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ON UNIFORM PARACOMPACTNESS

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0. Introduction

The idea of paracompactness is one of the most important in General Topology. There have been some attempts in defining paracompactness of uniform spaces ($\equiv$ uniform paracompactness). In connection with this we refer to the articles of M. D. Rice [4], Z. Frolík [3] and A. A. Borubaev [1]. In this article some other possible definitions of uniform paracompactness are suggested and the interrelations between these definitions and the already existing ones are investigated.

Throughout the paper by a space we understand a topological space, by a continuous map—a continuous map between spaces. For the system α of subsets of the set X and $x \in X$, $S \subset X$ we have:

$$St(x,\alpha) = \{A \in \alpha \colon x \in A\}, \quad \alpha(x) = \bigcup St(x,\alpha),$$
$$St(S,\alpha) = \{A \in \alpha \colon S \cap A \neq \emptyset\}, \quad \alpha(S) = \bigcup St(S,\alpha).$$

For the covers α and β of the set X, the symbols $\beta > \alpha$ and $\beta\star{>}\alpha$ mean respectively that the cover β is a refinement of the cover α and that $\{\beta(B) \colon B \in \beta\} > \alpha$. For the system α of subsets of the set X and an infinite cardinal number τ let $\alpha_\tau = \{\bigcup \beta \colon \beta \subset \alpha, |\beta| < \tau\}$. The system α will be called τ-additive if $\alpha_\tau = \alpha$ and ω- (respectively ω_1-) additive systems will be called finitely- (respectively countably-) additive.

Throughout the paper by a (pseudo)uniformity we understand a (pseudo)uniformity defined with the help of covers. For the (pseudo)uniformity $\mathscr{U}$ by τ_u we understand the topology generated by this (pseudo)uniformity. The system λ of subsets of the uniform space $(X,\mathscr{U})$ is called uniformly locally finite (in short, $\mathscr{U}$-locally finite), respectively uniformly discrete (in short, $\mathscr{U}$-discrete), if there exists $\alpha \in \mathscr{U}$ such that $|St(A,\lambda)| < \omega$, respectively $|St(A,\lambda)| \leqslant 1$, for every $A \in \alpha$.

We can now formulate the definitions of uniform paracompactness belonging to Rice, Frolík and Borubaev.

Definition 0.1. (M. D. Rice [4]) The uniform space $(X, \mathscr{U})$ is paracompact (in this situation we will use the term "R-paracompact"), if every open cover of $(X, \mathscr{U})$ admits a $\mathscr{U}$-locally finite open refinement.

Rice showed [4], that all R-paracompact uniform spaces are complete. From this follows that all metrizable, by a non complete metric, uniform spaces are not R-paracompact. Hence the class of R-paracompact uniform spaces turns out to be, in our opinion, too narrow.

Definition 0.2. (A. A. Borubaev [1]) The uniform space $(X, \mathscr{U})$ is paracompact (in this situation we will use the term "B-paracompact"), if for every finitely-additive open cover λ of $(X, \mathscr{U})$ there exists a sequence $\alpha_n \in \mathscr{U}$, $n \in \mathbb{N}$ such that

$$(*) \qquad \forall\, x \in X\ \exists\, n \in \mathbb{N} \text{ and } L \in \lambda \text{ with the property } \alpha_n(x) \subset L.$$

Note 0.3. It is clear that without loss of generality one can assume that in Definition 0.2 one can add,

$$(**) \qquad \alpha_{n+1} \star\!> \alpha_n, \quad \forall n \in \mathbb{N}.$$

Although the requirement of finite-additivity of the cover λ in definition 0.2 does not seem very natural, B-paracompactness has a series of good properties. It is worthwhile mentioning that all metrizable and all R-paracompact spaces are B-paracompact [1] and, consequently, the class (B) of all B-paracompact spaces is essentially wider than the class (R) of all R-paracompact spaces.

Definition 0.4. (Z. Frolík [3]) The uniform space $(X, \mathscr{U})$ is paracompact (in this situation we will use the term "F-paracompact"), if every open cover of $(X, \mathscr{U})$ admits a σ-$\mathscr{U}$-discrete (i.e. it is decomposed into a union of a countable number of $\mathscr{U}$-discrete subsystems) open refinement.

The class (F) of F-paracompact uniform spaces is essentially wider than class (B), as it is shown underneath. (In [1] p. 81 is mentioned that the classes (R), (B) and (F) are pairwise distinct.)

1. P-PARACOMPACTNESS

It is clear that the starting point for the definition of R-paracompact uniform spaces is the generally accepted definition of paracompact topological spaces. With this in mind any other criterion of paracompactness in topological spaces will do. This is seen, for example, in the definition of F-paracompactness. We are interested in the following criterion: a T_1-space X is paracompact if and only if every open cover of X is normal, i.e. for every open cover λ of X there exists a series of open covers α_n, such that $\alpha_1 > \lambda$ and $\alpha_{n+1} \star\!> \alpha_n$, $n \in \mathbb{N}$. (It is clear that in this criterion λ can be changed to λ_ω and with this point of view one looks at definition 0.2 and note 0.3.)

Definition 1.1. (B. A. Pasynkov) The (pseudo)uniform space $(X, \mathscr{U})$ is paracompact (in this situation we will use the term "P-paracompact"), if for every open cover λ of $(X, \mathscr{U})$ there exists a sequence $\alpha_n \in \mathscr{U}$, $n \in \mathbb{N}$, with property $(*)$.

Definition 1.2. The cover λ of a (pseudo)uniform space $(X, \mathscr{U})$ will be called weakly uniform, if there exists a sequence $\alpha_n \in \mathscr{U}$, $n \in \mathbb{N}$, such that properties $(*)$ and $(**)$ hold.

Note 1.3. It is clear that the (pseudo)uniform space $(X, \mathscr{U})$ is P-paracompact if and only if every open cover of $(X, \mathscr{U})$ is weakly uniform. Analogously, the (pseudo)uniform space $(X, \mathscr{U})$ is B-paracompact if and only if every finitely-additive open cover of $(X, \mathscr{U})$ is weakly uniform.

Proposition 1.4. *Every metrizable uniform space $(X, \mathscr{U})$ is P-paracompact.*

Proof. If $\{\alpha_n : n \in \mathbb{N}\}$ is a countable base for the uniformity $\mathscr{U}$ on X, then the sequence α_n, $n \in \mathbb{N}$, has property $(*)$ for every open cover λ of the space $(X, \mathscr{U})$. □

The following proposition is evident.

Proposition 1.5. *For a paracompactum X and the universal uniformity $\mathscr{U}$ on X, the uniform space $(X, \mathscr{U})$ is P-paracompact.*

Remember that the continuous map $f\colon X \to Y$ is called a λ-map for the open cover λ of the space X, if for every point $y \in Y$ there exists a neighbourhood Oy of y and $L \in \lambda$ such that $f^{-1}Oy \subset L$. It is known that a T_2-space is paracompact if and only if for every open cover λ of X there exists a λ-map of X onto a metrizable space. It happens that an analogous fact is true for P-paracompact uniform spaces.

Theorem 1.6. *The uniform space $(X, \mathscr{U})$ is P-paracompact if and only if for every open cover λ of $(X, \mathscr{U})$ there exists a uniformly continuous λ-map of $(X, \mathscr{U})$ onto a metrizable uniform space.*

Proof. Let $(X, \mathscr{U})$ be P-paracompact and λ its open cover. Then λ is weakly uniform and hence there exists a sequence $\alpha_n \in \mathscr{U}$, $n \in \mathbb{N}$, with properties $(*)$ and $(**)$. As for any normal sequence of uniform covers there exists a pseudometric ϱ on X such that

$$(\#)\qquad \alpha_{n+1}(x) \subset \left\{ y \in X : \varrho(x, y) < \frac{1}{2^{n+1}} \right\} \subset \alpha_n(x), \quad x \in X, \quad n \in \mathbb{N},$$

(see, for example, [2] or [1]).

For $x, y \in X$ let xEy if and only if $\varrho(x,y) = 0$. Let X_ϱ be the factor set of the set X relative to the equivalence relation E and $f\colon X \to X_\varrho$ the natural map that maps each point of X to its corresponding class in X_ϱ. Then $d(s,t) = \varrho(f^{-1}s, f^{-1}t)$, $s, t \in X_\varrho$, is a metric on X_ϱ. If $\mathscr{U}_d$ is the uniformity on X_ϱ induced from the metric d, then by (#) the map $f\colon (X, \mathscr{U}) \to (X_\varrho, \mathscr{U}_d)$ is uniformly continuous. From properties (#) and ($*$) follows that f is a λ-map.

The second part of the proof follows from the following lemma. □

Lemma 1.7. *If for every open cover λ of a uniform space $(X, \mathscr{U})$ there exists a uniformly continuous λ-map of $(X, \mathscr{U})$ onto a P-paracompact uniform space then the space $(X, \mathscr{U})$ is also P-paracompact.*

P r o o f. Let λ be an open cover of $(X, \mathscr{U})$ and f a uniformly continuous λ-map of $(X, \mathscr{U})$ onto a P-paracompact uniform space $(Y_\lambda, \mathscr{U}_\lambda)$. For every point $y \in Y$ there exists a neighbourhood Oy, the inverse image of which lies in some element of the cover λ. There exists a sequence $\beta_n \in \mathscr{U}_\lambda$, $n \in \mathbb{N}$, for which property ($*$) (changing X to Y_λ and λ to $\mu = \{Oy\colon y \in Y_\lambda\}$) holds. Then it is easy to see that the sequence $\alpha_n = f^{-1}\beta_n \in \mathscr{U}$, $n \in \mathbb{N}$, has property ($*$) for λ. □

From theorem 1.6 we get

Proposition 1.8. *If $(X, \mathscr{U})$ is a P-paracompact uniform space, then the space (X, τ_u) is paracompact.*

P r o o f. If $f\colon X \to Y$ is a λ-map for the open cover λ of the space X and the space Y is metrizable (or even paracompact), then there exists an open cover μ in Y such that $f^{-1}\mu > \lambda$. If ν is a locally finite open refinement of μ then the cover $f^{-1}\nu$ is open, locally finite and is a refinement of λ. □

Let us now investigate the interrelations between uniform P-paracompactness and uniform B- and R-paracompactness.

It is clear that every P-paracompact uniform space is B-paracompact.

Theorem 1.9. *Let the uniform space $(X_2, \mathscr{U}_2)$ have an uncountable uniform weight τ, and the discrete uniform space $(X_1, \mathscr{U}_1)$ have cardinality $\geqslant \tau$. Then the product $(X = X_1 \times X_2, \mathscr{U} = \mathscr{U}_1 \times \mathscr{U}_2)$ of these uniform spaces is not P-paracompact.*

P r o o f. Let the system of open uniform covers $\alpha(i)$, $i \in I$, of the space $(X_2, \mathscr{U}_2)$ be a base of cardinality τ. We decompose the set X_1 in a union of disjoint non-empty sets $X_1(i)$, $i \in I$.

Let $\lambda(i) = \{\{x_1\} \times A\colon x_1 \in X_1(i),\ A \in \alpha(i)\}$ and $\lambda = \bigcup\{\lambda(i)\colon i \in I\}$. Obviously, λ is an open cover of the space $(X, \mathscr{U})$. Take any sequence of uniform covers $\beta_n =$

$\{\{x_1\} \times A\colon x_1 \in X_1,\ A \in \alpha_n\}$, where $\alpha_n \in \mathscr{U}_2$, $n \in \mathbb{N}$, of the space $(X, \mathscr{U})$. Since the weight of the uniformity $\mathscr{U}_2$ is uncountable, there exists a point $x_0 \in X_2$ and its neighbourhood O such that $\alpha_n(x_0) \setminus O \neq \emptyset$ for all $n \in \mathbb{N}$. On the other hand there exists an index $j \in I$ such that $(\alpha(j))(x_0) \subset O$. Hence for $x_1 \in X_1(j)$ and all $n \in \mathbb{N}$ we have

$$\begin{aligned}\beta_n((x_1, x_0)) \setminus \lambda((x_1, x_0)) &= (\{x_1\} \times \alpha_n(x_0)) \setminus (\{x_1\} \times (\alpha(j))(x_0)) \\ &\supset \{x_1\} \times (\alpha_n(x_0) \setminus O) \neq \emptyset.\end{aligned}$$

□

Remember that the uniform space $(X, \mathscr{U})$ is uniformly locally compact, if there exists $\alpha \in \mathscr{U}$ such that $cl(A)$ is compact for every $A \in \alpha$. In [4] Rice showed that all uniformly locally compact spaces are R-paracompact. From theorem 1.9 we get

Corollary 1.10. *The product of a compact uniform space of uncountable uniform weight τ (for example, the Tychonoff cube I^τ or the Cantor cube D^τ of weight τ) and a discrete uniform space of cardinality $\geqslant \tau$ is uniformly locally compact (and so is R-paracompact and B-paracompact) but not P-paracompact.*

Hence, the class (B) of all B-paracompact uniform spaces is evidently larger than the class (P) of all P-paracompact uniform spaces. Also, from proposition 1.4 and corollary 1.10 we get the following relation $(R) \setminus (P) \neq \emptyset \neq (P) \setminus (R)$. Corollary 1.10 shows that P-paracompactness (like R-paracompactness) narrows excessively the class of "uniform paracompact" spaces and so, in our point of view, cannot be a full value uniform equivalent to topological paracompactness.

From corollary 1.10 follows:

Corollary 1.11. *The uniform space of a topological group, which is the product of τ copies of the two element discrete group, $\tau > \omega$, and a discrete group of cardinality $\geqslant \tau$, is uniformly locally compact but not P-paracompact.*

Let us now see what one needs to add to B-paracompactness to get P-paracompactness.

Definition 1.12. The open cover λ of the space X will be called a τ-decomposition, $\tau \geqslant \omega$, of the open cover μ of this space, if for every $M \in \mu$ there exists a system $\lambda_M \subset \lambda$ such that $M = \bigcup \lambda_M$ and $|\lambda_M| < \tau$. For $\tau = \omega$ (respectively $\tau = \omega_1$) a τ-decomposition of an open cover will be called its finite (respectively countable) decomposition.

Definition 1.13. A uniform space $(X, \mathscr{U})$ will be called τ-decomposable (respectively strongly τ-decomposable), $\tau \geqslant \omega$, if every τ-decomposition of every open uniform (respectively open weakly uniform) cover of this space is weakly uniform. A (strongly) τ-decomposable uniform space with $\tau = \omega$, respectively $\tau = \omega_1$, will be called (strongly) finitely-, respectively (strongly) countably-, decomposable.

Theorem 1.14. *The uniform space $(X, \mathscr{U})$ is P-paracompact if and only if it is B-paracompact and finitely-decomposable.*

Proof. Let $(X, \mathscr{U})$ be finitely-decomposable and B-paracompact. Take any open cover λ of the space $(X, \mathscr{U})$. The cover λ_ω (see Section 0) is finitely-additive and so there exists a sequence of open covers $\alpha_n \in \mathscr{U}$, $n \in \mathbb{N}$, with the property $(*)$. For every $M \in \lambda_\omega$ we fix $\mu(M) \subset \lambda$ such that $|\mu(M)| < \omega$ and $\bigcup \mu(M) = M$. For every $n \in \mathbb{N}$ we put $\alpha_n^* = \{A \in \alpha_n \colon A$ does not lie in any element of $\lambda_\omega\}$. For every $A \in \alpha_n \setminus \alpha_n^*$ we fix $M(A) \in \lambda_\omega$ such that $A \subset M(A)$ and put $\beta_n(A) = \{A \cap L \colon L \in \mu(M(A))\}$. Finally let $\beta_n = \alpha_n^* \cup \bigcup\{\beta_n(A) \colon A \in \alpha_n \setminus \alpha_n^*\}$, $n \in \mathbb{N}$.

Obviously, β_n is a finite decomposition of the uniform cover α_n. By hypothesis there exists a sequence of uniform covers β_{nm}, $m \in \mathbb{N}$, with property $(*)$ (changing λ to β_n and α_n to β_{nm}).

Let $x \in$ X. There exists $n \in \mathbb{N}$ and $M \in \lambda_\omega$ such that $\alpha_n(x) \subset M$. It follows that $St(x, \alpha_n) \subset \alpha_n \setminus \alpha_n^*$ and $St(x, \beta_n)$ is a refinement of the cover λ. If now we choose m such that the set $\beta_{nm}(x)$ lies in some element K of the cover β_n we get $K \in St(x, \beta_n)$ and so K lies in an element of the cover λ. We proved that $(X, \mathscr{U})$ is P-paracompact.

The second part of the theorem is evident. □

2. Countable- (and τ-) P-paracompactness

Definition 2.1. (B. A. Pasynkov) The pseudouniform space $(X, \mathscr{U})$ will be called τ-P-paracompact if for every open cover λ of $(X, \mathscr{U})$ the cover λ_τ (see Section 0) is weakly uniform.

Below ω_1-P-paracompact uniform spaces will be called countable-P-paracompact. We note that ω-P-paracompact uniform spaces coincide with B-paracompact spaces. It is clear that every B-paracompact uniform space is countable-P-paracompact. The following example shows that not every countable-P-paracompact space is B-paracompact.

Remember that a one point lindelöfication lX of the discrete space X is the set X together with a point $\Lambda \notin X$ and the topology consisting of all the subsets of the set X and all sets of the form $\{\Lambda\} \cup Y$, where $Y \subset X$ and $|X \setminus Y| \leqslant \omega$. It is worth mentioning that any finite power of such a lindelöfication is Lindelöf. From this follows that the free Abelian group of such a lindelöfication is Lindelöf.

Example 2.2. Let R be the group of all real numbers with the discrete topology, X an uncountable discrete space, lX is its one point lindelöfication and F the free Abelian group of the space lX. Then the topological group $G = R \times F$, with its natural uniformity $\mathscr{U}$, is a countable-P-paracompact, but not B-paracompact, uniform space.

Proof. Let e_1 and e_2 be the unit elements of the groups R and F respectively, $F_t = \{t\} \times F$, $t \in R$, and Λ is the only point of the set $lX \setminus X$.

Let us assume that the space $(G, \mathscr{U})$ is B-paracompact.

We prove that for every neighbourhood O of e_2 in F there exists a countable set $A_O \subseteq X$ such that $nx - ny \in O$ for all $x, y \in X \setminus A_O$ and $n \in \mathbb{N}$.

Fix any neighbourhood O of e_2. Then for every $n \in \mathbb{N}$ there exists a neighbourhood U_n of the point Λ in F with the property $nU_n + n(-U_n) \subseteq O$. Let $A_n = X \setminus U_n$. Then the set A_n is countable and for all $x, y \in X \setminus A_n \subseteq U_n$ the relation $nx - ny \in O$ holds. Evidently, $A_O = \bigcup\{A_n : n \in \mathbb{N}\}$ is the needed set.

Let $Y \subseteq X$, $|Y| = \omega_1$ and f is an injection of Y into R. The map $\varphi_x : lX \to R \times R$, which maps the point $x \in Y$ onto the point $(0,1)$ and the set $lX \setminus \{x\}$ onto the point $(1,0)$ can be extended to a continuous homomorphism $\bar{\varphi}_x : F \to R \times R$. The inverse image $O(x) = (\bar{\varphi}_x)^{-1}(0,0)$ is an open subgroup of the group F.

Let $\nu_x = \{\{fx\} \times (O(x) + ix + j\Lambda) : i, j \in \mathbb{Z}\}$, $x \in Y$ and let $\nu = (R \setminus fY) \times F \cup \bigcup\{\nu_x : x \in Y\}$. One can see that ν_x is an open cover of the set F_{fx}, $x \in Y$, and ν is an open cover of the group G. From B-paracompactness of the space $(G, \mathscr{U})$ follows the existence of a sequence of neighbourhoods $V(m)$ of the element e_2 of F, such that $2V(m+1) \subseteq V(m)$, $m \in \mathbb{N}$, and for every point $x \in Y$ one can find a number $m(x)$ with the property:

$$\{fx\} \times V(m(x)) \subseteq \{fx\} \times \bigcup\{O(x) + ix + j\Lambda : i, j = -m(x), \dots, -1, 0, 1, \dots, m(x)\}.$$

Hence,

$$V(m(x)) \subseteq \bigcup\{O(x) + ix + j\Lambda : i, j = -m(x), \dots, -1, 0, 1, \dots, m(x)\}.$$

From the uncountability of Y follows the existence of an uncountable set $Z \subseteq Y$ and a $k \in \mathbb{N}$, such that

$$V(k) \subseteq W(x) = \bigcup\{O(x) + ix + j\Lambda : i, j = -k, \dots, -1, 0, 1, \dots, k\}, \quad x \in Z.$$

In $Z \setminus A_{V(k)}$ there exist points x and $y, x \neq y$. For such points

$$(k+1)x - (k+1)y \in V(k) \subseteq W(x) \text{ and } \bar{\varphi}_x((k+1)x - (k+1)y) = (-(k+1), (k+1)) \\ \in \bar{\varphi}_x W(x) = \{(j,i) : i, j = -k, \dots, -1, 0, 1, \dots, k\},$$

which is impossible. Hence, the space $(G, \mathscr{U})$ is not B-paracompact. □

Countable-P-paracompactness of $(G, \mathscr{U})$ will be proved after proposition 2.4.

Definition 2.3. The uniform space $(X, \mathscr{U})$ will be called uniformly locally Lindelöf if there exists $\alpha \in \mathscr{U}$ such that the closure of each element of α is Lindelöf.

Proposition 2.4. *A uniformly locally Lindelöf uniform space is countable-P-paracompact.*

P r o o f. Let the closure of every element of the uniform cover α of the uniform space $(X, \mathscr{U})$ be Lindelöf. Then for every open cover λ of the space $(X, \mathscr{U})$ the relation $\alpha > \lambda_{\omega_1}$ holds. This implies that $\lambda_{\omega_1} \in \mathscr{U}$ and hence λ_{ω_1} is a weakly uniform cover. □

We now prove that the space $(G, \mathscr{U})$ of example 2.2 is countable-P-paracompact. Since the group F is Lindelöf, the space $(X, \mathscr{U})$ is uniformly locally Lindelöf and hence countable-P-paracompact.

Theorem 2.5. *Every countable-P-paracompact uniform space is F-paracompact.*

P r o o f. Let γ be an open cover of a countable-P-paracompact uniform space $(X, \mathscr{U})$. For the open cover $\delta = \gamma_{\omega_1}$ there exists a sequence of open covers $\alpha_n \in \mathscr{U}$, $n \in \mathbb{N}$, satisfying properties $(*)$ and $(**)$, and a pseudometric ϱ on X such that $(\#)$ holds (see theorem 1.6). This implies that $\langle\delta\rangle = \{\langle\Gamma\rangle_\varrho : \Gamma \in \delta\}$ is an open cover of the pseudometric space (X, ϱ), where $\langle\Gamma\rangle_\varrho$ is the interior of Γ in the topology τ_ϱ induced by ϱ. Hence there exists an open cover σ which is a refinement of δ such that $\sigma = \bigcup\{\sigma_n : n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ there exists an $\varepsilon(n) > 0$ such that for every $x \in X$, $O_{\varepsilon(n)}(x)$ meets at most one member of σ_n. This implies that σ is a uniformly σ-discrete open in τ_u (since $\tau_\varrho \subset \tau_u$) cover which is a refinement of δ.

Now for every $A \in \sigma_n$ fix $\Gamma(A) \in \delta$ and $\Gamma^j(A) \in \gamma$, $j \in \mathbb{N}$, for which $A \subset \Gamma(A) = \bigcup\{\Gamma^j(A) : j \in \mathbb{N}\}$. Let $\sigma_{nj} = \{A \cap \Gamma^j(A) : A \in \sigma_n\}$ for all $n, j \in \mathbb{N}$. It is easy to see that $\{\sigma_{nj}\}_{j=1,n=1}^{\infty}$ is a uniformly σ-discrete open cover which is a refinement of the cover γ. □

Theorem 2.6. *The uniform space $(X, \mathscr{U})$ is P-paracompact if and only if it is τ-P-paracompact and τ-decomposable.*

The p r o o f is analogous to the proof of theorem 1.14. □

Corollary 2.7. *The uniform space $(X, \mathscr{U})$ is P-paracompact if and only if it is countable-P-paracompact and countably-decomposable.*

Definition 2.8. (B. A. Pasynkov) The (pseudo)uniform space $(X, \mathscr{U})$ will be called paracompact if every open cover of $(X, \mathscr{U})$ admits a σ-$\mathscr{U}$-locally finite (i.e. it is decomposed into a union of a countable number of $\mathscr{U}$-locally finite subsystems) open refinement.

It is clear that every F-paracompact space is paracompact.

We will show that all definitions of uniform paracompactness introduced in this paper are equivalent for countably bounded uniform spaces. Remember that the uniform space $(X, \mathscr{U})$ is called τ-bounded if $\ell(\mathscr{U}) \leqslant \tau$, where the index of boundedness for $(X, \mathscr{U})$, $\ell(\mathscr{U})$, is the smallest infinite cardinal number τ such that $\mathscr{U}$ has a base consisting of covers of cardinality $\leqslant \tau$. Below ω-bounded uniform spaces will be called countably bounded.

Theorem 2.9. *For countably bounded uniform spaces $(X, \mathscr{U})$ the following properties are equivalent:*

(1) *P-paracompactness,*

(2) *B-paracompactness,*

(3) *countable-P-paracompactness,*

(4) *F-paracompactness,*

(5) *paracompactness,*

(6) *the space (X, τ_u) is Lindelöf.*

For the proof of theorem 2.9 it is enough to prove the following two lemmas:

Lemma 2.10. *If for the uniform space $(X, \mathscr{U})$ the topological space (X, τ_u) has the Lindelöf property, then $(X, \mathscr{U})$ is P-paracompact.*

P r o o f. Let λ be an open cover of $(X, \mathscr{U})$. For every $x \in X$ there exist a $\Gamma(x) \in \lambda$ and $\sigma_x \in \mathscr{U}$ such that $\sigma_x(x) \subset \Gamma(x)$. For every such $\sigma_x \in \mathscr{U}$ there exists an open cover $\tilde{\sigma}_x \in \mathscr{U}$ such that $\tilde{\sigma}_x \star\!> \sigma_x$.

Take the open cover $\gamma = \{\sigma_x(x) \colon x \in X\}$ of the space (X, τ_u). Since (X, τ_u) has the Lindelöf property there exists a countable subcover $\tilde{\gamma} = \{\sigma_{x(i)}(x(i)) \colon i \in \mathbb{N}\}$ of γ. Take the sequence $\tilde{\sigma}_{x(i)}$, $i \in \mathbb{N}$, of uniform covers.

For every $x \in X$ there exists an $i \in \mathbb{N}$ such that $x \in \tilde{\sigma}_{x(i)}(x(i))$. Then $\tilde{\sigma}_{x(i)}(x) \subset \sigma_{x(i)}(x(i)) \subset \Gamma(x(i)) \in \lambda$ and hence λ is weakly uniform. □

Lemma 2.11. *If the uniform space $(X, \mathscr{U})$ is paracompact and $\ell(\mathscr{U}) \leqslant \omega$ then (X, τ_u) has the Lindelöf property.*

P r o o f. Let γ be an open cover of $(X, \mathscr{U})$. There exists an open uniformly σ-locally finite cover α, which is a refinement of γ. Let $\alpha = \bigcup\{\alpha_i \colon i \in \mathbb{N}\}$, where for all $i \in \mathbb{N}$ there exists $\sigma_i \in \mathscr{U}$, $\sigma_i = \{U_{ij} \colon j \in \mathbb{N}\}$, such that for all $j \in \mathbb{N}$, U_{ij} meets at most a finite number of elements of α_i. Since for every $i \in \mathbb{N}$, every element of α_i meets some U_{ij} we conclude that $|\alpha_i| \leqslant \omega$. This implies that $|\alpha| \leqslant \omega$ and hence (X, τ_u) is Lindelöf. □

Corollary 2.12. *For a Tychonoff space (X, τ) the following are equivalent:*

(1) *The space (X, τ) is Lindelöf.*

(2) *The uniform space $(X, \mathscr{U})$ is P-paracompact for every such uniformity $\mathscr{U}$ on X for which $\tau_u = \tau$.*

(3) *The uniform space $(X, \mathscr{U})$ is B-paracompact for every such uniformity $\mathscr{U}$ on X for which $\tau_u = \tau$.*

(4) *The uniform space $(X, \mathscr{U})$ is countable-P-paracompact for every such uniformity $\mathscr{U}$ on X for which $\tau_u = \tau$.*

(5) *The uniform space $(X, \mathscr{U})$ is F-paracompact for every such uniformity $\mathscr{U}$ on X for which $\tau_u = \tau$.*

(6) *The uniform space $(X, \mathscr{U})$ is paracompact for every such uniformity $\mathscr{U}$ on X for which $\tau_u = \tau$.*

Proof. It is enough to prove the implication (6) $\Longrightarrow$ (1).

Let $(X, \mathscr{U})$ be any uniform space such that $\tau_u = \tau$. There exists a precompact uniformity $p\mathscr{U}$ for which $\tau_{pu} = \tau_u = \tau$. Since by hypothesis $p\mathscr{U}$ is paracompact and $\ell(p\mathscr{U}) \leqslant \omega$ it follows from lemma 2.11 that (X, τ) is Lindelöf. $\square$

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