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SOME CHARACTERISTICS OF THE EDGE DISTANCE BETWEEN GRAPHS

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1. PRELIMINARIES

A graph $G = (V, E)$ consists of a non-empty finite vertex set V and an edge set E. In this paper we consider undirected graphs without loops and multiple edges. A subgraph H of the graph G is a graph obtained from G by deleting some edges and vertices; notation: $H \subseteq G$. Every edge $x \in E$ can be written in the form $x = uv$, where $u, v \in V$ are vertices connected by x. By $\Delta(G)$ we denote the maximal degree of vertices of the graph G. A graph G is a common subgraph of graphs G_1, G_2 if there exist graphs H_1, H_2 such that $H_1 \subseteq G_1$, $H_2 \subseteq G_2$ and $H_1 \cong G$, $H_2 \cong G$. The maximal common subgraph is the common subgraph which contains the maximal number of edges.

A distance of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined by

$$d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + \big||V_1| - |V_2|\big| \tag{1}$$

where $|E_1|$, $|E_2|$, $|V_1|$, $|V_2|$ are the cardinalities of the edge sets and the vertex sets, respectively, and $|E_{1,2}|$ is the number of edges of the maximal common subgraph $G_{1,2}$ of the graphs G_1 and G_2 (by [1]).

If we identify isomorphic graphs then (1) defines a metric on the set of all (finite) graphs.

Throughout this paper, by $F_{p,q}$ we denote the set of all graphs with p vertices and q edges, $q \geqslant 1$. Further, $\operatorname{diam} F_{p,q} := \max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$. If $\operatorname{diam} F_{p,q} = d(G, H)$ and $c_{p,q}$ is the number of edges of the maximal common subgraph of the graphs G, H then

$$\operatorname{diam} F_{p,q} = 2q - 2c_{p,q}. \tag{2}$$

Obviously, $|E_{1,2}| \geqslant c_{p,q}$ for any $G_1, G_2 \in F_{p,q}$ ($c_{p,q}$is the minimal number of edges of the maximal common subgraph of two graphs from the class $F_{p,q}$).

2. Diameter of a family of graphs

Lemma 1. *For any classes $F_{p,q}$, $F_{p,q+1}$ the following inequalities are satisfied:*

$$c_{p,q} \leqslant c_{p,q+1} \leqslant c_{p,q} + 2.$$

P r o o f. a) First we prove $c_{p,q} \leqslant c_{p,q+1}$. Consider some graphs $G_1, G_2 \in F_{p,q+1}$. Deleting an arbitrary edge from each of these graphs we obtain graphs $G_1', G_2' \in F_{p,q}$. Evidently any common subgraph $G_{1,2}'$ of the graphs G_1', G_2' is also a common subgraph of the graphs G_1, G_2, hence

$$|E_{1,2}| \geqslant |E_{1,2}'| \geqslant c_{p,q}.$$

Since $|E_{1,2}| \geqslant c_{p,q}$, for any $G_1, G_2 \in F_{p,q+1}$ we get $c_{p,q+1} \geqslant c_{p,q}$.

b) We prove that $c_{p,q+1} \leqslant c_{p,q} + 2$. Let $G_1, G_2 \in F_{p,q}$ be graphs such that their maximal common subgraph $G_{1,2}$ satisfies $|E_{1,2}| = c_{p,q}$. To each of the graphs G_1, G_2 add an arbitrary edge. We obtain graphs $G_1', G_2' \in F_{p,q+1}$ with a maximal common subgraph $G_{1,2}'$. Thus, there is a subgraph H_1' of the graph G_1' and a subgraph H_2' of the graph G_2' such that $H_1' \cong G_{1,2}' \cong H_2'$. Obviously, there is at most one edge of the graph H_1' (H_2') not belonging to the graph G_1 (G_2). Hence we have

$$|E_{1,2}'| \leqslant c_{p,q} + 2,$$

which implies

$$c_{p,q+1} \leqslant c_{p,q} + 2.$$

□

Remark. Later on we will show that the inequalities in Lemma 1 cannot be strengthened.

Let G be an arbitrary graph from $F_{p,q}$ and let $\bar{q}$ denote the number of edges of the graph $\overline{G}$ complementary to G. Obviously,

$$\bar{q} = \frac{p(p-1)}{2} - q.$$

In this paper we will always denote by $\bar{q}$ the number of edges of the complementary graph of any graph with q edges and p vertices.

Lemma 2. *For any p, q ($q \geqslant 1$),*

$$c_{p,q+1} = c_{p,q} + 2 \qquad \textit{iff} \qquad c_{p,\overline{q+1}} = c_{p,\overline{q}}.$$

Proof. Let $c_{p,q+1} = c_{p,q} + 2$. Using (2), Theorem 5 from [4] and again (2), we get

$$2q - 2c_{p,q} = \operatorname{diam} F_{p,q} = \operatorname{diam} F_{p,\overline{q}} = 2\overline{q} - 2c_{p,\overline{q}},$$

i.e., $q - c_{p,q} = \overline{q} - c_{p,\overline{q}}$. Further, by (2) we have

$$\operatorname{diam} F_{p,q+1} = 2(q+1) - 2c_{p,q+1} = 2q - 2c_{p,q} - 2,$$

$$\operatorname{diam} F_{p,\overline{q+1}} = 2(\overline{q+1}) - 2c_{p,\overline{q+1}} = 2\overline{q} - 2c_{p,\overline{q+1}} - 2.$$

Since $\operatorname{diam} F_{p,q+1} = \operatorname{diam} F_{p,\overline{q+1}}$, we get $q - c_{p,q} = \overline{q} - c_{p,\overline{q+1}}$, hence $c_{p,\overline{q}} = c_{p,\overline{q+1}}$. The converse statement is now obvious. □

Theorem 3. *For any class $F_{p,q}$,*

$$\operatorname{diam} F_{p,q} = 2q - 4 \qquad \textit{iff} \qquad \frac{1}{2}p < q \leqslant p - 1.$$

Proof. If $\frac{1}{2}p < q \leqslant p - 1$ then $\operatorname{diam} F_{p,q} = 2q - 4$ (by [4, Theorem 3]). To prove the converse statement assume first that $q \geqslant p$. We will show that $|E_{1,2}| \geqslant 3$ for any graphs $G_1, G_2 \in F_{p,q}$, $p \geqslant 3$.

We distinguish two cases:

a) If $\Delta(G_1) \geqslant 3$ and $\Delta(G_2) \geqslant 3$ then both G_1 and G_2 contain a subgraph isomorphic to the graph in Figure 1 (in the sequel we briefly say that they contain the graph in Figure 1).

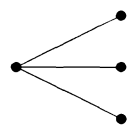


Fig. 1

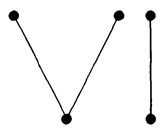


Fig. 2

b) Let $\Delta(G_1) = 2$. Then $q = p$ and G_1 is a regular graph of degree 2. If $p = 3$ then $G_1 \cong G_2$; if $p = 4$ or $p = 5$ then both the graphs contain a path of length three; if $p = 6$ then both the graphs G_1 and G_2 contain either the graph in Figure 2 or a path of length 3 or K_3. If $p \geqslant 7$ then they contain the graph in Figure 2.

It follows that $c_{p,q} \geqslant 3$, thus $\operatorname{diam} F_{p,q} \leqslant 2q - 6$. To complete the proof note that if $q \leqslant \frac{1}{2}p$ then by [4; Theorem 2], $\operatorname{diam} F_{p,q} = 2q - 2$. □

Corollary 4. *If* $3 \leqslant p \leqslant q$ *then* $\operatorname{diam} F_{p,q} \leqslant 2q - 6$.

Theorem 5. $\operatorname{diam} F_{p,p} = 2p - 6$ *for any* $p \geqslant 3$.

Proof. By Corollary 4 it suffices to find two graphs from the class $F_{p,p}$ whose maximal common subgraph has only 3 edges. Such graphs are depicted in Figure 3 (G_1 is a circle). □

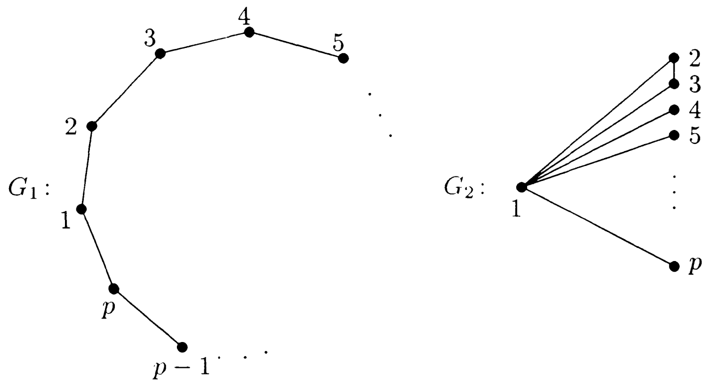


Fig. 3

Lemma 6. *Let* $q \geqslant p$ *and* $p \in \{6, 7, 8\}$. *If* $G \in F_{p,q}$ *and* $\Delta(G) \geqslant 4$ *then* G *contains the subgraph in Figure 4.*

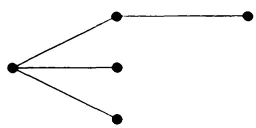


Fig. 4

P r o o f. Let v be a vertex of G of degree $\Delta(G) = k$, let $v_1, \ldots, v_k$ be vertices adjacent to v and $w_1, \ldots, w_{p-k-1}$ the other vertices of G (if they exist). If G contains no graph isomorphic to the graph in Figure 4 then it contains neither an edge of type v_iv_j nor an edge of type v_iw_j. Therefore the number of all edges of G is at most

$$s = k + \frac{1}{2}(p - k - 1)(p - k - 2). \tag{3}$$

For the values from the hypothesis we get $s < p$. □

Theorem 7. *If* $5 \leqslant p \leqslant 9$, *then* $\operatorname{diam} F_{p,p+1} = 2p - 6$.

P r o o f. 1) Case $p = 5$. By [4; Theorem 4] and Theorem 3 we have

$$\operatorname{diam} F_{5,6} = \operatorname{diam} F_{5,4} = 2.4 - 4 = 2.5 - 6.$$

In the remaining cases we first show that $|E_{1,2}| \geqslant 4$ for any graphs $G_1, G_2 \in F_{p,p+1}$ (i.e., $\operatorname{diam} F_{p,p+1} \leqslant 2p - 6$).

2) Case $p = 6$. If a graph $G \in F_{6,7}$ does not contain the graph in Fig. 4 then by Lemma 6, $\Delta(G) = 3$. Let its vertex v have degree 3, let v_1, v_2, v_3 be vertices adjacent to v and let w_1, w_2 be the remaining vertices. Since G contains no edge of type v_iw_j, G is isomorphic to the graph in Fig. 5.

Let $G_1, G_2 \in F_{6,7}$. If both the graphs contain the graph in Fig. 4 or are isomorphic to the graph in Fig. 5, then $|E_{1,2}| \geqslant 4$. Let G_1 contain the graph in Fig. 4 and let G_2 be isomorphic to the graph in Fig. 5. The graph G_1 contains three other edges and one can check that G_1 contains at least one of the graphs in Figs. 6, 7, 8. Hence again $|E_{1,2}| \geqslant 4$.

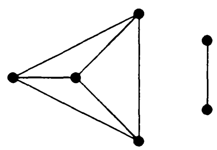


Fig. 5

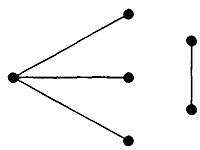


Fig. 6

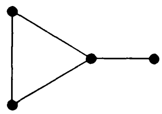


Fig. 7

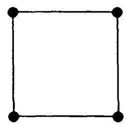


Fig. 8

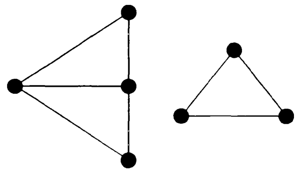


Fig. 9

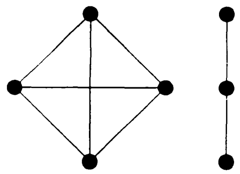


Fig. 10

3) Case $p = 7$. Similarly as in the previous case one can show that if a graph $G \in F_{7,8}$ does not contain the graph in Fig. 4 then G is isomorphic either to the graph in Fig. 9 or to the graph in Fig. 10.

If G_1 is the graph in Fig. 9 and G_2 is the graph in Fig 10 then evidently $|E_{1,2}| \geqslant 4$. Let G_1 be one of the graphs in Figs. 9, 10 and let G_2 contain the graph in Fig. 4. Since G_2 contains other four edges, G_2 again contains at least one of the graphs in Figs. 6, 7, 8. This yields $|E_{1,2}| \geqslant 4$.

4) Case $p = 8$. Let $G \in F_{8,9}$, $\Delta(G) \geqslant 4$. Obviously, G contains the graph in Fig. 6 or the graph in Fig. 7. By Lemma 6, G contains a subgraph isomorphic to the graph in Fig. 4, too.

Let $G \in F_{8,9}$ and $\Delta(G) = 3$. Let v be a vertex of degree 3, let v_1, v_2, v_3 be vertices adjacent to v and let the remaining vertices be w_1, w_2, w_3, w_4. If G does not contain the graph in Fig. 4 then at least two of the remaining six edges are of type v_iv_j or at least five of them are of type w_iw_j. In both cases G contains the graphs in Figures 7 and 8. It is obvious that G contains also the graph in Fig. 6. If the graph G contains the graph in Fig. 4 then it contains other five edges and one can verify that it contains at least one of the graphs in Figs. 6, 7, 8.

Let $G_1, G_2 \in F_{8,9}$; it follows from the previous part that G_1 and G_2 contain at least one of the graphs in Figs. 4, 6, 7, 8. Therefore $|E_{1,2}| \geqslant 4$.

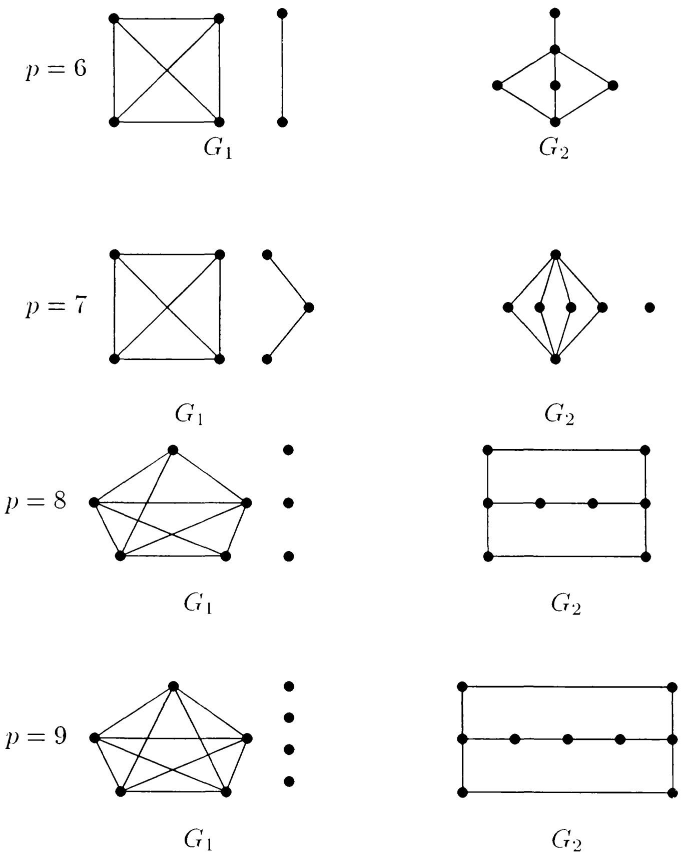
5) Case $p = 9$. Let $\Delta(G) \geqslant 4$ and let v be a vertex of degree greater than 3. Let the vertices adjacent to v be $v_1, \dots, v_k$. If G does not contain the graph in Fig. 6 then k=4 and the graph induced by the vertices v, v_1, v_2, v_3, v_4 is isomorphic to K_5.

Let $\Delta(G) = 3$ and let the vertex v have the adjacent vertices v_1, v_2, v_3. We denote the remaining vertices by $w_1, \dots, w_5$. Note that if the graph G did not contain the graph in Fig. 6 then it would contain at most nine edges (as v_1, v_2, v_3 have degree at most 3). So every graph $G \in F_{9,10}$ with $\Delta(G) = 3$ contains the graph in Fig. 6.

Let $G_1, G_2 \in F_{9,10}$. If $\Delta(G_1) \geqslant 4$ and $\Delta(G_2) \geqslant 4$ or if $\Delta(G_1) = \Delta(G_2) = 3$ then obviously $|E_{1,2}| \geqslant 4$.

Let $\Delta(G_1) \geqslant 4$ and $\Delta(G_2) = 3$ and let G_1 not contain the graph in Fig. 6. Then G_1 consists of K_5 and four isolated vertices. Since G_2 contains ten edges, G_2 evidently contains a subgraph with five vertices and at least four edges. Hence again $|E_{1,2}| \geqslant 4$.

Finally, it suffices to show that in each of the cases $p \in \{6, 7, 8, 9\}$, the equality $|E_{1,2}| = 4$ is possible. This is the case of the following graphs: $\square$



Theorem 8. *If $p \geqslant 16$ then* $\operatorname{diam} F_{p,p+1} = 2p - 8$.

Proof. Let $G \in F_{p,p+1}$ and let v be a vertex of degree $\Delta(G)$. We denote the vertices adjacent to v by $v_1, \ldots, v_k$ and the remaining vertices (if they exist) by $w_1, \ldots, w_{p-k-1}$.

1. Let $\Delta(G) = 3$. Then the subgraph H induced on the set $V - \{v, v_1, v_2, v_3\}$ has at least $p - 8$ edges. If $p - 8 > \frac{p-4}{2}$, i.e., $p > 12$, then H has a vertex of degree 2.

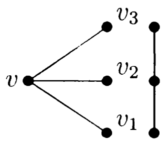


Fig. 11

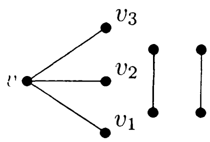


Fig. 12

This yields that G contains the graph in Fig. 11. If $p \geqslant 12$ then the subgraph H has at least four edges, hence G contains also the graph in Fig. 12.

2. a) If $\Delta(G) = 4$ and $p \geqslant 16$ then G contains a graph isomorphic to the graph in Fig. 13, since at most 16 edges can be incident with at least one of the vertices v, v_1, v_2, v_3, v_4. Obviously, the graph G contains also a graph isomorphic to the graph in Fig. 11 or 12.

b) If $\Delta(G) = 5$ then the subgraph induced by the set $\{v, v_1, \ldots, v_5\}$ contains at most 15 edges. Since G has at least 17 edges ($p \geqslant 16$) it obviously contains a subgraph isomorphic to the graph in Fig. 11 or 12 and also a subgraph isomorphic to the graph in Fig. 13.

c) If $\Delta(G) \geqslant 6$ then G evidently contains the graph in Fig. 13 and also (if $p \geqslant 9$) the graph in Fig. 11 or 12.

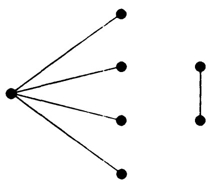


Fig. 13

3. It follows from the previous discussion that if $G_1, G_2 \in F_{p,p+1}$, $p \geqslant 16$, then

$$|E_{1,2}| \geqslant 5, \quad i.e., \quad \operatorname{diam} F_{p,p+1} \leqslant 2p - 8.$$

For the graphs in Fig. 14 we have $|E_{1,2}| = 5$, therefore $\operatorname{diam} F_{p,p+1} = 2p - 8$. $\square$

Remark. By Theorem 5, $\operatorname{diam} F_{p,p} = 2p - 6$ if $p \geqslant 3$ and by Theorem 8, $\operatorname{diam} F_{p,p+1} = 2p - 8$ if $p \geqslant 16$. Hence the answer to Problem 5 from [4] is negative, i.e.

$$q_1 \leqslant q_2 \leqslant \frac{1}{4}p(p-1) \text{ does not imply } \operatorname{diam} F_{p,q_1} \leqslant F_{p,q_2}.$$

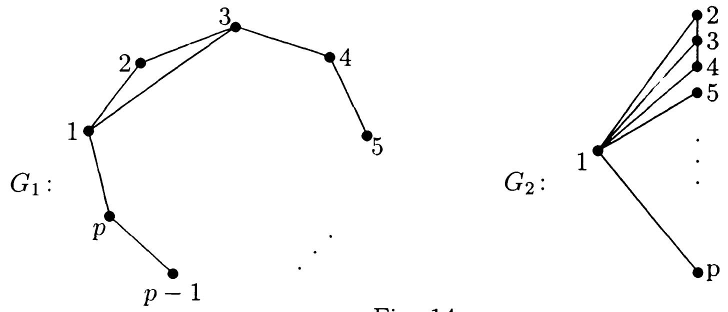
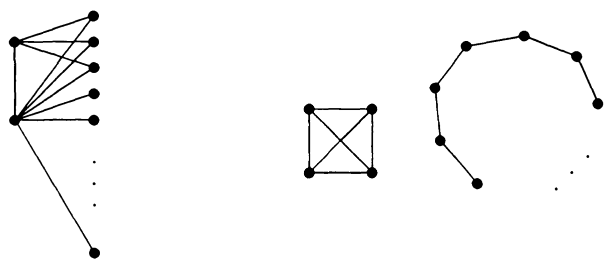


Fig. 14

Theorem 9. $\operatorname{diam} F_{p,p+2} = 2p - 6$ *if* $p \geqslant 16$.

Proof. By Theorem 8 and Lemma 1, it suffices to find two graphs $G_1, G_2 \in F_{p,p+2}$ such that $|E_{1,2}| = 5$. These graphs are depicted in the following figures: $\square$



G_1 $\qquad$ G_2

Theorem 10. *If* $G_1 \in F_{p_1,q_1}$ *and* $G_2 \in F_{p_2,q_2}$ *then*

$$d(G_1, G_2) = q_1 + q_2 + |p_1 - p_2| - 2$$

if and only if the graphs G_1, G_2 *satisfy one of the following two conditions:*

a) *One of the graphs* G_1, G_2 *is the graph in Fig.* 15 *and the other graph is arbitrary with at least one edge.*

b) *One of the graphs* G_1, G_2 *is the graph in Fig.* 16 *having at least* 2 *components* K_2 *and the other graph is either the graph in Fig.* 17 *or the graph in Fig.* 18 *with at least two edges.*

Proof. It is sufficient to take into account that each of the graphs G_1, G_2 must have at least one edge and at least one of the graphs G_1, G_2 cannot contain any vertex of degree at least 2. $\square$

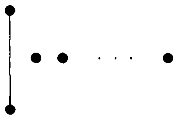


Fig. 15

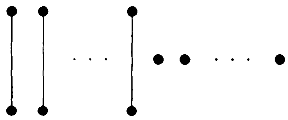


Fig. 16

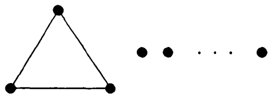


Fig. 17

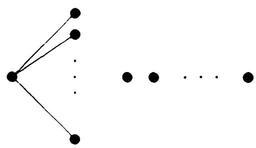


Fig. 18

Remark. Note that Theorem 10 gives the answer to Problems 2 and 6a from [4].

Lemma 11. $\operatorname{diam}(F_{5,3} \cup F_{5,7}) = \operatorname{diam} F_{5,3} + \operatorname{diam} F_{5,7}$.

Proof. By Theorems 5 and 3 from [4] we get

$$\operatorname{diam} F_{5,3} + \operatorname{diam} F_{5,7} = 2 \cdot \operatorname{diam} F_{5,3} = 2 \cdot (2 \cdot 3 - 4) = 4.$$

Now we show that $\operatorname{diam}(F_{5,3} \cup F_{5,7}) = 4$. This follows from the fact that each graph from $F_{5,3}$ is a subgraph of a graph from $F_{5,7}$, which is a consequence of the following facts. Firstly, the class $F_{5,3}$ contains the following four graphs.

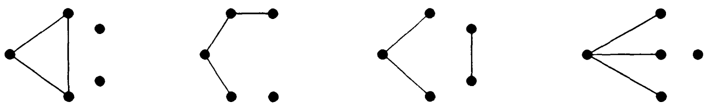


Fig. 19

Secondly, if a graph $G \in F_{5,7}$ has a vertex of degree four then it contains the graph in Fig. 20 and if it has no vertex of degree four then it is easy to show that it is isomorphic to the graph in Fig. 21. The graphs in Fig. 19 are subgraphs of each of the two graphs in Figs. 20, 21. □

Remark. Lemma 11 gives a partial answer to Problem 4 in [4].

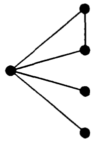


Fig. 20

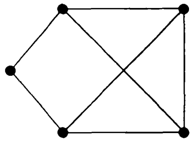


Fig. 21

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