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ON MINIMUM LOCALLY n-(ARC)-STRONG DIGRAPHS

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1. INTRODUCTION

Extensive studies have been devoted to the (global) connectedness in graphs and digraphs, one of the most important properties that a graph or digraph can possess (see, for instance, the surveys [2] and [8]). In 1974, G. Chartrand and R.E. Pippert [4] first defined locally connected and locally n-connected graphs and obtained some interesting results. Following [4], a variety of research [9–14] has been devoted to locally connected graphs. Recently, we first extended the study of local connectedness to digraphs (see [5] and [6]). In [5], we defined the locally n-strong digraphs and the locally n-arc-strong digraphs (See section 2 for definitions.), generalized some results of Chartrand and Pippert, and established relationships between local connectedness and global connectedness in digraphs, among which are the following theorems:

Theorem A. *Any weakly connected and locally n-arc-strong digraph is $(n+1)$-arc-strong.*

Theorem B. *Any weakly connected and locally n-strong digraph is $(n+1)$-strong.*

The aim of this paper is to further the study of locally n-(arc)-strong digraphs. We shall determine the minimum locally n-(arc)-strong digraphs and the minimum locally n-(arc)-strong oriented graphs. [Note: A minimum digraph with some property $\mathscr{P}$ is a digraph with minimum number of arcs in the digraphs with the property $\mathscr{P}$ which have minimum number of vertices.] Moreover, some results concerning tournaments are obtained, and the converses of the above Theorems A and B are shown to be not true.

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2. Definitions

We follow the standard terminology and notation. A digraph $D = \big(V(D), A(D)\big)$ is a finite nonempty set $V(D)$ of vertices together with a (possibly empty) set $A(D)$ of ordered pairs of distinct vertices of D called arcs. An ordered pair $(u, v) \in A(D)$ is also called an arc from u to v. A digraph D is said to be weakly connected if its underlying undirected graph is connected. If there is a dipath from u to v for any pair u and v of vertices in D, then the digraph D is said to be strongly connected, or simply said to be strong. The subdigraph induced by a nonempty subset $W \subset V(D)$ is denoted $\langle W\rangle_D$. Let u, $v \in V(D)$. We say u is a neighbor of v if $(u, v) \in A(D)$ or $(v, u) \in A(D)$. The set of neighbors of v in D is denoted $N_D(v)$. The induced subdigraph $\langle N_D(v)\rangle_D$ is said to be the neighborhood of v. The outdegree of v is denoted as $\operatorname{od} v$ and the indegree of v is denoted as $\operatorname{id} v$. Let $\delta(D) = \min\limits_{v\in V(D)} \{\operatorname{id} v, \operatorname{od} v\}$. If $\operatorname{id} v = \operatorname{od} v = \delta(D)$ for all $v \in V(D)$, D is said to be diregular. Let S and T be two disjoint proper subsets of $V(D)$. We use $(S, T)_D$ to denote the set of arcs (s, t) in D with $s \in S$ and $t \in T$. When there is no confusion, we may simply use $\langle W\rangle$, $\langle N(v)\rangle$ and (S, T) to denote the corresponding $\langle W\rangle_D$, $\langle N_D(v)\rangle_D$ and $(S, T)_D$, respectively.

Let $n \geqslant 1$. A digraph D is said to be n-strong [n-arc-strong, resp.] if the removal of fewer than n vertices [arcs, resp.] always results in a nontrivial strong digraph. Clearly, every n-strong digraph is n-arc-strong. Every n-strong [n-arc-strong, resp.] digraph is also m-strong [m-arc-strong, resp.] for $1 \leqslant m < n$. It should also be noted that D is 1-strong iff D is 1-arc-strong iff D is a nontrivial strong digraph. The trivial strong digraph consisting of a single vertex is the only digraph that is strong but not 1-strong (or not 1-arc-strong).

A digraph D is said to be locally strong [locally n-strong, locally n-arc-strong, resp.] if the neighborhood of every vertex of D is strong [n-strong, n-arc-strong, resp.].

The associated digraph of a graph G, denoted as $D(G)$, is the digraph obtained from G when each edge e of G is replaced by a pair of oppositely oriented arcs with the same ends as e.

For other terminologies not defined here we refer the reader to the book [3].

3. Main results

Theorem 1. *The associated digraph $D(K_{n+2})$ of the complete graph K_{n+2} is both the unique minimum locally n-strong digraph and the unique minimum locally n-arc-strong digraph.*

Before giving the proof of Theorem 1, we list some needed simple facts as the following propositions.

Proposition 1. *Let D be an n-(arc)-strong digraph. Then $\delta(D) \geqslant n$, $|V(D)| \geqslant n+1$, and $|A(D)| \geqslant n(n+1)$.*

The proof is easy and is omitted here.

From Proposition 1, we immediately get

Proposition 2. *The associated digraph $D(K_{n+1})$ is both the unique minimum n-strong digraph and the unique minimum n-arc-strong digraph.*

Proof. Clearly, $D(K_{n+1})$ is n-strong and n-arc-strong. Both the vertex number and the arc number reach the lower bounds given in Proposition 1. □

Proposition 3. *Let D be a locally n-(arc)-strong digraph. Then $\delta(D) \geqslant n+1$, $|V(D)| \geqslant n+2$, and $|A(D)| \geqslant (n+1)(n+2)$.*

Proof. By Theorem A and Proposition 1. □

Now the proof of Theorem 1 goes as follows.

Proof of Theorem 1. From Proposition 2, $D(K_{n+2})$ is locally n-strong and locally n-arc-strong. Since both the vertex number and the arc number of $D(K_{n+2})$ reach the lower bounds given in Proposition 3, $D(K_{n+2})$ is a minimum locally n-strong and minimum locally n-(arc)-strong digraph.

The uniqueness is easily seen from the following:

If D is a minimum locally n-(arc)-strong digraph, then by Proposition 3, $\delta(D) \geqslant n+1$. Note that $|V(D)|$ must be not greater than the vertex number of $D(K_{n+2})$. Then, $|V(D)| = n+2$. Thus we must have $\operatorname{od} v = \operatorname{id} v = n+1$ for all vertices in D. Therefore, $D = D(K_{n+2})$. □

Now we turn to determine the minimum locally n-(arc)-strong oriented graphs. Recall that a digraph is said to be an oriented graph if its underlying graph is a simple graph. Such digraphs are widely used in applications of graph theory.

Theorem 2. *A digraph D is a minimum locally n-arc-strong oriented graph if and only if D is a diregular tournament of $2n+3$ vertices.*

In the proof, we need the following lemmas where Lemma 1 is a rewritten version of a known result in [1].

Lemma 1. *Let D be an oriented graph. If $\delta(D) \geqslant \left\lfloor \frac{|V(D)|+2}{4} \right\rfloor$, then D is $\delta(D)$-arc-strong.*

Lemma 2. *Let D be a locally n-arc-strong oriented graph. Then $\delta(D) \geqslant n+1$, $|V(D)| \geqslant 2n+3$, and $|A(D)| \geqslant (n+1)(2n+3)$.*

Proof. By Propositions 3, $\delta(D) \geqslant n+1$. Then the other two inequalities immediately follow since D is an oriented graph. □

Now the proof of Theorem 2 goes as follows.

Proof of Theorem 2. We first prove the sufficiency. Let D be a diregular tournament of $2n+3$ vertices. By Lemma 1, it is easy to see that every neighborhood of a vertex in D is n-arc-strong. So, D is locally n-arc-strong. Since $|V(D)| = 2n+3$ and $A(D) = (n+1)(2n+3)$, D is a minimum locally n-arc-strong oriented graph by Lemma 2.

Now we prove the necessity. Let D be a minimum locally n-arc-strong oriented graph. Since we have proved that a diregular tournament of $2n+3$ vertices is a minimum locally n-arc-strong oriented graph, we have $|V(D)| = 2n+3$, $|A(D)| = (n+1)(2n+3)$. By Lemma 2, $\delta(D) \geqslant n+1$. Then we must have $\operatorname{id} v = \operatorname{od} v = n+1$ for any vertex v in D. Therefore, D is a diregular tournament of $2n+3$ vertices. □

For the minimum locally n-strong oriented graphs, we have the following result.

Theorem 3. *Every minimum locally n-strong oriented graph is a diregular tournament of $2n+3$ vertices.*

Before giving the proof we need to give a lemma, which also has its own interest.

Lemma 3. *Let D be a tournament. Then D is locally n-strong if and only if D is $(n+1)$-strong.*

Proof. The necessity is immediately seen from Theorem B. We only need to show the sufficiency.

Assume there is a tournament D which is $(n+1)$-strong but not locally n-strong. Then, there is a vertex v in D such that $\langle N(v)\rangle$ is not n-strong. Thus, we can find a proper subset S of $N(v)$ such that $|S| \leqslant n-1$ and $\langle N(v)\rangle - S$ is not strong. Let $S' = S \cup \{v\}$. Then $|S'| \leqslant n$, and $D - S' = \langle N(v)\rangle - S$ since D is a tournament. Thus, $D - S'$ is not strong, which contradicts the assumption that D is $(n+1)$-strong. It completes the proof of Lemma 3. □

Now we prove Theorem 3 as follows.

Proof of Theorem 3. First we claim that for any positive integer n, there exists a diregular tournament of $2n+3$ vertices which is locally n-strong. For instance, we may consider the right Cayley digraph $L(Z_{2n+3}, \{1, 2, \ldots, n+1\})$ which is a diregular tournament of $2n+3$ vertices. (Recall that for an additive group G and $S \subseteq G \setminus \{0\}$, the right Cayley digraph $L(G, S)$ is a digraph D with $V(D) = G$ and $A(D) = \{(x, x+y) \colon y \in S\}$.) By a result of Y. O. Hamidoune [7, Proposition 5.1], $L(Z_{2n+3}, \{1, 2, \ldots, n+1\})$ is $(n+1)$-strong. Then it is locally n-strong by Lemma 3. So, our claim is true.

Let D be a minimum locally n-strong oriented graph. By the above claim, $|V(D)| \leqslant 2n+3$ and $|A(D)| \leqslant (n+1)(2n+3)$. Then by Lemma 2, we must have $|V(D)| = 2n+3$ and $|A(D)| = (n+1)(2n+3)$. Moreover, from Lemma 2, $\delta(D) \geqslant n+1$. Then we must have $\operatorname{id} v = \operatorname{od} v = n+1$ for every vertex v in D. Therefore, D is a diregular tournament of $2n+3$ vertices. □

Remark 1. From Lemma 3, it seems natural to pose the following conjecture: Let D be a tournament. Then D is locally n-arc-strong if and only if D is $(n+1)$-arc-strong.

However, this conjecture is false, which can be seen from Proposition 4 given at the end of this paper.

Note that Theorem 3 only gives a result parallel to the necessity part of Theorem 2. In fact, the converse of Theorem 3 does not hold for $n \geqslant 3$. It can be seen from the following result.

Theorem 4. *For any integer $n \geqslant 3$, there exists a diregular tournament of $2n+3$ vertices which is not locally n-strong.*

Proof. We proceed in two steps.

Step 1. By induction on n, show that there is a diregular tournament D_{2n+3} of $2n+3$ vertices satisfying the following conditions: $V(D_{2n+3}) = X_n \cup Y_n \cup C$ where X_n, Y_n and C are pairwise disjoint, $|X_n| = |Y_n| = n$ and $\langle C \rangle$ is a dicycle of length 3; and $A(D_{2n+3}) \supset (X_n, C) \cup (C, Y_n)$.

For $n = 3$, the desired D_9 can be constructed as follows. Take three pairwise disjoint dicycles of length 3 and denote their vertex sets as X_3, Y_3 and C. Then add all arcs in $(X_3, C) \cup (C, Y_3) \cup (Y_3, X_3)$. It can be easily verified that this digraph is the desired D_9.

Now, assume that D_{2k+3} has been constructed for $k \geqslant 3$. We construct a new tournament of two more vertices as follows. First, we add two new vertices x and y and add arcs $(y, x) \cup (\{x\}, C) \cup (C, \{y\})$ so that we have $\operatorname{od} x - \operatorname{id} x = 2$, $\operatorname{id} y - \operatorname{od} y = 2$ and $\operatorname{od} v = \operatorname{id} v$ for every $v \in C$. Then, we arbitrarily take a subset $S \subset X_k \cup Y_k$ with $|S| = k+1$, and let $\overline{S} = (X_k \cup Y_k) - S$. Clearly, $|S| - |\overline{S}| = 2$. Then we add arcs $(S, \{x\}) \cup (\{x\}, \overline{S}) \cup (\{y\}, S) \cup (\overline{S}, \{y\})$. Let $X_{k+1} = X_k \cup \{x\}$ and $Y_{k+1} = Y_k \cup \{y\}$. Then it is easily seen that the obtained digraph is the desired D_{2n+3}. This completes the induction.

Step 2. Show that D_{2n+3} is not locally n-strong.

Let $D = D_{2n+3} - X_n$. Then $V(D)$ can be decomposed as two disjoint subsets C and Y_n. Since $(Y_n, C) = \emptyset$, D is not strong. Note that $|X_n| = n$. Then we see that D_{2n+3} is not $(n+1)$-strong. Hence, it is not locally n-strong by Theorem B.

It completes the proof of Theorem 4. □

Remark 2. The condition $n \geqslant 3$ in Theorem 4 is necessary since any diregular tournament of 5 (7, resp.) vertices is easily seen to be locally 1-strong (locally 2-strong, resp.). Therefore, the converse of Theorem 3 only holds for $n = 1, 2$.

Remark 3. It should be noted that the conclusion in Lemma 3 is not true for general digraphs, i.e., the converses of Theorems A and B are not true, which can be seen from the associated digraphs $D(K_{n+1,n+k})$ of the complete bipartite graphs $K_{n+1,n+k}$ (with $k \geqslant 1$).

It is easy to see the following facts:

(a) G is connected iff $D(G)$ is strong;

(b) G is n-connected iff $D(G)$ is n-strong;

(c) G is n-edge-connected iff $D(G)$ is n-arc-strong;

(d) G is locally n-connected iff $D(G)$ is locally n-strong;

(e) G is locally n-edge-connected iff $D(G)$ is locally n-arc-strong (Note: G is said to be locally n-edge-connected if the neighborhood of every vertex of G is n-edge-connected.)

From these relationships between G and $D(G)$, we can easily see that $D(K_{n+1,n+k})$ is $(n+1)$-strong and $(n+1)$-arc-strong but not locally n-(arc)-strong, since $K_{n+1,n+k}$ ($k \geqslant 1$) is $(n+1)$-connected and $(n+1)$-edge-connected but not locally n-(edge)-connected.

Finally, let us go back to the conjecture mentioned earlier. It is disproved by the following result:

Proposition 4. *For any integer $n \geqslant 1$, there is a tournament which is $(n+1)$-arc-strong but not locally n-arc-strong.*

Proof. Let D be a diregular tournament of $2n+3$ vertices. Let S be a subset of $V(D)$ with $|S| = n-1$, and let $\overline{S} = V(D) - S$. Then $|\overline{S}| = n+4$. Let D_1 be an isomorphic copy of D under the isomorphism $\varphi\colon V(D) \to V(D_1)$. Let $S_1 = \varphi(S)$ and $\overline{S}_1 = \varphi(\overline{S})$. Then we extend the digraph $D \cup D_1$ to a tournament H by adding arcs between $V(D)$ and $V(D_1)$ so that it satisfies the condition $\big(V(D), V(D_1)\big) = \big\{\big(x, \varphi(x)\big) | x \in S\big\}$. Then by Lemma 1 of [5] (which says that a digraph D is n-arc-strong if and only if $|(S, \overline{S})_D| \geqslant n$ for every nonempty proper subset S of $V(D)$ (where $\overline{S} = V(D) - S$), we see that H is not n-arc-strong since $\big|\big(V(D), V(D_1)\big)\big| =$

$|S| = n - 1 < n$. Now we construct the desired tournament T from H by adding a new vertex x and adding all the arcs in $(\{x\}, S \cup \overline{S}_1) \cup (\overline{S} \cup S_1, \{x\})$. It is easily seen that $\delta(T) = n + 2$.

Note that $\left\lfloor \frac{V(T)+2}{4} \right\rfloor = \left\lfloor \frac{(2(2n+3)+1)+2}{4} \right\rfloor = n + 2$. Then by Lemma 1, T is $\delta(T)$-arc-strong, implying that T is $(n+1)$-arc-strong. However, since $N_T(x) = H$, T is not locally n-arc-strong. $\square$

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