{0\}$. For each $n \in \mathbb{N}$ there exists a subsequence (b_n^m) of (a_n^m) such that $0 < b_n^m \in C_m$. In view of 2.2, the element $\alpha' = \bigvee_{m \in \mathbb{N}} \alpha_m$ does exist in $\operatorname{Conv} G$; moreover, according to 2.3 the element α' fails to be principal. Clearly $\alpha' \leqslant \alpha$. □

2.9. Corollary. *Let $\alpha \in \operatorname{Conv} G$ and assume that the interval $[d(G), \alpha]$ of $\operatorname{Conv} G$ is infinite. Then there is $\alpha' \in [d(G), \alpha]$ such that α' is not principal.*

Proof. This is an immediate consequence of 2.4, 2.7 and 2.8. □

Proof of (C). The relation (ii)⇒(i) is a consequence of 1.1. For the relation (i)⇒(ii) cf. 2.9. □

3

We shall now apply the results of the previous sections to obtain some insight into the structure of the partially ordered set $\operatorname{Conv} G$.

Let us denote by F the set of all $\alpha \in \operatorname{Conv} G$ such that the set $[d(G), \alpha]$ is finite.

3.1. Proposition. *Let α and β belong to F. Then $\alpha \vee \beta$ does exist in the partially ordered set $\operatorname{Conv} G$ and $\alpha \vee \beta \in F$.*

Proof. According to (B) there are convex subsets T_1 and T_2 of $\operatorname{Conv} G$ with $d(G) \in T_1 \cap T_2$ such that the direct decompositions

$$\operatorname{Conv} G = [d(G), \alpha] \times T_1, \tag{10}$$

$$\operatorname{Conv} G = [d(G), \beta] \times T_2 \tag{11}$$

hold. Denote

$$[d(G), \alpha] \cap [d(G), \beta] = V_1, \quad [d(G), \alpha] \cap T_2 = V_2,$$
$$T_1 \cap [d(G), \beta] = V_3, \quad T_1 \cap T_2 = V_4.$$

In view of [5], the direct decompositions (10) and (11) have a common refinement

$$\operatorname{Conv} G = V_1 \times V_2 \times V_3 \times V_4.$$

For each $\gamma \in \operatorname{Conv} G$ we denote by $\gamma(V_i)$ the component of γ in V_i, where $i \in \{1, 2, 3, 4\}$.

From the definition of V_1 it follows that $\alpha \wedge \beta$ is the greatest element in V_1. Hence for each $\gamma \in \operatorname{Conv} G$ we have $\gamma(V_1) = \alpha \wedge \beta \wedge \gamma$. Thus, in particular,

$$\alpha(V_1) = \alpha \wedge \beta = \beta(V_1). \tag{12}$$

It is easy to verify that

$$\alpha(V_3) = \beta(V_2) = \alpha(V_4) = \beta(V_4) = d(G). \tag{13}$$

There exists $\delta \in \operatorname{Conv} G$ such that

$$\delta(V_1) = \alpha \wedge \beta, \quad \delta(V_2) = \alpha(V_2),$$
$$\delta(V_3) = \beta(V_3), \quad \delta(V_4) = d(G).$$

The relations (12) and (13) yield that $\delta = \alpha \vee \beta$ is valid in $\operatorname{Conv} G$. Next, the cardinality of the set $[d(G), \delta]$ is the product of the cardinalities of the sets $[d(G), \alpha \wedge \beta], [d(G), \alpha(V_2)], [(d(G), \beta(V_3)]$. Since $\alpha \wedge \beta \leqslant \alpha, \alpha(V_2) \leqslant \alpha$ and $\beta(V_3) \leqslant \beta$, all the elements $\alpha \wedge \beta, \alpha(V_2)$ and $\beta(V_3)$ belong to F. Hence $\operatorname{card}[d(G), \delta]$ is finite. Thus $\delta \in F$, which completes the proof. $\square$

3.2. Corollary. *The set F is a lattice (under the inherited partial order).*

Thus if $\operatorname{Conv} G$ is finite, then $\operatorname{Conv} G = F$ and hence $\operatorname{Conv} G$ is a lattice; therefore (A) yields

3.3. Corollary. (Cf. [2], Theorems (B) and (C).) *If $\operatorname{Conv} G$ is finite and $\operatorname{card} \operatorname{Conv} G > 1$, then $\operatorname{Conv} G$ is a Boolean algebra.*

Let us denote by A the set of all atoms of $\operatorname{Conv} G$.

3.4. Proposition. *Let $A \neq \emptyset$. Then the element $\alpha_0 = \sup A$ does exist in $\operatorname{Conv} G$. Moreover, the interval $[d(G), \alpha_0]$ is a completely distributive complete Boolean algebra.*

Proof. The existence of α_0 is a consequence of [5], Theorem 2.2. Let $d(G) \neq \alpha \in \operatorname{Conv} G$, $\alpha \leqslant \alpha_0$. Further, let $A(\alpha) = \{\alpha_i \in A : \alpha_i \leqslant \alpha\}$. From the fact that the interval $[d(G), \alpha_0]$ is Brouwerian, we obtain that $\alpha = \sup A(\alpha)$. If $A(\alpha) = A$, then $\alpha = \alpha_0$; if $A(\alpha) \neq A$, then the element $\sup(A \setminus A(\alpha))$ is a complement of α in the interval $[d(G), \alpha_0]$. Thus $[d(G), \alpha_0]$ is a Boolean algebra. It is complete according to [1]. Moreover, being atomic, it is completely distributive. □

3.5. Remark. The first assertion of 3.1 (concerning the existence of $\alpha \vee \beta$) can be deduced also from 3.4 and from (A).

For $\emptyset \neq Y \subseteq \operatorname{Conv} G$ put $Y^\delta = \{\alpha \in \operatorname{Conv} G : \alpha \wedge \beta = d(G) \text{ for each } \beta \in Y\}$.

3.6. Lemma. *Assume that the set A is infinite. Then there exists $\alpha \in A^\delta$ such that $\alpha \neq d(G)$.*

Proof. There exists a sequence $(\alpha_m)_{m \in \mathbb{N}}$ such that $\alpha_m \in A$ for each $m \in \mathbb{N}$ and $\alpha_{m(1)} \neq \alpha_{m(2)}$ whenever $m(1)$ and $m(2)$ are distinct positive integers. For each α_m there is a sequence (a_n^m) in G such that this sequence generates α_m and $a_n^m > 0$ for each $n \in \mathbb{N}$. Next, in view of 1.2 there is a convex linearly ordered ℓ-subgroup C_m of G such that (a_n^m) ultimately deals on C_m. Thus for each $m \in \mathbb{N}$ there exists $n(m) \in \mathbb{N}$ such that $a_{n(m)}^m \in C_m$. Consider the sequence $(a_{n(m)}^m)_{m \in \mathbb{N}}$; this sequence is strictly disjoint. Thus there is $\alpha \in \operatorname{Conv} G$ such that α is generated by $(a_{n(m)}^m)_{m \in \mathbb{N}}$. Then clearly $\alpha \neq d(G)$. By applying Lemma 1.2 again we obtain that $\alpha \wedge \beta = d(G)$ for each $\beta \in A$. Therefore $\alpha \in A^\delta$. □

3.7. Proposition. *Put $A_0 = A \cup \{d(G)\}$. The following conditions are equivalent:*

(i) *$\operatorname{Conv} G$ is finite.*
(ii) *$A_0^\delta = \{d(G)\}$.*

P r o o f. Assume that (i) holds. Then in view of 3.3 the relation (ii) is valid. Conversely, suppose that (ii) holds true. By way of contradiction, assume that $\operatorname{Conv} G$ is infinite. We distinguish two cases.

(a) Assume that A_0 is infinite. Let α be as in 3.6. Then $d(G) \neq \alpha \in A^\delta = A_0^\delta$, which is a contradiction.

(b) Assume that A_0 is finite. If $A = \emptyset$, then $A_0^\delta = \operatorname{Conv} G \neq \{d(G)\}$, which is impossible. Let $A \neq \emptyset$ and let α_0 be as in 3.4. Then $[d(G), \alpha_0]$ is a finite Boolean algebra. Hence according to (B) there is a direct decomposition $\operatorname{Conv} G = [d(G), \alpha_0] \times Z$. Thus Z must be infinite and clearly $Z \subseteq A_0^\delta$; in this way we arrive at a contradiction. □

The following result improves Corollary 3.3.

3.8. Proposition. *Let* $\operatorname{card} \operatorname{Conv} G > 1$. *Then the following conditions are equivalent:*

(i) $\operatorname{Conv} G$ *is finite.*

(ii) $\operatorname{Conv} G$ *is an atomic Boolean algebra.*

P r o o f. The implication (i)⇒(ii) is expressed in Corollary 3.3. The relation (ii)⇒(i) follows from 3.7. □

3.9. Proposition. *Let* $\alpha \in \operatorname{Conv} G$, $\alpha \neq d(G)$. *Then the following conditions are equivalent:*

(i) α *is an atom of* $\operatorname{Conv} G$;

(ii) *the interval* $[d(G), \alpha]$ *of* $\operatorname{Conv} G$ *is a chain.*

P r o o f. The implication (i)⇒(ii) is obvious. Let (ii) be valid. First assume that the interval $[d(G), \alpha]$ is finite. Then in view of (A) this interval is a Boolean algebra. Now, because it is a chain, we have $\operatorname{card}[d(G), \alpha] = 2$, hence α is an atom of $\operatorname{Conv} G$. Next let us suppose that the interval $[d(G), \alpha]$ is infinite. Then according to 2.5, 2.7 and 2.8 the convergence α has the property (c), whence $[d(G), \alpha]$ fails to be a chain, which is a contradiction. □

The following questions remain open.

(1) Assume that $\operatorname{Conv} G$ has a greatest element γ and that γ is principal. Must $\operatorname{Conv} G$ be finite?

(2) Let $A \neq \emptyset$. Is the relation $A^{\delta\delta} = [d(G), \sup A]$ always valid?

(3) Let $A \neq \emptyset$. Is $A^{\delta\delta}$ a direct factor of $\operatorname{Conv} G$?

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