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ON THE EQUATION $x_{ap}^{(n)} = f(t,x)$

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The purpose of this paper is to prove an Aronszajn type theorem for the equation $(\mathrm{d}x/\mathrm{d}t)_{ap}^{(n)} = f(t,x)$ with the initial conditions, by using the Denjoy integral setting.

1. Introduction

The theory of the Denjoy-Perron integral (see [10]) makes it to possible to integrate an arbitrary derivative, i.e. for this type of integral the formula

$$\int_a^b f'(s)\,\mathrm{d}s = f(b) - f(a)$$

holds for every differentiable function $f\colon [a,b] \to \mathbb{R}$. Kurzweil [9], in 1957, and independently Henstock [7], in 1961, have showed that this integral can be defined by modifying Riemann's original definition.

The Denjoy-Perron integral has important applications in the theory of differential equations. In [9] Kurzweil used this type of integral to the study of generalized solutions of the Cauchy problem

$$x' = f(t,x), \qquad x(t_0) = x_0. \tag{1}$$

Recently Schwabik [11] showed that all known conditions for the existence of a generalized solution of (1) (cf. [4], [6], [8]) concern the case of a Carathéodory right hand side perturbed by a Denjoy-Perron integrable function.

On the other hand, in recent years papers have appeared (e.g. [3], [2]) concerning the problem (1) in which the usual derivative is replaced by the approximative one (see [10] for the definition). With this derivative the concept of the Denjoy integral (see [10]) is closely connected.

This paper is devoted to the study of the problem

$$(2)\qquad (\mathrm{d}x/\mathrm{d}t)^{(n)}_{ap} = f(t,x), \qquad x^{(i)}_{ap}(t_0) = x_i,\ i = 0,\dots,n-1,$$

where $I = [t_0, t_0 + a]$, $B = \{x \in \mathbb{R} \colon |x - x_0| \leqslant b\}$, $a, b > 0$, $f \colon I \times B \to \mathbb{R}$, and $(\mathrm{d}x/\mathrm{d}t)^{(n)}_{ap}$ denotes the n-th approximative derivative of x.

As a generalized solution of (2), defined on an interval $J \subset I$, we understand a function $x \colon J \to \mathbb{R}$ such that $x(t) \in B$ for $t \in J$, $x^{(n-1)}_{ap}$ is an ACG function (cf. [10]), $(\mathrm{d}x/\mathrm{d}t)^{(n)}_{ap} = f(t, x(t))$ for a.e. $t \in J$ and $x^{(i)}_{ap}(t_0) = x_i$, $i = 0, \dots, n-1$.

Equivalently, a function $x \colon J \to \mathbb{R}$ is a generalized solution of (2) if $x(t) \in B$ for $t \in J$ and

$$(3)\qquad x(t) = \sum_{i=0}^{n-1} \frac{(t-t_0)^i}{i!} x_i + \underbrace{(D)\int_{t_0}^{t} \mathrm{d}t\ (D)\int_{t_0}^{t} \mathrm{d}t \dots (D)\int_{t_0}^{t} f(t, x(t))\,\mathrm{d}t}_{n\text{-times}}$$

for every $t \in J$, where the sign "$(D)\int$" stands for the Denjoy integral.

In what follows we show that the set of all generalized solutions of (2) is R_δ, i.e. it is homeomorphic to the intersection of compact absolute retracts.

2. An Aronszajn type theorem

Let $f \colon I \times B \to \mathbb{R}$ be a function such that

(i) $t \to f(t,x)$ is a measurable function for every $x \in B$,

(ii) $x \to f(t,x)$ is a continuous function for a.e. $t \in I$,

(iii) there exist two Denjoy (shortly: D) integrable functions $m \colon I \to \mathbb{R}$, $M \colon I \to \mathbb{R}$ such that

$$m(t) \leqslant f(t,x) \leqslant M(t) \quad \text{for every } (t,x) \in I \times B.$$

Now, we prove the following

Theorem. *Under the above assumptions there exists an interval $J \subset I$ such the set of all generalized solutions of (2), defined on J, is R_δ.*

P r o o f. Our proof is based on the well known Vidossich theorem [12, Corollary 1.2].

First, we show that (3) is equivalent to the integral equation

$$(4)\qquad x(t) = \sum_{i=0}^{n-1} \frac{(t-t_0)^i}{i!} + \frac{1}{(n-1)!}(D)\int_{t_0}^{t} (t-s)^{n-1} f(s, x(s))\,\mathrm{d}s, \quad t \in I.$$

Let $n = 2$. For simplicity, denoting the variables of the integration by two different letters we can write (3) in the form

$$x(t) = \sum_{i=0}^{1} \frac{(t-t_0)^i}{i!} x_i + (D)\int_{t_0}^{t} \mathrm{d}t \ (D)\int_{t_0}^{t} f\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in J.$$

In view of [5, Th. 57, p. 69] we obtain

$$\begin{aligned} x(t) &= \sum_{i=0}^{1} \frac{(t-t_0)^i}{i!} x_i + (D)\int_{t_0}^{t} \mathrm{d}s \ (D)\int_{s}^{t} f\big(s, x(s)\big)\,\mathrm{d}t \\ &= \sum_{i=0}^{1} \frac{(t-t_0)^i}{i!} x_i + (D)\int_{t_0}^{t} f\big(s, x(s)\big)\,\mathrm{d}s \ (D)\int_{s}^{t} \mathrm{d}t \\ &= \sum_{i=0}^{1} \frac{(t-t_0)^i}{i!} x_i + (D)\int_{t_0}^{t} (t-s) f\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in I. \end{aligned}$$

Assume now that for $n-1$ the following formula is valid:

$$\begin{aligned} &\underbrace{(D)\int_{t_0}^{t} \mathrm{d}t \ (D)\int_{t_0}^{t} \mathrm{d}t \ldots (D)\int_{t_0}^{t} f\big(s, x(s)\big)\,\mathrm{d}s}_{(n-1)\text{-times}} \\ &\qquad = \frac{1}{(n-2)!} (D)\int_{t_0}^{t} (t-s)^{n-2} f\big(s, x(s)\big)\,\mathrm{d}s, \quad t \in I. \end{aligned}$$

Fix $t \in I$. It can be easily seen that the function

$$(s, w) \to \Phi_t^n(s, w) = \begin{cases} (w-s)^{n-2}, & t_0 \leqslant s \leqslant w, \\ & \\ 0, & w \leqslant s \leqslant t, \end{cases} \qquad w \in [t_0, t],$$

satisfies the inequality

$$\bigvee_{t_0}^{t} (\Phi_t^n) \leqslant P(w) \quad \text{for a.e. } w \in [t_0, t],$$

where $\bigvee_{t_0}^{t}(\Phi_t^n)$ denotes the variation of Φ_t^n on $[t_0, t]$ and P is the Lebesgue integrable function on $[t_0, t]$. Hence, again by [5, Th. 57, p. 69], we have

$$\begin{aligned}
&(D)\int_{t_0}^{t} \mathrm{d}t\,(D)\int_{t_0}^{t} \mathrm{d}t \ldots (D)\int_{t_0}^{t} f\big(s, x(s)\big)\,\mathrm{d}s \\
&\qquad = \frac{1}{(n-2)!}(D)\int_{t_0}^{t} \mathrm{d}t\,(D)\int_{t_0}^{t} (t-s)^{n-2} f\big(s, x(s)\big)\,\mathrm{d}s \\
&\qquad = \frac{1}{(n-2)!}(D)\int_{t_0}^{t} f\big(s, x(s)\big)\,\mathrm{d}s\,(D)\int_{s}^{t} (t-s)^{n-2}\,\mathrm{d}t \\
&\qquad = \frac{1}{(n-1)!}(D)\int_{t_0}^{t} (t-s)^{n-1} f\big(s, x(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Thus (3) and (4) are equivalent for n and, consequently, using mathematical induction we conclude that this equivalence is valid for each $n \geqslant 2$.

Choose a positive number d in such a way that $d \leqslant a$,

$$\begin{gathered}
-\frac{b}{2} \leqslant \frac{1}{(n-1)!}(D)\int_{t_0}^{t} (t-s)^{n-1} m(s)\,\mathrm{d}s, \\
\frac{1}{(n-1)!}(D)\int_{t_0}^{t} (t-s)^{n-1} M(s)\,\mathrm{d}s \leqslant \frac{b}{2}
\end{gathered}$$

and

$$-\frac{b}{2} \leqslant \sum_{i=1}^{n-1} \frac{(t-t_0)^i}{i!} x_i \leqslant \frac{b}{2}$$

for $t_0 \leqslant t \leqslant t_0 + d$.

Define

$$F(z)(t) = \sum_{i=0}^{n-1} \frac{(t-t_0)^i}{i!} x_i + \frac{1}{(n-1)!}(D)\int_{t_0}^{t} (t-s)^{n-1} f\big(s, z(s)\big)\,\mathrm{d}s,$$

$t \in J$, $z \in \tilde{B}$, where $\tilde{B} = \{z \in C(J, \mathbb{R})\colon |z(t) - x_0| \leqslant b, t \in J\}$ and $C(J, \mathbb{R})$ denotes the space of all continuous functions $J \to \mathbb{R}$ with the topology of uniform convergence.

The inequalities

$$\begin{aligned}
x_0 - b \leqslant x_0 &+ \sum_{i=1}^{n-1} \frac{(t-t_0)^i}{i!} x_i + \frac{1}{(n-1)!}(D)\int_{t_0}^{t} (t-s)^{n-1} m(s)\,\mathrm{d}s \leqslant F(z)(t) \\
&\leqslant x_0 + \sum_{i=1}^{n-1} \frac{(t-t_0)^i}{i!} x_i + \frac{1}{(n-1)!}(D)\int_{t_0}^{t} (t-s)^{n-1} M(s)\,\mathrm{d}s \leqslant x_0 + b,
\end{aligned}$$

$$
\begin{aligned}
&F(z)(t_1) - F(z)(t_2) \\
&\quad = \sum_{i=0}^{n-1} \frac{(t_1 - t_0)^i}{i!} x_i + \frac{1}{(n-1)!} (D)\int_{t_0}^{t_1} (t_1 - s)^{n-1} f\big(s, z(s)\big)\, \mathrm{d}s \\
&\quad\quad - \sum_{i=0}^{n-1} \frac{(t_2 - t_0)^i}{i!} x_i - \frac{1}{(n-1)!} (D)\int_{t_0}^{t_2} (t_2 - s)^{n-1} f\big(s, z(s)\big)\, \mathrm{d}s \\
&\quad = \sum_{i=0}^{n-1} \frac{x_i}{i!} [(t_1 - t_0)^i - (t_2 - t_0)^i] \\
&\quad\quad + \frac{1}{(n-1)!} (D)\int_{t_0}^{t_2} [(t_1 - s)^{n-1} - (t_2 - s)^{n-1}] f\big(s, z(s)\big)\, \mathrm{d}s \\
&\quad\quad + \frac{1}{(n-1)!} (D)\int_{t_2}^{t_1} (t_1 - s)^{n-1} f\big(s, z(s)\big)\, \mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=0}^{n-1} \frac{x_i}{i!} [(t_1 - t_0)^i - (t_2 - t_0)^i] \\
&\quad + \frac{1}{(n-1)!} (D)\int_{t_0}^{t_2} [(t_1 - s)^{n-1} - (t_2 - s)^{n-1}] m(s)\, \mathrm{d}s \\
&\quad + \frac{1}{(n-1)!} (D)\int_{t_2}^{t_1} (t_1 - s)^{n-1} m(s)\, \mathrm{d}s \\
&\leqslant F(z)(t_1) - F(z)(t_2) \leqslant \sum_{i=0}^{n-1} \frac{x_i}{i!} [(t_1 - t_0)^i - (t_2 - t_0)^i] \\
&\quad + \frac{1}{(n-1)!} (D)\int_{t_0}^{t_2} [(t_1 - s)^{n-1} - (t_2 - s)^{n-1}] M(s)\, \mathrm{d}s \\
&\quad + \frac{1}{(n-1)!} (D)\int_{t_2}^{t_1} (t_1 - s)^{n-1} M(s)\, \mathrm{d}s, \quad t, t_1, t_2 \in J,\ t_1 > t_2,\ z \in \tilde{B},
\end{aligned}
$$

imply that $F(\tilde{B}) \subset \tilde{B}$ and the family is equiuniformly continuous.

Now, we verify that F is continuous. Let $z_0 \in \tilde{B}$ and let (z_m) be a sequence such that $z_m \in \tilde{B}$ for $m \in \mathbb{N}$ and $z_m \to z_0$ as $m \to \infty$. Fix $t \in J$. Put $\varphi_t^m(s) = (t-s)^{n-1} f\big(s, z_m(s)\big)$, $\varphi_t^0(s) = (t-s)^{n-1} f\big(s, z_0(s)\big)$ for $s \in [0, t]$. Obviously $\varphi_t^m(s) \to \varphi_t^0(s)$ for a.e. $s \in [0, t]$ as $m \to \infty$. By the well known Dominated Convergence Theorem, we infer that $\lim_{m\to\infty} F(z_m)(t) = F(z)(t)$. Since $F(\tilde{B})$ is equiuniformly continuous, we deduce that the mapping F is continuous.

In view of the above it is clear that F satisfies the conditions of Vidossich's theorem and therefore the set S is R_δ. □

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