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Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 2, 331–333

Persistent URL: http://dml.cz/dmlcz/127295

Czechoslovak Mathematical Journal, 46 (121) 1996, Praha

CONSTRUCTION OF ALL HOMOMORPHISMS OF MONO-n-ARY ALGEBRAS

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(Received December 16, 1994)

In [3] a construction of all homomorphisms of a groupoid into another one is described. In the present paper we present a generalization of this result, i.e., a construction of all homomorphisms of an algebra with one n-ary operation into another algebra of the same type. The proofs are omitted because they may be easily obtained from the proofs of [3]. Our generalized construction is needed, e.g., if constructing all strong homomorphisms of a structure with one $n+1$-ary relation into another structure of the same type as described in Corollary 2 of [2].

Let n be an integer such that $n \geqslant 2$. If A is an arbitrary set, we denote by A^n the Cartesian product $\times\{A_i\,;\ 1 \leqslant i \leqslant n\}$ where $A_i = A$ for any i with $1 \leqslant i \leqslant n$.

Suppose that A, A' are sets and $n \geqslant 2$ an integer. A mapping f of A^n into $(A')^n$ is said to be *n-decomposable* if there exists a mapping h of A into A' such that $f(x_1, \ldots, x_n) = (h(x_1), \ldots, h(x_n))$ for any $(x_1, \ldots, x_n) \in A^n$. Then we write $f = h^n$.

Let $n \geqslant 1$ be an integer. We denote by (A, o) an algebraic structure where o is an n-ary operation on the set A. This structure will be called a *mono-n-ary algebra*. Furthermore, we denote by $\mathbf{ALG}n$ the category whose objects are mono-n-ary algebras and whose morphisms are homomorphisms of these algebras. (The symbol $\mathbf{ALG}n$ in [2] has a different meaning!)

Let A be a set, $n \geqslant 2$ an integer. A unary operation w on the set A^n is said to be *binding* if for any $(x_1, \ldots, x_n) \in A^n$ the condition $w(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ implies that $x_i = y_{i-1}$ for any i with $2 \leqslant i \leqslant n$. An algebra (A^n, w) with a binding unary operation w will be called a *binding unary n-algebra*.

We now define a category $\mathbf{MAP}n$. Its objects are binding unary n-algebras, its morphisms are n-decomposable homomorphisms of these n-algebras.

We now present a functor F of the category $\mathbf{ALG}n$ into $\mathbf{MAP}n$ by presenting the object mapping Fo and the morphism mapping Fm.

If (A, o) is an object in the category **ALG**n, we define $\mathbf{un}[o](x_1, \dots, x_n) = (x_2, \dots, x_n, o(x_1, \dots, x_n))$ for any $(x_1, \dots, x_n) \in A^n$. Clearly, $(A^n, \mathbf{un}[o])$ is an object in the category **MAP**n. We put

$$Fo(A, o) = (A^n, \mathbf{un}[o]).$$

Let (A, o), (A', o') be objects in **ALG**n, h a homomorphism of (A, o) into (A', o'). It is easy to see that h^n is a morphism of $Fo(A, o)$ into $Fo(A', o')$ in the category **MAP**n. We put

$$Fm(h) = h^n.$$

Similarly as Theorem 5 in [3] we obtain

Theorem. *Let $n \geqslant 2$ be an integer. The functor F is an isomorphism of the category* **ALG**n *onto the category* **MAP**n.

A generalization of Corollary 3 in [3] reads as follows.

Corollary. *Let $n \geqslant 2$ be an integer, (A, o), (A', o') mono-n-ary algebras.*

(i) *For any homomorphism h of (A, o) into (A', o') there exists an n-decomposable homomorphism f of $(A^n, \mathbf{un}[o])$ into $((A')^n, \mathbf{un}[o'])$ such that $f = h^n$.*

(ii) *If f is an n-decomposable homomorphism of $(A^n, \mathbf{un}[o])$ into $((A')^n, \mathbf{un}[o'])$, then $f = h^n$ and h is a homomorphism of (A, o) into (A', o').*

Construction from [3] may be generalized as follows.

Construction. Let $n \geqslant 2$ be an integer, let mono-n-ary algebras (A, o), (A', o') be given.

Construct the mono-unary algebras $(A^n, \mathbf{un}[o])$ and $((A')^n, \mathbf{un}[o'])$.

Construct all homomorphisms of $(A^n, \mathbf{un}[o])$ into $((A')^n, \mathbf{un}[o'])$ using the construction described in [1].

Test the constructed homomorphisms and reject all of them that are not n-decomposable.

For any n-decomposable homomorphism f of $(A^n, \mathbf{un}[o])$ into $((A')^n, \mathbf{un}[o'])$ construct the mapping h such that $f = h^n$.

By Corollary, we obtain that any constructed mapping h is a homomorphism of (A, o) into (A', o') and that any homomorphism of (A, o) into (A', o') can be constructed in this way.

Application. Let $n \geqslant 1$ be an integer, A, A' sets, t a relation of arity $n+1$ on A, t' a relation of the same arity on A'. In Corollary 2 of [2] a construction of all strong homomorphisms of the structure (A,t) into (A',t') is described: We construct mono-n-ary algebras $(\boldsymbol{P}(A), \mathbf{R}[t])$ and $(\boldsymbol{P}(A'), \mathbf{R}[t'])$ where $\boldsymbol{P}(A) = \{X; X \subseteq A\}$, $\mathbf{R}[t](X_1, \ldots, X_n) = \{x \in A; (x_1, \ldots, x_n, x) \in t, x_1 \in X_1, \ldots, x_n \in X_n\}$ for any $X_1, \ldots, X_n$ in $\boldsymbol{P}(A)$; $\boldsymbol{P}(A')$, $\mathbf{R}[t']$ are defined in a similar way. Furthermore, we construct all homomorphisms of the first algebra into the other using [1] or the presented Construction. Then we choose all of them that are totally additive and atom-preserving in the sense of [2]. Any of them defines a strong homomorphism of (A,t) into (A',t') and any strong homomorphism of (A,t) into (A',t') can be obtained in this way.

It is easy to see that the above constructed isomorphism F of the category $\mathbf{ALG}n$ onto $\mathbf{MAP}n$ is not the only possible isomorphism of $\mathbf{ALG}n$ onto a category of mono-unary algebras. The other isomorphisms define a relationship between mono-n-ary algebras and mono-unary algebras that is different from the relationship that has been presented here.

References

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