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A FORBIDDEN SUBGRAPHS CHARACTERIZATION AND A POLYNOMIAL ALGORITHM FOR RANDOMLY DECOMPOSABLE GRAPHS

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1. Introduction

In what follows the graphs considered are finite, without loops or multiple edges. For a given graph H, a graph G is said to be H-decomposable if there is a collection π of subgraphs of G, each of which is isomorphic to H, and whose edge-sets form a partition of the edge-set $E(G)$ of G. Let $D(H)$ denote the family of graphs which are H-decomposable. For a given graph H, a graph G is said to have H-factor if there is a collection π of vertex-disjoint subgraphs of G, each of which contains a spanning graph isomorphic to H and whose vertex sets form a partition of the vertex-set $V(G)$ of G. Let $F(H)$ be the family of all the graphs having an H-factor.

The problems of characterizing $D(H)$ or $F(H)$ are, by now, classical. In fact, Volume 1 of Journal of Graph Theory 9 (1985) is entierely devoted to factors and decompositions. Hence we refer the reader to this source [JGT] for a comprehensive survey of these problems. We shall mention here only a few results concerning the algorithmic complexity of the membership in $D(H)$ or $F(H)$.

Theorem A. (Hell and Kirkpatrick [JGT, p. 34].) *Let H be a graph having at least one component with more than two vertices. Then the problem "Does $G \in F(H)$" is NP-complete.*

The related decomposition problem has been solved only recently (June 1991) by M. Tarsi and D. Dor from Tel-Aviv University.

Theorem B. (Tarsi-Dor [TAD].) *Let H be a graph having at least one component with more than two edges. Then the problem "Does $G \in D(H)$" is NP-complete.*

[1] Professor Sergio Ruiz died tragically in an accident on 1.12. 1991.

As to the case where all the components of H have at most two vertices, respectively, two edges the following is known.

For such a graph H the problem "Does $G \in F(H)$" is in the class P, as readily seen using the $O(n^{2.5})$ algorithm to find the maximum matching in a graph, (see e.g. [EVK]).

The problem "Does $G \in D(H)$" is yet an intriguing open problem. Due to these facts about the NP-completeness of both $D(H)$ and $F(H)$ Sergio Ruiz [RU] introduced in 1985 the concept of random-decomposition which we extend here to cover also random-factors.

A graph $G \in D(H)$ is said to be randomly H-decomposable if any H-decomposition of a subgraph of G can be extended to an H-decomposition of G. Such graphs form the family $RD(H)$. A graph $G \in F(H)$ is said to have random H-factor if any H-factor of a subgraph of G can be extended to an H-factor of G. Such graphs form the family $RF(H)$. Much efforts have been done in the last years to characterize $RD(H)$ for various graphs, and as a result $RD(H)$ is known for $H \in \{K_{1,n}, nK_2, K_n, P_k,\ 3 \leqslant k \leqslant 6$ and $P_3 \cup K_2\}$. The details can be found in the works of Barrientos, Bernasconi, Jeltech, Ruiz, Smith, Kabell, Beineke, Goddard and Hamburger mentioned in the References. The only known result concerning $RF(H)$ is a 1979 result of Sumner [SU] who showed that the only connected graphs in $RF(K_2)$ are K_{2n} and $K_{n,n}$. His proof was rather technical and we shall give here a much simpler proof based on our forbidden subgraph technique. A closer inspection of the known cases of $RD(H)$ reveals that $RD(H)$ consists of graphs having simple structure, such as nK_2, $K_{1,n}$, K_n, $K_{n,n}$ and some finite exceptions. However, the following construction shows that in general this is not the case.

Construction. Let H be a 2-connected graph on n vertices. Let G be a graph with girth $g(G) > n$. Extend every edge to a copy of H in such a way that apart from vertices of G the copies of H are pairwise disjoint. Denote the resulting graph $G[H]$.

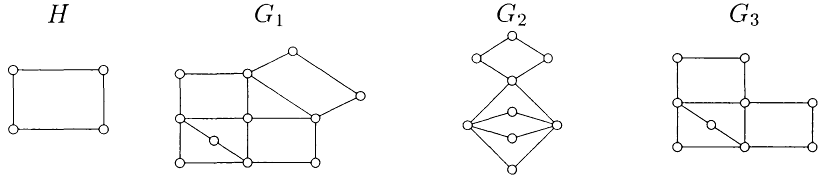
Clearly $G[H] \in RD(H)$ and the structure of $G[H]$ might be far from trivial. This fact convinced us that the first step to be taken is to consider the algorithmic complexity of deciding a membership in $RD(H)$ or $RF(H)$. Fortunately this happened to be polynomial as we shall see later, and the proof of this statement constitutes the main part of this paper.

2. The characterization theorem

We begin with some observations before presenting the main result. For any graph $G \in D(H) \setminus RD(H)$ there is at elast one minimal subgraph which also belongs to

$D(H) \setminus RD(H)$. We denote by $MD(H)$ the family of these minimal graphs for all such graphs G.

In Fig. 1, we exhibit examples of graphs in $D(H), RD(H)$ and $MD(H)$ where H is the 4-cycle. We note also that for any graph $G \in F(H) \setminus RF(H)$ there is at least one minimal subgraph which also belongs to $F(H) \setminus RF(H)$, this time minimality with respect to the number of vertices. We denote by $MF(H)$ the family of these minimal graphs for all such graphs G.



$G_1 \in D(H)$ $G_1 \notin RD(H)$ $G_1 \notin MD(H)$ $\quad G_2 \in RD(H) \quad G_3 \in MD(H)$

Fig. 1

In Fig. 2, we exhibit the family $MF(K_2)$.

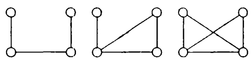
$MF(K_2) = \{$  $\}$

Fig. 2

Let's now present the main theorem.

Theorem 2.1. *Let H be a graph of size $q > 1$. Then there is a finite family $\mathscr{F}_H$ of graphs, each of which has size at most q^2, such that a graph $G \in D(H)$ is randomly H-decomposable if and only if G does not contain a member of $\mathscr{F}_H$ as a subgraph. Moreover, the problem "Does $G \in RD(H)$" is solvable in time $O(e^{q^2})$ where $e = |E(G)|$.*

Similarly to our proof of Theorem 2.1 one can prove the following result which we state as Theorem 2.2, without proof.

Theorem 2.2. *Let H be a graph on $m > 1$ vertices. Then there is a finite family $\mathscr{F}_H$ of graphs, each of which has order at most n^2, such that a graph $G \in F(H)$ has random H-factor if and only if G does not contain a member of $\mathscr{F}_H$ as an induced subgraph. Moreover the problem "Does $G \in RF(H)$" is solvable in time $O(n^{m^2})$ where $n = |V(G)|$.*

We shall give a detailed proof of Theorem 2.1.

We split the proof of theorem 2.1 into several lemmas.

Lemma 2.3. *Let $G \in RD(H)$. Then any H-decomposable subgraph of G is randomly H-decomposable.*

Proof. Let G' be a subgraph of G and $G' \in D(H)$. Take an arbitrary H-decomposition $\mathscr{F}'$ of a subgraph of G'. Since $G \in RD(H)$ and $G' \in D(H), G - G'$ has an H-decomposition $\mathscr{F}''$. Now using again the fact that $G \in RD(H)$, $\mathscr{F}' \cup \mathscr{F}''$ can be extended to an H-decomposition $\mathscr{F}' \cup \mathscr{F}'' \cup \mathscr{F}'''$ of G. Thus $\mathscr{F}' \cup \mathscr{F}'''$ is an H-decomposition of G' which extends $\mathscr{F}'$. This proves that $G' \in RD(H)$. □

An immediate consequence is this.

Corollary 2.4. *A graph G in $D(H)$ is not randomly H-decomposable iff G contains a subgraph in $D(H) - RD(H)$.*

Lemma 2.5. *A graph G in $D(H)$ is randomly H-decomposable if and only if $G - H' \in RD(H)$ for any subgraph $H' \cong H$ of G.*

Proof. The necessity follows from Lemma 2.3 to show the sufficiency take any subgraph G' of G which has an H-decomposition $\mathscr{F}'$. If $G' \cong H$ then $\mathscr{F}'$ extends to an H-decomposition of G by hypothesis. If G' contains more than one copy of H, say, $\mathscr{F}' = \{H_1, . . , H_k\}$, where $k > 1$, then $\mathscr{F}' - \{H_1\}$ extends to an H-decomposition $\{\mathscr{F}' - \{H_1\}) \cup \mathscr{F}''$ of $G - H_1 \in RD(H)$. Thus $\mathscr{F}'' \cup \mathscr{F}'$ is an H-decomposition of G. Therefore $G \in RD(H)$. □

Lemma 2.6. *A graph $G \in D(H)$ is randomly H-decomposable if and only if G does not contain a subgraph isomorphic to a member of $MD(H)$.*

This lemma can be proved with a straightforward argument using Lemma 2.3 and the definition of $MD(H)$.

Lemma 2.7. *Every graph G in $MD(H)$ has a subgraph H_0 isomorphic to H such that $G - H_0 \notin D(H)$.*

Proof. Suppose to the contrary that for any subgraph $H_0 \cong H$ of $G, G - H_0 \in D(H)$. The minimality of G implies that $G - H_0 \in RD(H)$. Thus, by Lemma 2.5 $G \in RD(H)$. A contradiction. □

We say that for a graph $G \in D(H)$ a subgraph $H' \cong H$ is a bad copy of H in G if $\{H'\}$ cannot be extended to an H-decomposition of G. Any other copy of H belonging to an H-decomposition is called a good copy of H in G.

Lemma 2.8. *Let $G \in MD(H)$ and let H_0 be a bad copy of H in G. Then any good copy of H has an edge in common with H_0.*

Proof. Assume that there is a bad copy H_0 of H, a good copy H_1 of H and that H_0 and H_1 share no edge. Since H_1 belongs to an H-decomposition of $G, G - H_1 \in D(H)$. Now, the minimality of G implies that $G - H_1 \in RD(H)$. Thus, $G - H_1 - H_0 \in D(H)$. Let $\mathscr{F}$ be an H-decomposition of $G - H_1 - H_0$. Note that $\{H_0\} \cup \{H_1\} \cup \mathscr{F}$ is an H-decomposition of G, contradicting that H_0 is a bad copy of H. □

Lemma 2.9. *Let H be a graph with q edges and let $G \in MD(H)$ then G has at most q^2 edges.*

Proof. Let H_0 be a bad copy of H in G and let $\mathscr{D}$ be an H-decomposition of G. All members of $\mathscr{D}$ are good copies of H and by the above lemma, they share edges with H_0. But as the members of $\mathscr{D}$ are edge-disjoint, $\mathscr{D}$ has at most q copies of H. Therefore G has at most q^2 edges. □

Proof of Theorem 2.1. Take $\mathscr{F}_H$ as the family $MD(H)$. This family is finite since by the former lemma, each member of $MD(H)$ has size q^2 at most. Now the theorem follows from Lemma 2.6. It remains to present a polynomial algorithm to decide membership in $RD(H)$.

Algorithm for $RD(H)$

Input: a fixed graph H on q edges, and a graph G on m edges to be tested for membership in $RD(H)$.

Step 1. Construct the family $MD(H)$, of minimal forbidden subgraphs. As $MD(H)$ is finite this would take $O(1)$ time.

Step 2. Construct the family $I(G : H)$ of all the copies of H in G. This would take $O(m^q)$ time.

Step 3. Verify for all subgraphs of G, of size at most q^2 their membership in $MD(H)$. This would take at most $O\Big(\sum_{j=1}^{q} \binom{m}{jq}\Big) = O\Big(\sum_{j=1}^{q} m^{jq}\Big) = O(m^{q^2})$ time.

Step 4. If any of the subgraphs in step 3 is in $MD(H)$ then clearly $G \notin RD(H)$.

Step 5. Use $I(G : H)$ of step 2, to delete one by one copies of H from G. This would take at most $O(m^q)$ time, (in fact much faster). If we get stuck in the process before accomplishing a full decomposition of G, then by definition $G \notin RD(H)$. Otherwise $G \in D(H)$ but contains no members of $MD(H)$ and hence by the first part of theorem 2.1 $G \in RD(H)$. Hence the overall complexity of this algorithm is at most $O(m^{q^2})$. □

Several remarks are in order now.

1. The content of theorems 2.1–2.2 can be generalized to cover the following situations.

a: $RD(H)$ and $RF(H)$, when H is a hypergraph of finite rank.

b: $RD(Q)$ and $RF(Q)$, where Q is a finite family of finite graphs with the obvious modification of the concept of decomposition and Q-factor.

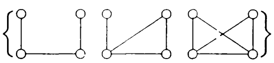
c: $RD(\vec{H})$ and $RF(\vec{H})$, where $\vec{H}$ is a directed graph and we deal with directed-graph decomposition.

Surely there are many more cases in which our method works with some minor changes.

2. One may hope that $MD(H)$ might not contain a graph of size q^2, which would imply an improvement in the running time of the algorithm. This is however not the case when H is connected. Just take a copy $H_0 \simeq H$ and on each of its edges construct a copy $H_i \simeq H$, $1 \leqslant i \leqslant q$, to form a graph G. Clearly $G \in MD(H)$ and $e(G) = q^2$. In fact it is also not hard to show that $|MD(H)|$ grows rather fast.

3. In 1979 Sumner [SU] characterized $RF(K_2)$. His proof is technical and there is no use of forbidden family of graphs. We shall present a proof using $MF(K_2)$ which is rather short and elegant.

Theorem [SU]. *The only connected graphs in $RF(K_2)$ and K_{2n} and $K_{n,n}$.*

Proof. Observe first that $MF(K_2) = \{$  $\}$.

Suppose first $\chi(G) = k \geqslant 3$. Let us consider a coloring in which $|V_1| \leqslant |V_2| \leqslant \ldots \leqslant |V_k|$, such that $|V_k|$ is as large as possible, then $|V_{k-1}|$ is as large as possible, etc. etc.

Consider $u_1 \in V_1$, u_1 must be adjacent to vertices $u_i \in V_i$, $2 \leqslant i \leqslant k$ for otherwise we can move u_1 to other class V_i, which is already as large as possible, which is a contradiction to our particular choice. If $|V_k| = 1$, we are done as $\chi(G) = k$ implies $G = K_k$. Hence assume $|V_k| \geqslant 2$. Consider $u_2 \in V_2$, it must be adjacent to some vertex $v \in V_k$ for the same reason as before. If $v \neq u_k$ then it follows that the graph induced by $\{u_1, u_2, v, u_k\}$ is forbidden. Hence u_2 is connected only to u_i in V_i, $3 \leqslant i \leqslant k$. Hence for $1 \leqslant j \leqslant k$ we showed that u_j is connected only to u_i, $1 \leqslant i \leqslant k$, $i \neq j$. Thus $\{u_1, u_2, \ldots, u_k\}$ must form a component which is a clique in G, but as G is connected $G = K_k$ and as $G \in RF(K_2)$ it follows that $G = K_{2n}$.

If $\chi(G) = 2$ consider a bipartition of G with classes A and B. Suppose $u \in A$, $v \in B$ are not adjacent. But as G is connected there is a shortest path from u to v which is an induced path of length at least 3, and G must contain an induced P_4 which is forbidden. Hence G is complete bipartite and it follows that $G = K_{n,n}$, proving the theorem. □

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