Czechoslovak Mathematical Journal

Roman R. Zapatrin

A characterization of polarities whose polar lattice is orthomodular

Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 4, 625–627

Persistent URL: http://dml.cz/dmlcz/127324

Czechoslovak Mathematical Journal, 46 (121) 1996, Praha

A CHARACTERIZATION OF POLARITIES WHOSE POLAR LATTICE IS ORTHOMODULAR

ROMAN R. ZAPATRIN, St. Petersburg

(Received May 6, 1994)

In this note the pairs (X, δ) where X is a set and δ is a symmetric irreflexive relation on X are studied. The relation δ endows X with the closure operation **Cl** in the following way. For any $A \subseteq X$ define its *polar* $A^\delta \subseteq X$ as

$$A^\delta = \{x \in X \mid \forall a \in A \quad a\delta x\}.$$

Define the operation **Cl** on subsets of X as

$$\mathbf{Cl}A = A^{\delta\delta}$$

and call it the closure *induced* on X by the relation δ.

Theorem 1. *The operation* **Cl** *is a closure on subsets of* X, *namely*

$$\begin{aligned} &A \subseteq \mathbf{Cl}A, \\ &\mathbf{Cl}\mathbf{Cl}A = \mathbf{Cl}A, \\ &A \subseteq B \Rightarrow \mathbf{Cl}A \subseteq \mathbf{Cl}B. \end{aligned}$$

Proof. See [1], Section V.7. □

The collection of all closed with respect to **Cl** subsets of X always forms the complete orthocomplemented lattice $\Gamma_\delta(X)$ with set intersections serving as meets and the operation $A \mapsto A^\delta$ as orthocomplements (see also [1]).

The necessary and sufficient condition for the relation δ to make the lattice $\Gamma_\delta(X)$ Boolean was established in [2]. In this note the necessary and sufficient condition for δ to provide the *orthomodularity* of $\Gamma_\delta(X)$ is given.

Definition. An orthocomplemented lattice $\mathcal{L}$ is called *orthomodular* if for any $a, b \in \mathcal{L}$

$$b \leqslant a \quad \Rightarrow \quad a = b \vee (b' \wedge a)$$

or, in the form convenient for further purposes

$$(1) \qquad b \leqslant a \quad \Rightarrow \quad b = a \wedge (a \wedge b')' .$$

To set up the orthomodularity condition for $\Gamma_\delta(X)$, some further definitions are to be introduced. Let $A \subseteq X$ be a closed subset of X, $A \in \Gamma_\delta(X)$. Consider the pair (A, δ_A) where δ_A is the restriction of the relation δ on A. The pair (A, δ_A) can be, in turn, considered polarity: for any $B \subseteq A$ define

$$B^{\delta_A} = \{a \in A \mid \forall b \in B \quad a\delta b\} = A \cap B^\delta .$$

Define the *induced closure* $\mathbf{Cl}_A$ on subsets of A as

$$(2) \qquad \mathbf{Cl}_A B = B^{\delta_A \delta_A} .$$

Since A is closed with respect to the initial closure $\mathbf{Cl}$ on X, it possesses one more closure operation: the restriction of $\mathbf{Cl}$ onto A, let us call it the *relative closure*. The following lemma evidently holds:

Lemma 2. (i) *Any* $\mathbf{Cl}_A$*-closed subset of* A *is* $\mathbf{Cl}$*-closed.*
(ii) *For any* $B \subseteq A$ *we have* $\mathbf{Cl} B \subseteq \mathbf{Cl}_A B$.

Now everything is ready to prove the main result of this note.

Theorem 3. *The lattice* $\Gamma_\delta(X)$ *is othomodular if and only if on each closed subset* $A \in \Gamma_\delta(X)$ *the operations of relative and induced closure coincide:*

$$(3) \qquad \mathbf{Cl}_A = \mathbf{Cl}\,|_A .$$

P r o o f. Let A be a $\mathbf{Cl}$-closed subset of X, and $B \subseteq A$ an arbitrary subset of A. By virtue of Lemma 2(i) it suffices to verify (3) only for $\mathbf{Cl}$-closed subsets of A. Thus (3) reads

$$(4) \qquad \forall A, B \in \Gamma_\delta(X) \qquad B \subseteq A \Rightarrow B = \mathbf{Cl}_A B .$$

Now note that by definition (2)

$$\mathbf{Cl}_A B = B^{\delta_A \delta_A} = A \cap (A \cap B^\delta)^\delta$$

since set intersections are meets and $(\cdot)^\delta$ is the orthocomplement in $\Gamma_\delta(X)$. Therefore (4) is exactly the orthomodularity condition (1). □

Acknowledgment. The author is indebted to Prof. F. Šik for sending the reprints of his papers.

References

[1] *G. Birkhoff*: Lattice Theory. Providence, Rhode Island, 1967.

[2] *F. Šik*: A characterization of polarities whose lattice of polars is Boolean. Czechoslovak Mathematical Journal *98 (106)* (1981), 31.

Author's address: Department of Mathematics, SPb UEF, Griboyedova 30/32, 191023, St-Petersburg, Russia.