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SOME REMARKS ON LORENZEN r-GROUP OF PARTLY ORDERED GROUPS

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1. INTRODUCTION

The investigation of arithmetical properties of partly ordered groups (po-groups) has its origin in the study of arithmetics of integral domains. Especially the notion of a Kronecker function ring is of great importance in the theory of divisibility of integral domains. This notion was introduced by W. Krull in order to study the arithmetics of integral domains. The principal advantage of the extension process which leads from an integrally closed domain A to its Kronecker function ring $K(A)$ is the fact that $K(A)$ is a Bezout domain, i.e. any finitely generated ideal is principal.

Another historical source of a study of arithmetics of po-groups was the work of S.I. Borewicz and I.R. Shafarevicz [4], where the concept of an integral domain with the theory of divisors was introduced. It was observed for the first time by L. Skula [22] that an integral domain has a theory of divisors if and only if it is a Krull domain.

In the course of time it has become more and more clear that all this arithmetical notions in integral domains have their purely multiplicative analogues in (commutative) semigroups with cancellation law. One of the first such observations was done again by Skula who defined the notion of a *semigroup with divisor theory* [21]. Since divisibility properties of integral domains and semigroups with cancellation are mostly represented by properties of order relations of their groups of divisibility, a very natural and fruitful generalization was to investigate arithmetical properties of po-groups. The principal tool for the investigation of these properties in po-groups seems to be the notion of an r-ideal which has its origin in a paper of Lorenzen [13]. We will not be dealing here with a history of this notion, see e.g. [2], [12]. We recall only that by an r-system of ideals in a directed po-group G we mean a map $X \mapsto X_r$

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(X_r is called an r-ideal) from the set of all lower bounded subsets X of G into the power set of G which satisfies the following conditions:

(1) $X \subseteq X_r$,

(2) $X \subseteq Y_r \Longrightarrow X_r \subseteq Y_r$,

(3) $\{a\}_r = a.G^+ = (a)$ for all $a \in G$,

(4) $a.X_r = (a.X)_r$ for all $a \in G$.

The theory of r-ideals of *po*-groups seems to be a tool which enables us to establish relationships between arithmetical properties of integral domains and the theory of *po*-groups. There exists a lot of works justifying these processes, one of the last result in this direction being a paper of A. Geroldinger and the first author ([6]) where it is proved that properties of being a PVMD (Prüfer v-multiplication domain), a domain of Krull type, or an independent domain of Krull type, are purely multiplicative ones and can be expressed by using r-ideals in the corresponding groups of divisibility. For example, a domain A is a PVMD iff $G(A)$ has a theory of quasi-divisors (see [2], [16] for definitions).

From the point of view of the r-ideals theory the notion of a Kronecker function ring, or a domain of Krull type, or PVMD, have the same common background in the r-ideal theory, namely the notion of a *Lorenzen r-group*. Recall that for any r-closed *po*-group G with an r-system (i.e. A_r: $A_r \subseteq G^+$ for any finite $A \subseteq G$), to an r-system r we can associate another r-system denoted by r_a such that

$$A_{r_a} = \{g \in G \colon g.K_r \subseteq A_r \times K_r, \text{ for some finite } K \subset G\}$$

whenever $A \subset G$ is finite. The principal property of this r-system r_a is that the monoid of finitely generated r_a-ideals (under r_a-multiplication, see e.g. [2],[12]) satisfies the cancellation law and hence possesses a quotient group $\Lambda_r(G)$ which is called the *Lorenzen r-group of G*. This group is a lattice ordered group if we set $\Lambda_r(G)^+ = \{A_{r_a}/B_{r_a} \colon A_{r_a} \subseteq B_{r_a}\}$ and, moreover, $\Lambda_r(G)$ contains G as an ordered subgroup. Then this common background of the above mentioned notions can be described as follows:

(1) A domain A is a PVMD iff the embedding $G(A) \to \Lambda_t(G(A))$ is a theory of quasi-divisors ([6]).

(2) A domain A is of a Krull type iff the same embedding as in (1) is a theory of quasi-divisors of a finite character ([6]).

(3) $G(K(A)) \cong \Lambda_t(G(A))$.

Hence it seems worthwhile to investigate other properties of Lorenzen r-groups since these properties can reflect some arithmetical properties of integral domains.

In this paper we investigate some of these properties of Lorenzen r-groups of *po*-groups. First we describe relationships between the structure of o-ideals in a *po*-

group and some l-ideals of its Lorenzen r-group. If a *po*-group G has a theory of quasi-divisors of a finite character, there exists a defining family W of t-valuations $w: G \to G_w$ of G such that W satisfies some approximation theorem (see [16]). Then there exists a special subset $\mathcal{K}(W)$ of $\prod_{w \in W} G_w$ (called the set of *compatible elements*) and we show that even in a rather general case (G is only defined by a family of r-valuations of a finite character) the set $\mathcal{K}(W)$ is an l-group which is isomorphic to $\Lambda_r(G)$ for rather general r-systems. Finally, we show that a *po*-group G with a theory of quasi-divisors (more generally, r-Prüfer *po*-groups) may be defined as a *po*-group with G/H an o-group for any minimal r-local o-ideal H of G, and it is shown that in this case G and $\Lambda_r(G)$ behave analogously with respect to the property v-system $= t$-system.

2. Lorenzen r-group

Let G be a directed commutative partly ordered group (*po*-group) and let r be an r-system of ideals defined on G. Recall that an r-system is called a v-system, if

$$X_v = \bigcap_{X \subseteq (y), y \in G} (y),$$

and it is called a t-system, if

$$X_t = \bigcup_{Y \subseteq X, Y \text{ finite}} Y_v.$$

An r-system r is said to be of *a finite character*, if

$$X_r = \bigcup_{Y \subseteq X, Y \text{ finite}} Y_r.$$

An r-ideal X_r is *finitely generated* if $X_r = Y_r$ for some finite subset Y. Clearly, any t-system is of finite character and for any r-system r of a finite character on G, $X_r \subseteq X_t$ ($r \leqslant t$, in symbol). On the set $\mathcal{I}_r(G)$ of r-ideals we may define an ordering by $X_r \leqslant Y_r$ iff $Y_r \subseteq X_r$ and a multiplication $X_r \times Y_r = (X_r Y_r)_r = (X.Y)_r$. A *po*-group G with an r-system r is r-closed if $(X_r : X_r) \subseteq G^+$ for any X_r. As we have mentioned in the introduction, for any r-closed *po*-group G we may construct another r-system in G, denoted by r_a, such that r_a is *regularly closed*, i.e. in the semigroup $(\mathcal{I}_r^f(G), \times)$ of all finitely generated r-ideals the cancellation law holds. The quotient group $\Lambda_r(G)$ of $\mathcal{I}_r^f(G)$ is called the *Lorenzen r-group of G* and a map $h: G \to \Lambda_r(G)$ defined by $h(g) = (g)$ is an o-isomorphism into.

An o-homomorphism φ from a *po*-group G_1 with an r-system r_1 into a *po*-group G_2 with an r-system r_2 is an (r_1, r_2)*-morphism* if $\varphi(X_{r_1}) \subseteq (\varphi(X))_{r_2}$ for any lower bounded subset X. If G_2 is totally ordered (i.e. an *o-group*) and φ is surjective, then φ is called an r_1*-valuation* if it is an (r_1, t)-morphism. Moreover, an o-homomorphism $\varphi\colon G_1 \to G_2$ is called *essential* if it is an o-epimorphism and $\ker\varphi$ is a directed convex subgroup of G_1 (i.e. an *o-ideal* of G_1). Finally, we say that a *po*-group G admits a *theory of quasi-divisors* if there exists an l-group Γ and an o-isomorphism h of G into Γ such that for any $\alpha \in \Gamma$ there exist $g_1, \ldots, g_n \in G$ such that $\alpha = h(g_1) \wedge \ldots \wedge h(g_n)$. In [6]; Th. 3.8, it is proved that the existence of a theory of quasi-divisors of a finite character is equivalent to the existence of a family W of essential t-valuations such that

(1) $\forall g \in G,\ g \geqslant 1 \Leftrightarrow (\forall w \in W) w(g) \geqslant 1$
(2) $\forall g \in G,\ g \neq 1,\ \{w \in W : w(g) \neq 1\}$ is finite.

In this case W is called a *defining family of a finite character*.

In this section we want first to modify some constructions used originally for t-valuations. We need the following simple lemma.

Lemma 2.1. *Let G be a directed po-group and let H_1, H_2 be o-ideals of G. Then in the set of all convex subgroups of G there exists the smallest one containing H_1, H_2, denoted by $[H_1, H_2]$. Moreover, $[H_1, H_2]$ is also an o-ideal.*

Proof. Let $S = \{g \in G\colon \exists h_i \in H_i^+ \text{ such that } 1 \leqslant g \leqslant h_1 h_2\}$. Then S is a convex subgroup in G^+, $H_i^+ \subseteq S$. It follows that the quotient group $[H_1, H_2]$ of S in G is an o-ideal which possesses the required properties. □

Lemma 2.2. *Let $\varphi\colon G \to \Gamma$ be an essential o-homomorphism of a po-group G into Γ. Then φ is a (t, t)-morphism.*

Proof. Let X be a lower bounded subset in G and let $g \in X_t$. Then there exists a finite subset $K \subseteq X$ such that $g \in K_t$. Let α be a lower bound of $\varphi(X)$ in Γ. Then, since φ is an o-epimorphism, for any $k \in K$ there exists $b_k \in \ker\varphi$ such that $a \leqslant b_k.k$, where $\varphi(a) = \alpha$. Since $\ker\varphi$ is directed, there exists $b \in \ker\varphi$ such that $b \leqslant b_k^{-1}$ for all $k \in K$. Hence, $a.b \leqslant k$ for all $k \in K$ and it follows that $g \geqslant a.b$. Therefore, $\varphi(X_t) \subseteq (\varphi(X))_t$. □

Let w, v be essential o-homomorphisms of G with value groups G_w, G_v, respectively. Then the canonical o-homomorphism $G \to G/[\ker w, \ker v]$ is essential and there are essential o-homomorphisms d_{vw}, d_{wv} such that $d_{vw}.v = d_{wv}.w$. This common essential o-homomorphism will be denoted by $v \wedge w$. Now, elements $(g_1, g_2) \in G_w \times G_v$ are called *compatible*, if $d_{wv}(g_1) = d_{vw}(g_2)$. Moreover, if W is a set of essential o-homomorphisms, an element $(g_w)_w \in \prod_{w\in W'} G_w$ (where $W' \subseteq W$) is called *compatible* if any pair (g_w, g_v) from this element is compatible. It is clear that all these notions are generalizations of analogous notions for t-valuations as introduced e.g. in [16]. Finally, we say that an element $(g_w)_w \in \prod_{w\in W} G_w$ is *W'-complete* for $W' \subseteq W$, if $\bigcup_{w\in W'} W(g_w) \subseteq W'$, where $W(g_w) = \{v \in W : d_{wv}(g_w) \neq 1\}$. We set $W(1) = \emptyset$.

Now, let W be a defining family of essential o-homomorphisms of G. For $W' \subseteq W$ we set

$$\mathcal{K}(W') = \{(g_w)_w \in \sum_{w\in W'} G_w : (g_w)_w \text{ is compatible}\},$$

$$\mathcal{K}^+(W') = \mathcal{K}(W') \cap \sum_{w\in W'} G_w^+,$$

$$\mathcal{K} = \mathcal{K}(W), \mathcal{K}^+ = \mathcal{K}^+(W).$$

Lemma 2.3. *Let W be a defining family of essential o-homomorphisms of G and let $W' \subseteq W$. Then $\mathcal{K}(W')$ is a subgroup in $\prod_{w\in W'} G_w$. If for $a, b \in \mathcal{K}(W')$ there exists $a \wedge b$ in $\prod_{w\in W'} G_w$, then $a \wedge b \in \mathcal{K}(W')$ and dually for supremum.*

Proof. For $a = (a_w)_w, b = (b_w)_w \in \mathcal{K}(W')$ let there exist $c = (c_w)_w = a \wedge b$ in $H = \prod_{w\in W'} G_w$. Since H is ordered componentwise, $c_w = a_w \wedge b_w$ in G_w. Now let $\varphi\colon G \to G'$ be an essential o-homomorphism and let $x, y \in G$ be such that $x \wedge y$ exists in G. Then $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$. In fact, if $\alpha \leqslant \varphi(x), \varphi(y)$, then for $z \in G$ such that $\varphi(z) = \alpha$ there exists $h \in \ker\varphi$ such that $zh \leqslant x, y$ and it follows that $\alpha \leqslant \varphi(x \wedge y)$. Therefore, for $w, v \in W'$ we have

$$d_{wv}(a_w \wedge b_w) = d_{wv}(a_w) \wedge d_{wv}(b_w) =$$
$$= d_{vw}(a_v) \wedge d_{vw}(b_v) = d_{vw}(a_v \wedge b_v)$$

since d_{wv}, d_{vw} are also essential. The rest of the proof may be done analogously. □

The investigation of approximation theorems for a po-group G with a defining family W of t-valuations is especially complicated in the case $\mathcal{K}(W) \subset \prod_{w\in W} G_w$, or $\mathcal{K}(W) \subset \sum_{w\in W} G_w$ if W is of finite character. On the other hand, there are very special cases when $\mathcal{K}(W) = \sum_{w\in W} G_w$. Geroldinger and Halter-Koch [11] investigated conditions under which the following approximation theorem holds for a po-group G with a defining family W of o-homomorphisms $w\colon G \to G_w$ (here G_w is only a directed po-group) of a finite character:

(AT). *For any $W' \subseteq W$ finite and any $g_w \in G_w$, $w \in W'$, there exists $g \in G$ such that*

$$w(g) = g_w, \quad w \in W',$$
$$w(g) \geqslant 1, \quad w \in W \setminus W'.$$

We show that this approximation theorem represents really a very special case, since $\mathcal{K}(W) = \sum_{w \in W} G_w$ in this case.

Proposition 2.4. *If (AT) holds for a po-group G with a defining family W of o-homomorphisms of a finite character, then any $w \in W$ is essential and $\mathcal{K}(W) = \sum_{w \in W} G_w$.*

Proof. We show first that any $w \in W$ is essential. Let $a \in H_w = \ker w$ and let $W' = \{w \in W \colon w(a) \neq 1\}$. Since G_w is a directed po-group, there exists $a_w \in G_w$ such that $a_w \geqslant w(a), 1$. Then according to (AT) there exists $g \in G$ such that

$$v(g) \begin{cases} = 1, & \text{for } v = w, \\ = a_v, & \text{for } v \in W', \\ \geqslant 1, & \text{for } v \in W \setminus W'. \end{cases}$$

Then $g \geqslant 1$, a and $g \in H_w$. Applying (AT) in an analogous way we may prove that w is an o-epimorphism. Hence, w is essential and for any $w, v \in W$ we can construct $w \wedge v$ according to 2.1. Let us assume that there are $w, v \in W$ such that $w \leqslant v$. Then for any $g \in G$ with $v(g) = 1$ it follows that $w(g) = 1$, but according to (AT) there should exist $g \in G$ such that $w(g) > 1$, $v(g) = 1$, a contradiction. Hence, elements of W are incomparable. Finally, let $v_1, v_2 \in W$. We show that $[\ker v_1, \ker v_2] = G$. In fact, let $g \in G$, $g \geqslant 1$ and let $W_1 = \{w \in W \colon w(g) \neq 1\}$. We may assume that $v_i \in W_1$, otherwise $g \in \ker v_i$. According to (AT) there exist $a, b \in G$ such that

$$w(a) \begin{cases} = w(g), & w \neq v_1, w \in W_1 \\ = 1, & w = v_1, \\ \geqslant 1, & \text{otherwise}, \end{cases}$$

$$w(b) \begin{cases} = 1, & w \in W_1, w \neq v_1, \\ = v_1(g), & w = v_1, \\ \geqslant 1, & \text{otherwise}. \end{cases}$$

Then it is clear that for any $w \in W$ we have $1 \leqslant w(g) \leqslant w(a.b)$ and $a \in \ker v_1$, $b \in \ker v_2$. Therefore, $g \in [\ker v_1, \ker v_2]$. Hence, $v_1 \wedge v_2$ is trivial and it follows that $\mathcal{K}(W) = \sum_{w \in W} G_w$. □

Let G be a *po*-group with an r-system r of a finite character. We say that this r-system is defined by a family W of r-valuations, if for any finite set A of G,

$$g \in A_r \Leftrightarrow (\forall w \in W) w(g) \in (w(A))_t.$$

An example of such an r-system is a t-system which is defined by any defining family of t-valuations. P. Jaffard [12] proved (Th. 5, Chap. II, par. 2) that in this case the r-system r is regularly closed and there exists a Lorenzen r-group $\Lambda_r(G)$. Moreover, the r_a-system coincides with r in this case.

If $h\colon G \to \Lambda_t(G)$ is a theory of quasi-divisors, then $\Lambda_t(G)$ may be identified with the group $(\mathcal{I}_t^f(G), \times)$ of finitely generated t-ideals of G with t-multiplication. But even in the general case, any r-valuation $w\colon G \to G_w$ (where r is of finite character) may be extended onto an o-homomorphism $\hat{w}\colon (\mathcal{I}_t^f(G), \times, \leqslant) \to G_w$ such that

$$\hat{w}(A_r) = \inf w(A).$$

Using this extension we can obtain another characterization of an r-system defined by a family of r-valuations.

Proposition 2.5. *Let r be an r-system of a finite character on G and let W be a defining family of r-valuations of G. Let $\varphi\colon \mathcal{I}_r^f(G) \to \prod_{w \in W} G_w$ be a product of $\hat{w}, w \in W$. Then φ is injective if and only if r is defined by W.*

Proof. Let r be defined by W and let $\varphi(A_r) = \varphi(B_r)$. Let $g \in A_r$. Then for any $w \in W$ there exists $a \in A$ such that $w(g) \geqslant w(a) \geqslant \hat{w}(A_r) = \hat{w}(B_r)$. If $g \notin B_r$ then there exists $w \in W$ such that for all $b \in B$ we have $w(b) > w(g)$. Hence, $w(g) < \hat{w}(B_r) = \hat{w}(A_r)$, a contradiction.

Conversely, let φ be injective and let x be an r-system defined by W. Then $x \geqslant r$. In fact, let $g \in A_r$ and $w \in W$. Then $w(g) \in w(A_r) \subseteq (w(A))_t$ and it follows that $g \in A_x$. But in this case $A_x \in \mathcal{I}_r^f(G)$ and clearly $\varphi(A_r) = \varphi(A_x)$. Hence, $x = r$. □

Even in the case that φ is not injective, the map from the previous proposition may be useful. For a finite set $A \subseteq G$ we put

$$A_x = \bigcup_{\substack{\varphi(B_r)=\varphi(A_r) \\ B \subseteq G \text{ finite}}} B_r$$

where we use a defining family W of r-valuations of G. It is clear that x is an r-system of a finite character, $x \geqslant r$. Moreover, since any r-valuation is an x-valuation, it may be proved simply that x is defined by W, i.e. x is regularly closed. Hence, $x \geqslant r_a$.

In the next theorem we investigate relationships between $\Lambda_t(G)$ and an l-group $\mathcal{K}(W)$ of compatible elements, where W is a defining family of r-valuations of G of a finite character. By $\hat{w}$ we denote a canonical extension of an r-valuation $w \in W$ onto a t-valuation of $\Lambda_r(G)$; let $\tilde{w}$ be a w-projection from $\mathcal{K} = \mathcal{K}(W)$ onto G_w.

Theorem 2.6. *Let G be a po-group with an r-system r of a finite character and let r be defined by a family W of r-valuations of a finite character. Then there is an o-isomorphism ψ such that the following diagram commutes for all $w \in W$:*

$$\begin{array}{ccc} \Lambda_r(G) & \xrightarrow{\ \psi\ } & \mathcal{K}(W) \\ {\scriptstyle \hat{w}}\downarrow & & \downarrow{\scriptstyle \tilde{w}} \\ G_w & = & G_w \end{array}$$

P r o o f. Let $\varphi\colon G \to \mathcal{K}$ be an embedding defined by $\varphi(g) = (w(g))_w$. Then φ is an (r,t)-morphism. In fact, if $a \in X_r$, then since r is of finite character, there exists a finite subset $K \subseteq X$ such that $a \in K_r$. Then $w(a) \in (w(K))_t = (\wedge w(K))$ and it follows that $\varphi(a) \in (\varphi(K))_t$. The universality of $\Lambda_r(G)$ (see [2]; Th. 1) implies that φ may be extended onto an l-homomorphism $\varrho\colon \Lambda_r(G) \to \mathcal{K}$ such that for $A_r/B_r \in \Lambda_r(G)$ (here $r_a = r$) we have $\varrho(A_r/B_r) = (\wedge\varphi(A)).(\wedge\varphi(B))^{-1}$, where A, B are finite. Then $\wedge\varphi(A) = (w(A))_w \in \mathcal{K}$ according to 2.3. It is clear that $\tilde{w}$ is a t-valuation of $\mathcal{K}$ and that the set $\widetilde{W} = \{\tilde{w}\colon w \in W\}$ is a defining family of a finite character of $\mathcal{K}$. We further show that a (t,t)-morphism ϱ is an embedding. In fact, if $\varrho(A_r/B_r) = \varrho(C_r/D_r)$, then for any $w \in W$ we have

$$(w(A.D))_t = (w(A))_t \times (w(D))_t = (w(B))_t \times (w(C))_t = (w(B.C))_t$$

and since the r-system r is defined by W, we have $A_r \times D_r = B_r \times C_r$. Hence, ϱ is injective. It is clear that ϱ is an o-isomorphism into. Moreover, we show that ϱ: $\Lambda_r(G) \to \mathcal{K}$ is a strong theory of quasi-divisors. Let $a, b \in \mathcal{K}_+$, $a = (a_w)$, $b = (b_w)$, and let $W_1 = \{w \in W : a_w.b_w > 1\}$. Then W_1 is a finite set and since any l-group admits a strong theory of quasi-divisors (namely an identical map), any of its defining families of t-valuations of a finite character satisfies the positive weak approximation theorem ([16], Th. 3.5). Since a is clearly W_1-complete and compatible, there exists $\gamma \in \Lambda_r(G)$ such that

$$\hat{w}(\gamma) = a_w, \quad w \in W_1,$$
$$\hat{w}(\gamma) \geqslant 1, \quad w \in W \setminus W_1.$$

Hence, $\varrho(\gamma) \geqslant a$ in $\mathcal{K}$ and there exists $c \in \mathcal{K}_+$ such that $\varrho(\gamma) = a.c \in \varrho(\Lambda_r(G))$. Moreover, if $b_w > 1$ then $w \in W_1$ and $c_w = 1$. Hence, $b_w \wedge c_w = 1$ and it follows that $b \wedge c = 1$ in $\mathcal{K}$. Therefore, ϱ is a strong theory of quasi-divisors and according to [16], ϱ is a theory of quasi-divisors as well. Hence, according to [2]; Th. 4, there exists an l-isomorphism $\mathcal{K} \cong \Lambda_r(\Lambda_r(G)) = \Lambda_r(G)$. □

From this theorem we can obtain the following very general form of an approximation theorem which holds in any *po*-group defined by a family of t-valuations of a finite character.

Corollary. *Let G be a po-group and let W be a defining family of t-valuations of G of a finite character. Let $g = (g_w) \in \mathcal{K}(W)$. Then there exist $a_1, \dots, a_n$, $b_1, \dots, b_m \in G$ such that*

$$(w(a_1) \wedge \dots \wedge w(a_n)).(w(b_1) \wedge \dots \wedge w(b_m)) = g_w, \text{ if } g_w \neq 1,$$
$$(w(a_1) \wedge \dots \wedge w(a_n)).(w(b_1) \wedge \dots \wedge w(b_m)) \geqslant 1, \text{ if } g_w = 1.$$

It is clear that the classical approximation theorem is a special case of this general approximation theorem for $n = m = 1$.

3. Structural properties of $\Lambda_t(G)$

In this part of the paper we will investigate some structural properties of the Lorenzen t-group. We recall first a method which enables us to define another r-system on a factor group G/H if an r-system is defined on G. If x is an r-system on G of a finite character and H is an o-ideal of G, for any lower bounded subset $\mathcal{A} \subseteq G/H$ there exists a lower bounded subset $A \subseteq G$ such that $\varphi(A) = \mathcal{A}$, where $\varphi\colon G \to G/H$ is a canonical homomorphism. Then we set $\mathcal{A}_{x_H} = A_x/H = \varphi(A_x)$. In [16] it was proved that x_H is an r-system of a finite character.

Lemma 3.1. *Let G be an x-closed po-group, where x is an r-system of a finite character. If H is an o-ideal of G, then G/H is x_H-closed.*

P r o o f. Let $\mathcal{A} \subseteq G/H$ be a finite set and let $\alpha \in G/H$ be such that $\alpha.\mathcal{A}_{x_H} \subseteq \mathcal{A}_{x_H}$. Let A be a finite set such that $\mathcal{A}_{x_H} = A_x/H$, $A_x = (a_1, \dots, a_n)_x$, $\alpha = a.H$. Since $aa_iH \in A_x/H$, for any i there exist $g_i \in A_x$ and $h_i \in H$ such that $aa_i = g_ih_i$. Since H is directed, there exists $h \in H$ such that $h \leqslant h_i$ for all i. Then we have

$$\begin{aligned}(aa_1, \dots, aa_n)_x &\subseteq (g_1h_1, \dots, g_nh_n)_x \subseteq (g_1h, \dots, g_nh)_x \\ &= h(g_1, \dots, g_n)_x \subseteq h(a_1, \dots, a_n)_x\end{aligned}$$

and it follows that $ah^{-1} \geqslant 1$. Hence, $\alpha \geqslant 1$. □

Lemma 3.2. *Let x be an r-system on G of a finite character and let x be regularly closed. If H is an o-ideal of G, then x_H is regularly closed in G/H.*

P r o o f. According to [12]; Lemma, par. 2, Chapt. 2, we have only to prove that for any finitely generated x_H-ideals $\mathcal{A}_{x_H}, \mathcal{B}_{x_H}$ in G/H from $\mathcal{A}_{x_H} \subseteq \mathcal{B}_{x_H} \times \mathcal{A}_{x_H}$ it follows that $1_{G/H} \in \mathcal{B}_{x_H}$. Let $\mathcal{A}_{x_H} = A_x/H$, $\mathcal{B}_{x_H} = B_x/H$. If $\mathcal{A}_{x_H} \subseteq (\mathcal{A}.\mathcal{B})_{x_H}$, then for any $a \in A$ we have $aH \in (A.B)_x/H$ and for any $a_i \in A = \{a_1, \ldots, a_n\}$ there exist $c_i \in (A.B)_x$ and $h_i \in H$ such that $a_i = c_i h_i$. Since H is directed there exists $h \in H$ such that $h \leqslant h_i$ for all i. Then

$$\begin{aligned} A_x = (c_1 h_1, \ldots, c_n h_n)_x \subseteq (c_1, \ldots, c_n)_x h \subseteq (A.B)_x h \\ = A_x \times (B.h)_x. \end{aligned}$$

Since x is regularly closed, we have $1 \in (Bh)_x$ and it follows that $1_{G/H} \in B_x/H$. □

Now, let G be a *po*-group which is t-closed and let $h\colon G \to \Lambda_t(G)$ be the embedding. Then an l-ideal $\mathcal{H}$ in $\Lambda_t(G)$ is called *G-dense* if for any $\alpha \in \mathcal{H}$ there exists $g \in G$ such that $\alpha \leqslant h(g)$ and $h(g) \in \mathcal{H}$.

Theorem 3.3. *Let G be a po-group which is t-closed. Then there exists a bijection between the set of o-ideals of G and the set of G-dense l-ideals of $\Lambda_t(G)$ such that if H from G corresponds to Δ from $\Lambda_t(G)$, then*

$$\Lambda_{t_H}(G/H) \cong \Lambda_t(G)/\Delta.$$

P r o o f. Since G is t-closed, G/H is t_H-closed according to 3.1, and hence there exists the Lorenzen t_H-group of G/H. Since the canonical morphism $\varphi\colon G \to G/H$ is a (t, t_H)-morphism (see [16]), there exists a (t, t)-morphism $\hat{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & G/H \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_H} \\ \Lambda_t(G) & \xrightarrow{\ \hat{\varphi}\ } & \Lambda_{t_H}(G/H) \end{array}$$

According to 3.2, the r-system t_H is regularly closed in G/H and it follows from [2]; Th. 3, that t_H coincides with the r-system $(t_H)_a$, i.e. the Lorenzen t_H-group $\Lambda_{t_H}(G/H)$ consists of quotients $\mathcal{A}_{t_H}/\mathcal{B}_{t_H}$, where $\mathcal{A}$, $\mathcal{B}$ are finite sets in G/H. Then $\hat{\varphi}$ is surjective. Indeed, let $\mathcal{X} = \mathcal{A}_{t_H}/\mathcal{B}_{t_H} \in \Lambda_{t_H}(G/H)$, $\mathcal{A}_{t_H} = A_t/H$, $\mathcal{B}_{t_H} = B_t/H$. Then $A_t/B_t \in \Lambda_t(G)$ and $\hat{\varphi}(A_t/B_t) = \mathcal{X}$. Hence, $\hat{\varphi}$ is an l-epimorphism and $\Delta = \ker\hat{\varphi}$ is an l-ideal, $\Lambda_t(G)/\Delta \cong \Lambda_{t_H}(G/H)$. We show that Δ is G-dense. Indeed, let

$A_t/B_t \in \Delta$, where A, B are finite. Since $A_t/H = B_t/H$, for any $b \in B$ there exist $c_b \in A_t$ and $h_b \in H$ such that $b = c_b.h_b$. Since H is directed, there exists $g \in H$ such that $g^{-1} \leqslant h_b$ for all $b \in B$. Thus,

$$B_t = \{c_b.h_b : b \in B\}_t \subseteq \{c_b : b \in B\}_t.g^{-1} \subseteq A_t.g^{-1}$$

and we have $A_t/B_t \leqslant (g)_t = h(g)$ in $\Lambda_t(G)$. Clearly $h(g) \in \Delta$.

Therefore, we have obtained a map

$$\psi : \mathcal{O}(G) \to \mathcal{L}_d(\Lambda_t(G))$$

where $\mathcal{O}(G)$ is the set of o-ideals of G and $\mathcal{L}_d(\Lambda_t(G))$ is the set of G-dense l-ideals of the Lorenzen t-group.

Conversely, let $\Delta \in \mathcal{L}_d(\Lambda_t(G))$. Let $H = h^{-1}(\Delta)$. It is clear that H is a convex subgroup of G. We have to show that it is directed. Let $g \in H$. Since Δ is directed, there exists $\alpha \in \Delta$ such that $\alpha \geqslant h(g), 1$. Since Δ is G-dense, we can find an element $g' \in G$ such that $h(g') \in \Delta$ and $h(g') \geqslant \alpha \geqslant h(g), 1$. Hence, H is an o-ideal in G. Then $\Lambda_{t_H}(G/H) \cong \Lambda_t(G)/\Delta$. Indeed, let us consider the same diagram as in the previous part of the proof. It suffices to prove that $\ker\hat{\varphi} = \Delta$. Let $A_t/B_t \in \ker\hat{\varphi}$. Then $A_t/H = B_t/H$ and for any $a \in A (b \in B)$ there exists $c_a \in B_t (d_a \in A_t)$ and $h_a \in H (g_b \in B$, respectively) such that $a = c_a.h_a, b = d_b.g_b$. Since H is an o-ideal, there exist $u, v \in H$ such that $u \leqslant h_a$ for all $a \in A$ and $v \leqslant g_b$ for all $b \in B$. Hence,

$$A_t = \{c_a.h_a : a \in A\}_t \subseteq \{c_a : a \in A\}.u \subseteq B_t.u,$$
$$B_t = \{d_b.g_b : b \in B\}_t \subseteq \{d_b : b \in B\}_t.v \subseteq A_t.v$$

and it follows that $h(v^{-1}) \geqslant A_t/B_t \geqslant h(u)$. Hence, $A_t/B_t \in \Delta$. Conversely, if $A_t/B_t \in \Delta$ then since Δ is G-dense, there exist $g_1, g_2 \in H$ such that $h(g_1) \leqslant A_t/B_t \leqslant h(g_2)$. Since $\hat{\varphi}$ is an o-homomorphism, we have $A_t/B_t \in \ker\hat{\varphi}$. □

Proposition 3.4. *If a po-group G admits a theory of quasi-divisors then any prime l-ideal of $\Lambda_t(G)$ is G-dense.*

Proof. Let $h : G \to \Lambda_t(G)$ be a theory of quasi-divisors and let Δ be a prime l-ideal of $\Lambda_t(G)$. Then the canonical map $\hat{w} : \Lambda_t(G) \to \Lambda_t(G)/\Delta$ is a t-valuation. Let $\alpha \in \Delta$. There exist $g_1, \ldots, g_n \in G$ such that $\alpha = h(g_1) \wedge \ldots \wedge h(g_n) \leqslant h(g_1)$ and we have $1 = \hat{w}(\alpha) = \hat{w}h(g_1) \wedge \ldots \wedge \hat{w}h(g_n) = \hat{w}h(g_i)$ for some i. Then $h(g_i) \in \Delta$ and $\alpha \leqslant h(g_i)$. Hence, Δ is G-dense. □

Proposition 3.5. *If a po-group G admits a strong theory of quasi-divisors of a finite character then any finite intersection of prime l-ideals of $\Lambda_t(G)$ is G-dense.*

P r o o f. Let $h\colon G \to \Lambda_t(G)$ be a strong theory of quasi-divisors of a finite character, let W_1 be a defining family of t-valuations of a finite character of G and let $\widehat{W}_1$ be canonical extensions of elements from W_1 onto t-valuations of $\Lambda_t(G)$. Let $\Delta = \bigcap_{i=1,\dots,n} \Delta_i$, where Δ_i are prime l-ideals of $\Lambda_t(G)$ and let $\hat{v}_i\colon \Lambda_t(G) \to \Lambda_t(G)/\Delta_i$ be a canonical t-valuation. Then $\widehat{W} = \widehat{W}_1 \cup \{\hat{v}_1, \dots, \hat{v}_n\}$ is a defining family of a finite character of $\Lambda_t(G)$ and the set W of restrictions of elements from $\widehat{W}$ onto G is a defining family of a finite character of G (see [16]). Let $\alpha \in \Delta$. Then $a = (\hat{w}(a))_{\hat{w}} \in \mathcal{K}(W)$ and if we set $W' = \{w \in W_1 : \hat{w}(\alpha) \neq 1\} \cup \{v_1, \dots, v_n\}$, then W' is finite and a is W'-complete. Since W satisfies the approximation theorem ([16]; Th. 3.5), there exists $g \in G$ such that

$$w(g) = \hat{w}(\alpha), \quad w \in W',$$
$$w(g) \geqslant 1, \quad w \in W \setminus W'.$$

Hence, $h(g) \geqslant \alpha$ and $h(g) \in \Delta$. □

If $h\colon G \to \Lambda_t(G)$ is a theory of quasi-divisors, then the Lorenzen t-group reflects many of the algebraic properties of G. In the following proposition we show that it reflects even property which is not of a finite character.

Proposition 3.6. *Let $h\colon G \to \Gamma$ be a theory of quasi-divisors. If a complete v-system coincides with a complete t-system in G, the same is true in Γ.*

P r o o f. Let $\mathcal{A} \subseteq \Gamma$ be a lower bounded set. Then for any $\alpha \in \mathcal{A}$ there exists a finite subset $A_\alpha \subseteq G$ such that $\alpha = \wedge_{a \in A_\alpha} h(a)$. We put $A = \bigcup_{\alpha \in \mathcal{A}} A_\alpha$. Then A is lower bounded. Indeed, if $\beta \leqslant \mathcal{A}$ in Γ, then there exist $r_1, \dots, r_n \in G$ such that $\beta^{-1} = h(r_1) \wedge \dots \wedge h(r_n)$ and we have $\beta \geqslant h(r_i^{-1})$. Then for any $g \in G$ such that $h(g) \leqslant \beta \leqslant \mathcal{A}$ we have $h(g) \leqslant \alpha \leqslant h(a)$, $a \in A_\alpha$ and it follows that $g \leqslant A$.

Then $(h(A))_v = \mathcal{A}_v$. Indeed, let $\omega \in \mathcal{A}_v$ and let $\beta \leqslant h(A)$. Then $\beta \leqslant h(A_\alpha)$ and it follows that $\beta \leqslant \alpha$ for all $\alpha \in \mathcal{A}$. Hence, $\omega \geqslant \beta$ and $\omega \in (h(A))_v$. Conversely, let $\omega \in (h(A))_v$ and let $\beta \leqslant \mathcal{A}$. Then $\beta \leqslant \alpha \leqslant h(a)$ for all $\alpha \in \mathcal{A}, a \in A_\alpha$ and we have $\omega \geqslant \beta$ and $\omega \in \mathcal{A}_v$. Now, let $\alpha \in \mathcal{A}_v = (h(A))_v$, $\alpha = h(g_1) \wedge \dots \wedge h(g_m)$. Then $h(g_i) \in (h(A))_v$ and $g_i \in A_v$ for all i. Since $A_v = A_t$ in G, for any i there exists a finite subset $B^i \subseteq A$ such that $g_i \in B^i_t$. Let $B = \bigcup_i B^i \subseteq A$ and let $b \in B$. We set $\mathcal{A}_b = \{\beta \in \mathcal{A} : b \in A_\beta\}$, where $A_\beta \subseteq G$ is a finite subset such that $\beta = \wedge h(A_\beta)$. Then $\mathcal{A}_b \neq \emptyset$. Indeed, since $b \in A$, then $b \in A_\alpha$ for some $\alpha \in \mathcal{A}$ and $\alpha \in \mathcal{A}_b$. In any

$\mathcal{A}_b$ we choose one element β_b and set $\mathcal{B} = \{\beta_b : b \in B\}$. Then $\mathcal{B} \subseteq \mathcal{A}$ is a finite set. We show that $\alpha \in \mathcal{B}_t$. Let $\varrho \leqslant \mathcal{B}$ in Γ. Then $\varrho \leqslant h(B)$. Indeed, if $b \in B$ then $\beta_b \in \mathcal{B}$ and $\varrho \leqslant \beta_b = \wedge h(A_{\beta_b}) \leqslant h(b)$, since $b \in A_{\beta_b}$. Hence, $\varrho \leqslant h(B^i)$ for any i and since $g_i \in B_t^i$ and h is a (t,t)-morphism, we have $h(g_i) \geqslant \varrho$ for any i. Therefore, $\varrho \leqslant \alpha$ and $\alpha \in \mathcal{B}_t$. Hence, $\mathcal{A}_v = \mathcal{A}_t$ in Γ. □

P. Lorenzen introduced the notion of a group $\Lambda_r(G)$ in order to clarify a construction of the Kronecker function ring and to emphasize the multiplicative basis of this construction. Although both these constructions are widely used it seems to us that the explicit relationships between the Kronecker function ring and the Lorenzen group has not been published yet. Let R be an integral domain with the quotient field K and let R be defined by a family W of valuations. Then any valuation $w \in W$ may be extended onto an o-homomorphism (denoted again by w) from the group of divisibility $G(R)$ of R onto the o-group G_w of w. Let r be an r-system in $G(R)$ defined by these extended o-homomorphisms. Then any $w \in W$ is an r-valuation and r is regularly closed. Hence, the Lorenzen r-group $\Lambda_r(G(R))$ exists and any element of this group is of the form A_r/B_r, where A, B are finite subsets in $G(R)$. Let $K(R)$ be the Kronecker function ring of R.

Proposition 3.7. *The group of divisibility of $K(R)$ is o-isomorphic to $\Lambda_r(G(R))$.*

Proof. A full description of $G(K(R))$ was done by J. Ohm [20] who proved (in our notation) that if $d\colon G(R) \to \prod_{w\in W} G_w$ is a canonical embedding (and hence a (t,t)-morphism) then $G(K(R))$ is an l-ideal in $\prod_{w\in W} G_w$ generated by $d(G(R))$. Let $h\colon G(R) \to \Lambda_r(G(R))$ be an (r,t)-embedding, then d may be extended onto a (t,t)-morphism $\hat{d}\colon \Lambda_r(G(R)) \to \prod_{w\in W} G_w$. We show that $\hat{d}$ is injective. Indeed, if for a finite subsets A, B, C, D in $G(R)$ we have $\hat{d}(A_r/B_r) = \hat{d}(C_r/D_r)$, then we have $(d(A))_t.(d(D))_t = (d(C))_t.(d(B))_t$ and it follows that $(w(A))_t.(w(D))_t = (w(C))_t.(w(B))_t$ for all $w \in W$. Since r is defined by W, we have $A_r.D_r = C_r.B_r$ and $\hat{d}$ is injective. Then $\hat{d}(\Lambda_r(G(R)))$ is an l-group containing $d(G(R))$ and it follows that $G(K(R)) \subseteq \hat{d}(\Lambda_r(G(R)))$. Let $\alpha \in \Lambda_r(G(R))$, $\alpha = A_r/B_r$ where $A = \{a_1, \dots, a_n\}, B = \{b_1, \dots, b_m\}$. Then

$$\alpha = \frac{(a_1) \wedge \dots \wedge (a_n)}{(b_1) \wedge \dots \wedge (b_m)}$$

and

$$\hat{d}(\alpha) = \frac{d(a_1) \wedge \dots \wedge d(a_n)}{d(b_1) \wedge \dots \wedge d(b_m)}$$

is an element of an l-ideal generated by $d(G(R))$. Therefore, $\hat{d}(\Lambda_r(G(R))) = G(K(R))$ and $\Lambda_r(G(R)) \cong G(K(R))$. □

A Lorenzen r-group plays an important role even for r-Prüfer *po*-groups which were introduced by Aubert [2] and under a different name by P. Jaffard [12]. Recall that G with an r-system r of a finite character is called *r-Prüfer* if $(\mathcal{I}_r^f(G), \times)$ is a group. In [16] and [2] some characterizations of these *po*-groups were proved. In the next proposition we extend these results.

Proposition 3.8. *Let G be a po-group with an r-system of a finite character. Then the following conditions are equivalent.*

(1) *G is an r-Prüfer po-group.*
(2) *G/H is an o-group for any r-local o-ideal H of G.*
(3) *G/H is an o-group for any minimal r-local o-ideal H of G.*

Proof. (1)$\Longrightarrow$(2). Let H be an r-local o-ideal of G. Then according to [17]; 2.9, there exists a prime l-ideal Δ of $\Lambda_r(G)$ such that $G/H \cong \Lambda_r(G)/\Delta$. Hence, G/H is an o-group.

(2)$\Longrightarrow$(3). Trivial.

(3)$\Longrightarrow$(1). Let $a, b \in G$ and let H be a minimal r-local o-ideal of G and let, for example, $aH \leqslant bH$. Then according to [17]; 2.5, the r_H-system is a t-system and we have

$$(bH) = (aH) \cap (bH) = ((a) \cap (b))/H = ((a) \cap (b))_r/H,$$
$$(a,b)_r/H = (aH, bH)_{r_H} = (aH, bH)_t = (aH)$$

concluding

$$\begin{aligned} (a.b)_r/H &= (a.bH) = (aH) \times (bH) = (aH.bH)_{r_H} \times ((a) \cap (b))/H \\ &= (a,b)_r/H \times ((a) \cap (b))/H = [(a,b)_r \times ((a) \cap (b))]/H. \end{aligned} \tag{1}$$

Now, let H' be an arbitrary r-local o-ideal of G. According to [17]; 2.4, $G_+ \setminus H'$ is a prime r-ideal in G_+ and there exists a maximal r-ideal M containing this r-ideal. Let H be the quotient group of $G_+ \setminus M$ in G. Then H is the minimal r-local o-ideal of G, $H \subseteq H'$. It follows that (1) holds even for any r-local o-ideal of G. According to [17]; 2.8, we obtain that $(a.b)_r = (a,b)_r \times ((a) \cap (b))$ and it follows that $(a.b)_r$ is r-invertible. Hence, G is an r-Prüfer *po*-group. □

M. Griffin [9]; Th. 5. proved that an integral domain R is a PVMD (Prüfer v-multiplication domain) iff R_M is a valuation domain for each maximal t-ideal of R. In [6] it was proved that a notion of a PVMD is a purely multiplicative one, i.e. it may be defined by using the group of divisibility $G(R)$ of R only. Using the previous proposition we can give another proof of this Griffin result.

Proposition 3.9. (M.Griffin). *An integral domain R is a PVMD if and only if R_M is a valuation domain for each maximal t-ideal M of R.*

Proof. If R is a PVMD, according to [6]; 5.2, $G = G(R)$ admits a theory of quasi-divisors and is a t-Prüfer *po*-group. Let M be a maximal t-ideal of R. Then according to [6]; 4.7, $\mathcal{M} = w_R(M \setminus \{0\})$ is a maximal t-ideal of G, where w_R is a semi-valuation associated with R. Then according to 3.8, $G(R_M) \cong G/H$ is an o-group, where H is a t-local o-ideal generated by $G_+ \setminus \mathcal{M}$. The converse implication may be proved analogously. □

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