Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 2, 291–308

Persistent URL: `http://dml.cz/dmlcz/127291`

Czechoslovak Mathematical Journal, 46 (121) 1996, Praha

RETRACT IRREDUCIBILITY OF CONNECTED MONOUNARY ALGEBRAS I

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(Received July 11, 1994)

For some types of mathematical structures the relations between retracts and direct product decompositions have been studied (cf., e.g., [1] for the case of ordered sets, [2] and [5] for the case of graphs and [7] for the case of metric spaces). In the present paper we deal with a question concerning these relations for the case of connected monounary algebras.

Let (A, f) be a monounary algebra. As usual, a nonempty subset M of A is said to be a retract of (A, f) if there is a mapping h of A onto M such that h is an endomorphism of (A, f) and $h(x) = x$ for each $x \in M$. The mapping h is then called a retraction endomorphism corresponding to the retract M. Further, let $R\Big(A, f\Big)$ be the system of all monounary algebras (B, g) such that (B, g) is isomorphic to (M, f) for some retract M of (A, f).

In Section 1 (Theorem 1.3) we characterize retracts of a monounary algebra (A, f) by means of properties of degrees of elements of A.

In the remaining sections we deal with the notion of retract irreducibility of a connected monounary algebra. It is defined as follows. A connected monounary algebra $\mathcal{A}$ will be said to be retract irreducible if, whenever $\mathcal{A} \in R\Big(\prod_{i \in I} \mathcal{A}_i\Big)$ for some connected monounary algebras $\mathcal{A}_i$, then there exists $j \in I$ such that $\mathcal{A} \in R\mathcal{A}_j$. If this condition is not satisfied, then $\mathcal{A}$ will be called retract reducible.

The following result will be proved:

(R). *Let $\mathcal{A} = (A, f)$ be a connected monounary algebra possessing a one element cycle $\{c\}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is retract irreducible;*

(ii) *if a and b are elements of A such that $f(a) = f(b)$, then either $a = b$ or $c \in \{a, b\}$.*

The case when $\mathcal{A}$ has no one-element cycle will be dealt within Part II.

In some proofs we essentially apply the results and methods of M. Novotný [8], [9] concerning homorphisms of monounary algebras. Homomorphisms of monounary algebras were investigated also in [3], [4], [6].

1. Retracts

Let (A, f) be a monounary algebra. The aim of this section is to describe all retracts of (A, f).

Let us remark that if M is a retract of (A, f), then (M, f) is a subalgebra of (A, f).

The notion of degree $s_f(x)$ of an element $x \in A$ was introduced in [8] (cf. also [6] and [4]) as follows. Let us denote by $A^{(\infty)}$ the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $A^{(0)} = \{x \in A\colon f^{-1}(x) = \emptyset\}$. Now we define a set $A^{(\lambda)} \subseteq A$ for each ordinal λ by induction. Assume that we have defined $A^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$A^{(\lambda)} = \Big\{x \in A - \bigcup_{\alpha<\lambda} A^{(\alpha)}\colon f^{-1}(x) \subseteq \bigcup_{\alpha<\lambda} A^{(\alpha)}\Big\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal λ with $x \in A^{(\lambda)}$. In the former case we put $s_f(x) = \infty$, in the latter we set $s_f(x) = \lambda$. We put $\lambda < \infty$ for each ordinal λ.

The following assertions are consequences of the definition of $s_f(x)$ (cf. also [9]) and we will use them without further reference:

1) If $s_f(x) \neq \infty$, then $s_f(f(x)) > s_f(x)$.

2) If h is a homomorphism of (A, f) into (B, f), then $s_f(h(x)) \geqslant s_f(x)$ for each $x \in A$.

3) Let $\{(A_i, f)\colon i \in I\}$ be a system of monounary algebras. If $y, z \in \prod_{i\in I} A_i$, $s_f(y(i)) \leqslant s_f(z(i))$ for each $i \in I$, then $s_f(y) \leqslant s_f(z)$.

1.1. Lemma. *Let (A, f) be a monounary algebra and let M be a retract of (A, f). If $y \in f^{-1}(M)$, then there is $z \in M$ with $f(y) = f(z)$, $s_f(y) \leqslant s_f(z)$.*

Proof. Let $x \in M, y \in f^{-1}(x)$. The set M is a retract; let h be the corresponding retraction endomorphism. Then $h(x) = x$. Put $h(y) = z$. We obtain

$$f(z) = f(h(y)) = h(f(y)) = h(x) = x.$$

Further, $z = h(y) \in M$ and $s_f(y) \leqslant s_f(h(y)) = s_f(z)$. □

1.2.1. Lemma. *Let (A, f) be a connected monounary algebra and let (M, f) be a subalgebra of (A, f). Suppose that if $y \in f^{-1}(M)$, then there is $z \in M$ with $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$. For $n \in \mathbb{N}$ we denote by Y_n the set of all $y \in A$ such that $y \in f^{-n}(M) - M$ and $y \notin f^{-m}(M)$ for any $m \in \mathbb{N}$, $m < n$. Let $Y = \bigcup_{n \in \mathbb{N}} Y_n$. There exists a mapping $\varphi\colon Y \to M$ such that, whenever $n \in \mathbb{N}$, $y \in Y_n$, then*

(i) $f^n(y) = f^n(\varphi(y))$,

(ii) $s_f(y) \leqslant s_f(\varphi(y))$,

(iii) $\varphi(f^k(y)) = f^k(\varphi(y))$ *for each* $k \in \mathbb{N}$, $k < n$.

Proof. If $n = 1$, $y \in f^{-1}(M) - M$, then there is $z \in M$ with $f(y) = f(z)$, $s_f(y) \leqslant s_f(z)$; we can put $\varphi(y) = z$.

Let $n \in \mathbb{N}$, $n > 1$, $y \in Y_n$. Then

$$y \in f^{-n}(M) - M,\ y \notin f^{-m}(M) \quad \text{for each } m \in \mathbb{N},\ m < n, \tag{1}$$

which implies

$$f(y) \in f^{-(n-1)}(M) - M,\ f(y) \notin f^{-m}(M) \quad \text{for each } m \in \mathbb{N},\ m < n - 1, \tag{2}$$

$$f(y) \in Y_{n-1}. \tag{3}$$

Analogously,

$$f^2(y) \in Y_{n-2},\ \ldots,\ f^{n-1}(y) \in Y_1. \tag{4}$$

Suppose that if $m \in \mathbb{N}$, $m < n$, $t \in Y_m$, then $\varphi(t) \in M$ is defined and

(a) $f^m(t) = f^m(\varphi(t))$,

(b) $s_f(t) \leqslant s_f(\varphi(t))$,

(c) $\varphi(f^k(t)) = f^k(\varphi(t))$ for each $k \in \mathbb{N}$, $k < m$.

Take $y' = f(y)$. By the induction hypothesis and (3),

$$f^{n-1}(y') = f^{n-1}(\varphi(y')), \tag{5}$$

$$s_f(y') \leqslant s_f(\varphi(y')), \tag{6}$$

$$\varphi(f^k(y')) = f^k(\varphi(y')) \quad \text{for each } k \in \mathbb{N},\ k < n - 1. \tag{7}$$

Put $\varphi(y') = z'$. Let

$$S = \{x \in f^{-1}(z')\colon s_f(y) \leqslant s_f(x)\}.$$

If $s_f(y) = \infty$, then $s_f(y') = \infty$ and (6) yields that $s_f(z') = \infty$. Then there is $x \in f^{-1}(z')$ with $s_f(x) = \infty$, i.e., $x \in S$. Let $s_f(y) < \infty$ and suppose that $S = \emptyset$. We obtain one of the following relations:

$$s_f(y') > s_f(y) \geqslant \sup\{s_f(x)\colon x \in f^{-1}(z')\} = s_f(z'), \tag{8.1}$$

$$s_f(y') > s_f(y) > \max\{s_f(x)\colon x \in f^{-1}(z')\} = s_f(z') - 1, \tag{8.2}$$

a contradiction to (6). Thus

$$S \neq \emptyset. \tag{9}$$

Further, $S \cap M \neq \emptyset$, since if $x \in S - M$, then $z' \in M$ implies that $x \in f^{-1}(M)$ and there is (by the assumption) $t \in M$ with

$$f(x) = f(t) \quad \text{and} \quad s_f(x) \leqslant s_f(t).$$

Hence $S \cap M \neq \emptyset$; take $x \in S \cap M$ and put $\varphi(y) = x$. Then (5) implies

(i) $f^n(y) = f^{n-1}(f(y)) = f^{n-1}(y') = f^{n-1}(z') = f^{n-1}(f(x)) = f^n(x) = f^n(\varphi(y))$. Since $x \in S$, we have

(ii) $s_f(y) \leqslant s_f(x)$.

According to (7),

(iii) $\varphi(f^k(y)) = \varphi(f^{k-1}(y')) = f^{k-1}(\varphi(y')) = f^{k-1}(z') = f^k(x) = f^k(\varphi(y))$ for each $k \in \mathbb{N}$, $k < n$. □

1.2.2. Lemma. *Let the assumption of* 1.2.1 *be valid. Then M is a retract of (A, f).*

P r o o f. Let $a \in A$. Since (M, f) is a subalgebra of (A, f), we obtain that either $a \in M$ or there is $n \in \mathbb{N}$ such that

$$a \in f^{-n}(M) - M \text{ and } a \notin f^{-m}(M) \text{ for any } m \in \mathbb{N}, m < n,$$

i.e., $a \in Y_n$. Put

$$h(a) = \begin{cases} a & \text{if } a \in M, \\ \varphi(a) & \text{if } a \in Y_n,\ n \in \mathbb{N}. \end{cases}$$

According to 1.2.1, h is a mapping of A onto M. If $a \in M$, then obviously $h(f(a)) = f(h(a))$. Let $a \in Y_1$. Then $f(a) \in M$. By 1.2.1(i), $f(a) = f(\varphi(a))$ and we obtain

$$h(f(a)) = f(a) = f(\varphi(a)) = f(h(a)).$$

If $a \in Y_n$, $n > 1$, then $f(a) \in Y_{n-1}$ and 1.2.1(iii) yields

$$h(f(a)) = \varphi(f(a)) = f(\varphi(a)) = f(h(a)).$$

Therefore M is a retract of (A, f). □

1.2. Corollary. *Let (A, f) be a connected monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following condition is satisfied:*

(1) *if $y \in f^{-1}(M)$, then there is $z \in M$ with $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$.*

Proof. The assertion is obtained by virtue of 1.1 and 1.2.2. □

1.3. Theorem. *Let (A, f) be a monounary algebra and let (M, f) be a subalgebra of (A, f). Then M is a retract of (A, f) if and only if the following conditions are satisfied:*

(a) *If $y \in f^{-1}(M)$, then there is $z \in M$ such that $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$.*

(b) *For any connected component K of (A, f) with $K \cap M = \emptyset$, the following conditions are satisfied.*

  (b1) *If K contains a cycle with d elements, then there is a connected component K' of (A, f) with $K' \cap M \neq \emptyset$ and there is $n \in \mathbb{N}$ such that n/d and K' has a cycle with n elements.*

  (b2) *If K contains no cycle and x_0 is a fixed element of K, then there is $y_0 \in M$ such that $s_f(f^k(x_0)) \leqslant s_f(f^k(y_0))$ for each $k \in \mathbb{N} \cup \{0\}$.*

Proof. Let M be a retract of (A, f). By 1.1, the condition (a) is fulfilled. Suppose that h is the corresponding retraction endomorphism. Let K be a connected component of (A, f) such that $K \cap M = \emptyset$. If K contains a cycle with d elements, then there is a connected component K' of (A, f) such that $h(K) \subseteq K'$, K' contains a cycle with n elements, n/d. Obviously, $h(K) \subseteq M$, thus $K' \cap M \neq \emptyset$. If K contains no cycle and $x_0 \in K$, $y_0 = h(x_0)$, then $y_0 \in M$ and from the fact that h is an endomorphism of (A, f) we get

$$s_f(f^k(x_0)) \leqslant s_f(f^k(y_0)) \text{ for each } k \in \mathbb{N} \cup \{0\}.$$

Conversely, suppose that the conditions (a),(b) are satisfied. We will construct a retraction endomorphism h of (A, f) corresponding to M. Consider a connected component K of (A, f). We have to define a homomorphism of (K, f) onto (M, f).

A) Let $K \cap M \neq \emptyset$. Put $M' = K \cap M$. Then we proceed as in 1.2.2, only with M' instead of M and (K, f) instead of (A, f). The obtained mapping $h\colon K \to M'$ is an endomorphism, $h(a) = a$ for each $a \in M'$.

B) Let $K \cap M = \emptyset$. If K contains a cycle, then (b1) and [8], Thm. 2.14 imply that there is a connected component K' of (A, f) with $K' \cap M \neq \emptyset$ and there is a homomorphism g of (K, f) into (K', f). If K contains no cycle, then the existence of such K' and g follows from (b2) and [8], Thm. 2.14.

We have $K' \cap M \neq \emptyset$, thus (by A) there exists a homomorphism $h\colon K' \to K' \cap M$. Then $g \circ h$ is a homomorphism of (K, f) onto $(K' \cap M, f)$.

Therefore M is a retract of irreducibility as introduced above. □

2. Retract irreducible (A, f)

We apply the notion of retract irreducibility as introduced above.

Assumption. In what follows in the present Part I suppose that (A, f) is a connected monounary algebra possessing a cycle $\{c\}$.

2.0. Lemma. *Let (A, f) consist of a one-element cycle. Then (A, f) is retract irreducible.*

Proof. Suppose that $(A, f) \in R\Big(\prod_{i\in I}(B_i, f)\Big)$, where (B_i, f) is a connected monounary algebra for each $i \in I$. Then (A, f) is isomorphic to a subalgebra of $\prod_{i\in I}(B_i, f)$, thus there is $b \in \prod_{i\in I} B_i$ such that $f(b) = b$. This implies that $f(b(i)) = b(i)$ for each $i \in I$ and then $\{b(i)\}$ is a retract of (B_i, f). Therefore $(A, f) \in R(B_i, f)$ and (A, f) is retract irreducible. □

2.1. Lemma. *Let (E, f) be a connected monounary algebra and let (M, f) be a subalgebra of (E, f) such that $\operatorname{card} M = n > 1$, $M = \{e_1, \dots, e_n\}$, $f(e_n) = e_{n-1}, \dots, f(e_2) = e_1 = f(e_1)$. Then M is a retract of (E, f) if and only if $f^{-(n-1)}(e_2) = \emptyset$.*

Proof. Let M be a retract of (E, f) and suppose that $x \in f^{-(n-1)}(e_2)$. Let h be a corresponding retraction endomorphism and let $h(x) = e_j$, $j \in \{1, \dots, n\}$. Then

$$\begin{aligned} e_2 &= h(e_2) = h(f^{n-1}(x)) = f^{n-1}(h(x)) \\ &= f^{n-1}(e_j) = f^{n-j}(f^{j-1}(e_j)) = f^{n-j}(e_1) = e_1, \end{aligned}$$

which is a contradiction.

Conversely, suppose that $f^{-(n-1)}(e_2) = \emptyset$. If $x \in E$, then either

$$f^k(x) \neq e_2 \quad \text{for each } k \in \mathbb{N} \cup \{0\}, \tag{1.1}$$

or

$$f^k(x) = e_2 \quad \text{for some } k \in \mathbb{N} \cup \{0\}. \tag{1.2}$$

If (1.2) holds, then $k < n - 1$ and k is uniquely determined. In the first case put $h(x) = e_1$; in the second case let $h(x) = e_{2+k}$. If $x \in E$ and (1.1) is valid, then $f^k(f(x)) \neq e_2$ for each $k \in \mathbb{N} \cup \{0\}$ and

$$h(f(x)) = e_1 = f(e_1) = f(h(x)).$$

Let $x \in E$ and suppose that (1.2) holds. If k is as in (1.2) and $k \geqslant 1$, then $f^{k-1}(f(x)) = e_2$ and

$$h(f(x)) = e_{2+(k-1)} = f(e_{2+k}) = f(h(x)).$$

If $x \in E$ and $x = e_2$, then

$$h(f(x)) = h(e_1) = e_1 = f(e_2) = f(h(e_2)) = f(h(x)).$$

Therefore h is a homomorphism of (E, f) into (M, f). If $e_j \in M$, then $e_2 = f^{j-2}(e_j)$ and $h(e_j) = e_{2+(j-2)} = e_j$. Thus M is a retract of (E, f). □

2.2. Corollary. *Let $n \in \mathbb{N}$, $n > 1$ and let (A, f) be a connected monounary algebra such that* $\operatorname{card} A = n$, $A = \{a_1, \dots, a_n\}$, $f(a_n) = a_{n-1}, \dots, f(a_2) = a_1 = f(a_1)$. *Suppose that (E, f) is a connected monounary algebra. The following conditions are equivalent:*

(i) $(A, f) \in R(E, f)$;

(ii) *there exist distinct elements $e_1, \dots, e_n \in E$ such that $f(e_n) = e_{n-1}, \dots, f(e_2) = e_1 = f(e_1)$ and that $f^{-(n-1)}(e_2) = \emptyset$.*

2.3. Lemma. *Let the relations $a, b \in A$, $f(a) = f(b)$ imply that either $a = b$ or $c \in \{a, b\}$. If A is a finite set, then (A, f) is retract irreducible.*

Proof. Let $\operatorname{card} A = n \in \mathbb{N}$. If $n = 1$, then (A, f) is retract irreducible in view of 2.0. Suppose that $n > 1$ and that $(A, f) \in R\Bigl(\prod\limits_{i \in I}(B_i, f)\Bigr)$, where (B_i, f) is a connected monounary algebra for each $i \in I$. Put $(B, f) = \prod\limits_{i \in I}(B_i, f)$. Then (A, f) is isomorphic to a subalgebra (M, f) of (B, f), M is a retract of (B, f). We obtain that there are distinct elements $\{b_1, \dots, b_n\} = M$ such that $f(b_n) = b_{n-1}, \dots, f(b_2) = b_1 = f(b_1)$. If $i \in I$, then

$$f(b_k(i)) = (f(b_k))(i) = b_{k-1} \quad \text{for each } k \in \{2, \dots, n\}, \tag{1}$$

$$f(b_1(i)) = (f(b_1))(i) = b_1(i). \tag{2}$$

Assume that

$$(A, f) \notin R(B_i, f) \quad \text{for each } i \in I. \tag{3}$$

If $i \in I$, then 2.2 implies that one of the following conditions is satisfied:

(4.1) if $e_1, e_2, \dots, e_n \in B_i$, $f(e_n) = e_{n-1}, \dots, f(e_2) = e_1 = f(e_1)$, then the elements $e_1, \dots, e_n$ are not distinct (and then $e_2 = e_1$, since $e_k = e_l$ for $k < l$ implies $e_2 = f^{l-2}(e_l) = f^{l-2}(e_k) = f^{l-k-1}(f^{k-1}(e_k)) = f^{l-k-1}(e_1) = e_1$);

(4.2) if there are distinct elements $e_1, \dots, e_n \in B_i$ such that $f(e_n) = e_{n-1}, \dots, f(e_2) = e_1 = f(e_1)$, then $f^{-(n-1)}(e_2) \neq \emptyset$.

Let $I_1 = \{i \in I \colon b_1(i), \dots, b_n(i) \text{ are not distinct}\}$, $I_2 = I - I_1$. If $I_2 = \emptyset$, then $b_2(i) = b_1(i)$ for each $i \in I$, thus $b_2 = b_1$, which is a contradiction. Thus $I_2 \neq \emptyset$. Let $i \in I_2$. Then (4.1) is not valid for (B_i, f), hence (4.2) holds and $f^{-(n-1)}(b_2(i)) \neq \emptyset$. Take $t_i \in f^{-(n-1)}(b_2(i))$. Let $x \in B$ be such that

$$x(j) = \begin{cases} b_2(j) & \text{if } j \in I_1, \\ t_j & \text{if } j \in I_2. \end{cases}$$

We obtain

$$(f^{n-1}(x))(j) = b_1(j) = b_2(j) \text{ if } j \in I_1,$$
$$(f^{n-1}(x))(j) = f^{n-1}(t_j) = b_2(j) \text{ if } j \in I_2,$$

i.e., $x \in f^{-(n-1)}(b_2)$. Since M is a retract of (B, f), this is a contradiction in view of 2.1. □

2.4. Proposition. *Let the relations $a, b \in A$, $f(a) = f(b)$ imply that either $a = b$ or $c \in \{a, b\}$. Then (A, f) is retract irreducible.*

Proof. If A is finite, then (A, f) is retract irreducible in view of 2.3. Let A be infinite. Suppose that $(B, f) = \prod_{i \in I} (B_i, f)$ for connected monounary algebras (B_i, f), $i \in I$, where $(A, f) \in R\Big(\prod_{i \in I} (B_i, f)\Big)$. Then there are distinct elements $b_k \in B$ for $k \in \mathbb{N}$ such that

(1) $$f(b_1) = b_1, f(b_k) = b_{k-1} \quad \text{for each } k \in \mathbb{N},\ k > 1.$$

Let $i \in I$. By (1),

(2) $$f(b_1(i)) = b_1(i),$$

(3) $$f(b_k(i)) = b_{k-1}(i) \quad \text{for each } k \in \mathbb{N},\ k > 1,$$

therefore either

(4.1) $$b_k(i) = b_1(i) \quad \text{for each } k \in \mathbb{N}$$

or

(4.2) there is $n \in \mathbb{N}$ such that the elements $b_k(i), k \in \mathbb{N},\ k > n$ are mutually distinct and $b_1(i) = b_2(i) = \ldots = b_n(i)$.

If (4.1) holds for each $i \in I$, then $b_2 = b_1$, which is a contradiction. Thus (4.2) is valid for some $i \in I$. Let us denote $M = \{b_k(i) : k \in \mathbb{N}, k > n\}$. We have

$$(M, f) \cong (A, f). \tag{5}$$

Let $y \in f^{-1}(M)$. Then $f(y) = b_k(i)$ for some $k \in \mathbb{N}$, $k > n$. Put $z = b_{k+1}(i)$. We obtain

$$f(z) = b_k(i) = f(y), \quad s_f(z) = s_f(b_{k+1}(i)) = \infty \geqslant s_f(y). \tag{6}$$

According to 1.2, M is a retract of (B_i, f) and (5) implies that $(A, f) \in R(B_i, f)$. Hence (A, f) is retract irreducible. □

3. Condition (C3)

In 3.1–3.6 suppose that the following condition is satisfied:

(C3) (A, f) contains a cycle $\{c\}$ and there are $a, b \in A - \{c\}$ with $a \neq b$ and $f(a) = f(b) = c$.

3.1. Construction. Let $\{a_i : i \in I\}$ be the set of all elements $x \in A - \{c\}$ with $f(x) = c$ (assume $a_i \neq a_j$ for $i \neq j$). According to (C3), $\operatorname{card} I > 1$. For $i \in I$ put

$$A_i = \{c\} \cup \{x \in f^{-k}(a_i) : k \in \mathbb{N} \cup \{0\}\}.$$

Then (A_i, f) is a subalgebra of (A, f). Let

$$(B, f) = \prod_{i \in I} (A_i, f).$$

3.2. Lemma. *If $i \in I$, then $(A, f) \notin R(A_i, f)$.*

Proof. Suppose that $(A, f) \in R(A_i, f)$ for some $i \in I$. Then (A, f) is isomorphic to a subalgebra of (A_i, f). Since

$$\operatorname{card}\{x \in A - \{c\} : f(x) = c\} \geqslant 2,$$
$$\operatorname{card}\{x \in A_i - \{c\} : f(x) = c\} = \operatorname{card}\{a_i\} = 1,$$

we arrive at a contradiction. □

3.3. Notation. If $i \in I$, then denote

$$T_i = \{b \in B\colon b(j) = c \text{ for each } j \in I - \{i\}, b(i) \in A_i\}.$$

Further put,

$$T = \bigcup_{i \in I} T_i.$$

Define a mapping $\nu\colon T \to A$ as follows. If $b \in T_i$ for some $i \in I$, then $\nu(b) = b(i)$.

Notice that $b \in T_i \cap T_j$ for $i \neq j$ iff $b(k) = c$ for each $k \in I$ and then $\nu(b) = c = b(i) = b(j)$, thus the mapping ν is defined correctly.

3.4. Lemma. *(T, f) is a monounary algebra and ν is an isomorphism of (T, f) onto (A, f).*

P r o o f. Suppose that $b, t \in T$, $\nu(b) = \nu(t)$. Then there is $i \in I$ with $\{b, t\} \subseteq T_i$. We obtain $b(j) = t(j) = c$ for each $j \in I - \{i\}$, $b(i) = \nu(b) = \nu(t) = t(i)$, thus the mapping ν is injective.

If $x \in A$, then $x \in A_i$ for some $i \in I$ and then $x = \nu(b)$, where $b(i) = x, b(j) = c$ for each $j \in I - \{i\}$. The mapping ν is surjective.

Let $b \in T$. Then there is $i \in I$ such that $b \in T_i$ and $f(b) \in T_i$. Thus

$$(\nu(f(b))) = (f(b))(i) = f(b(i)) = f(\nu(b)).$$

Therefore ν is an isomorphism, $(A, f) \cong (T, f)$. □

3.5. Lemma. *If $y \in f^{-1}(T)$, then there is $z \in T$ such that $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$.*

P r o o f. Let $y \in f^{-1}(T)$, $f(y) = t \in T_i$. Then $t(i) \in A_i$ and $t(j) = c$ for each $j \in I - \{i\}$. Take $z \in B$ such that $z(i) = y(i)$, $z(j) = c$ for each $j \in I - \{i\}$. We get

(1) $$z \in T_i \text{ and } f(z) = t = f(y).$$

Further,

$$s_f(z(i)) = s_f(y(i)),$$
$$s_f(z(j)) = \infty \geqslant s_f(y(j)) \quad \text{for each } j \in I - \{i\},$$

which implies that $s_f(z) \geqslant s_f(y)$. □

3.6. Lemma. *T is a retract of (B, f).*

P r o o f. (a) of 1.3 is valid in view of 3.5. Further, (T, f) contains a one-element cycle by 3.4, thus 1.3(b1) and 1.3(b2) are satisfied. Hence T is a retract of (A, f). □

3.7. Proposition. *If (A, f) satisfies (C3), then (A, f) is retract reducible.*

Proof. We get the assertion by virtue of 3.1, 3.2, 3.4 and 3.6. □

4. Condition (C4)

In 4.1–4.7 suppose that the following condition is satisfied:

(C4) (A, f) contains a cycle $\{c\}$, (C3) is not valid, there are $a, b \in A$ with $a \neq b$, $f(a) = f(b) \neq c$ and $s_f(x) = \infty$ for each $x \in A$.

Hence, A is infinite.

4.1. Construction. Let I be an index set, $\operatorname{card} I = \lambda = \operatorname{card} A$ and let $(\mathbb{N}, f)$ be a monounary algebra with $f(n) = n - 1$ for each $n \in \mathbb{N}$, $n > 1$, $f(1) = 1$. For $i \in I$ put

$$(B_i, f) = (\mathbb{N}, f),$$
$$(B, f) = \prod_{i \in I} (B_i, f).$$

4.2. Lemma. *If $i \in I$, then $(A, f) \notin R(B_i, f)$.*

Proof. The assertion is obvious, (A, f) is not isomorphic to any subalgebra of $(\mathbb{N}, f)$. □

4.3. Lemma. *Let $R = \{x \in B \colon \{i \in I \colon x(i) \neq 1\}$ is finite$\}$. Then*

(i) *R contains a one-element cycle $\{r\}$, where $r(i) = 1$ for each $i \in I$;*
(ii) *(R, f) is a connected subalgebra of (B, f);*
(iii) *$s_f(x) = \infty$ for each $x \in R$;*
(iv) *$\operatorname{card} f^{-1}(x) \geqslant \lambda$ for each $x \in R$.*

Proof. (i) It is obvious that $r \in R$ and that $f(r) = r$.

(ii) Let $x \in R$. The set $\{i \in I \colon x(i) \neq 1\}$ is finite, thus there is $m = \max\{x(i) \colon x(i) \neq 1\}$. Then, for $j \in I$,

$$(f^m(x))(j) = f^m(x(j)) = 1 = r(j),$$

i.e., $f^m(x) = r$ and (ii) is valid.

(iii) Let $x \in R$. If $x = r$, then $s_f(x) = \infty$. Let $x \neq r$. For $k \in \mathbb{N} \cup \{0\}$ define an element $y_k \in R$ as follows:

$$y_k(i) = \begin{cases} x(i) + k & \text{if } x(i) \neq 1, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that $y_k \in R$ for each $k \in \mathbb{N} \cup \{0\}$. Further, $y_k \neq y_l$ for $k, l \in \mathbb{N} \cup \{0\}$, $k \neq l$ and if $k \in \mathbb{N}$, then

$$(f(y_k))(i) = f(y_k(i)) = \begin{Bmatrix} x(i) + k - 1 & \text{if } x(i) \neq 1, \\ 1 & \text{otherwise} \end{Bmatrix} = y_{k-1}(i),$$

i.e., $f(y_k) = y_{k-1}$. Clearly $y_0 = x$. Hence $s_f(x) = \infty$.

(iv) Let $x \in R$, $y \in f^{-1}(x)$. If $x(i) \neq 1$, then $y(i) = x(i) + 1$. If $x(i) = 1$, then $y(i) \in \{1, 2\}$. The assumption of the lemma implies

$$\operatorname{card}\{i \in I : x(i) = 1\} = \lambda,$$

therefore card $f^{-1}(x) = 2^{\lambda}$. □

4.4. Construction. Let us define a mapping $\nu : A \to R$ as follows. Consider $x \in A$. There is a unique $n(x) \in \mathbb{N} \cup \{0\}$ such that $f^{n(x)}(x) = c$ and if $m \in \mathbb{N} \cup \{0\}$, $m < n(x)$, then $f^m(x) \neq c$.

The relation $n(x) = 0$ implies $x = c$; put $\nu(c) = r$.

Let $n \in \mathbb{N}$, $n > 0$. Suppose that if $y \in A$, $n(y) < n$, then $\nu(y)$ is defined, and that $y_1 \neq y_2$, $n(y_1) = n(y_2) < n$ yield $\nu(y_1) \neq \nu(y_2)$. Let $n(x) = n$. Put $y = f(x)$. Then $n(y) = n - 1 < n$ and $\nu(y) = y' \in R$. In view of 4.3(iv) we get

$$\operatorname{card} f^{-1}(y) \leqslant \operatorname{card} A = \lambda,$$
$$\operatorname{card} f^{-1}(y') \geqslant \lambda,$$

therefore there is an injective mapping ν of $f^{-1}(y)$ into $f^{-1}(y')$. Thus, $\nu(x)$ is defined.

Let us have $x_1, x_2 \in A$, $x_1 \neq x_2$, $n(x_1) \leqslant n$, $n(x_2) \leqslant n$. Put $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_1' = \nu(y_1)$, $y_2' = \nu(y_2)$. Then $n(y_1) < n$, $n(y_2) < n$. If $y_1 \neq y_2$, the induction hypothesis implies that $\nu(y_1) \neq \nu(y_2)$. This entails that $f^{-1}(y_1') \cap f^{-1}(y_2') = \emptyset$ and the conditions $\nu(x_1) \in f^{-1}(y_1')$, $\nu(x_2) \in f^{-1}(y_2')$ imply $\nu(x_1) \neq \nu(x_2)$. If $y_1 = y_2$, then the injectivity of the mapping ν of $f^{-1}(y_1)$ into $f^{-1}(y_1')$ implies that $\nu(x_1) \neq \nu(x_2)$. Thus, $\nu(x)$ is defined for any $x \in A$ with $n(x) < n + 1$ and $x_1 \neq x_2$, $n(x_1) < n + 1$, $n(x_2) < n + 1$ yield $\nu(x_1) \neq \nu(x_2)$.

4.5. Lemma. *ν is an isomorphism of (A, f) into (R, f).*

Proof. By 4.4, ν is an injective mapping and a homomorphism. □

4.6. Lemma. *If $T = \nu(A)$ and $y \in f^{-1}(T)$, then there is $z \in T$ with $f(y) = f(z)$ and $s_f(y) \leqslant s_f(z)$.*

Proof. Let $T = \nu(A)$, $y \in f^{-1}(T)$. There is $t \in T$ with $f(y) = t$. Since $(T, f) \cong (A, f)$ by 4.5 and $s_f(x) = \infty$ for each $x \in A$, we obtain $s_f(t) = \infty$. Then there is $z \in T$ with $f(z) = t$ and $s_f(z) = \infty \geqslant s_f(y)$. □

4.7. Lemma. *$\nu(A)$ is a retract of (B, f).*

Proof. We get the assertion by virtue of 1.3. In fact, (a) of 1.3 is valid in view of 4.6. Further, if K is a connected component of (B, f) with $K \cap T = \emptyset$, then (b1) and (b2) are satisfied, because there is a cycle $\{r\} = \{\nu(c)\} \in T$, $s_f(r) = \infty$. □

4.8. Proposition. *If (A, f) satisfies (C4), then (A, f) is retract reducible.*

Proof. It follows from 4.1, 4.2, 4.5 and 4.7. □

5. Condition (C5)

In 5.1-5.6 suppose that the following condition is satisfied:

(C5) (A, f) contains a cycle $\{c\}$, there are $a, b \in A$ such that $a \neq b$ and $f(a) = f(b) \neq c$ and (A, f) fulfils neither (C3) nor (C4).

5.0. Lemma. *If (A, f) is a connected monounary algebra, $M \subseteq A$, $x \in M$ such that $f^{-1}(x) \neq \emptyset$ and $f^{-1}(x) \cap M = \emptyset$,then M is not a retract of (A, f).*

Proof. Suppose that M is a retract of (A, f) and let the assumption hold. Then there is $y \in f^{-1}(x)$ and, by 1.1, there exists $z \in M$ with $f(z) = f(y)$. Hence $z \in f^{-1}(x) \cap M$, which is a contradiction. □

5.1. Construction. (A, f) does not satisfy (C4), thus the set $L = \{a \in A\colon f^{-1}(a) = \emptyset\}$ is nonempty. By (C5), for $a \in L$ we have

$$\{k \in \mathbb{N}\colon \operatorname{card} f^{-1}(f^k(a)) > 1\} \neq \emptyset;$$

put

$$k(a) = \min\{k \in \mathbb{N}\colon \operatorname{card} f^{-1}(f^k(a)) > 1\}.$$

Further let,

$$m = \min\{k(a)\colon a \in L\},$$
$$J = \{a \in L\colon k(a) = m\},$$
$$V = \{f^m(a)\colon a \in J\}.$$

Since (C3) is not valid, $c \notin V$. For each $v \in V$ such that $f^{-m}(v) \subseteq J$ we choose a fixed element of the set $f^{-m}(v)$ and denote it by $\overline{v}$. Then we define

$$I = \{a \in J\colon f^{-m}(f^m(a)) \nsubseteq J\} \cup \{a \in J\colon f^{-m}(f^m(a)) \subseteq J, a \neq \overline{f^m(a)}\}.$$

If $a \in I$, then put

$$A_a = \{a, f(a), \ldots, f^{m-1}(a)\},$$
$$B_a = A_a \cup \{c\},$$
$$g(x) = \begin{cases} f(x) & \text{if } x \in A_a - \{f^{m-1}(a)\}, \\ c & \text{if } x \in \{c, f^{m-1}(a)\}. \end{cases}$$

Denote

$$B_0 = A - \bigcup_{a \in I} A_a.$$

If $x \in B_0$, then $f(x) \in B_0$, because in the opposite case $f(x) \in \{a, f(a), \ldots, f^{m-1}(a)\}$ for some $a \in I$ and then

$$x \in \{a, f(a), \ldots f^{m-2}(a)\} \subseteq A_a.$$

Thus (B_0, f) is a subalgebra of (A, f). Put

$$(B_0, g) = (B_0, f),$$
$$(B, g) = \prod_{a \in I \cup \{0\}} (B_a, g).$$

5.2. Lemma. *If $a \in I \cup \{0\}$, then $(A, f) \notin R(B_a, g)$.*

Proof. Let $a \in I$. In (B_a, g) there are no distinct elements x, y with $g(x) = g(y) \neq c$, thus (A, f) is not isomorphic to any subalgebra of (B_a, g) and hence $(A, f) \notin R(B_a, g)$.

Suppose that $(A, f) \in R(B_0, g)$. Then there is an isomorphism ε of (A, f) onto a subalgebra (M, g) of (B_0, g), where M is a retract of (B_0, g); let h be the corresponding retraction endomorphism. Take $a \in I$, $b = \varepsilon(a)$. First suppose that $g^{-1}(b) \neq \emptyset$. According to 5.0, there is $z \in g^{-1}(b) \cap M$. This implies that $z = \varepsilon(d)$ for some $d \in A$. We have

$$\varepsilon(f(d)) = g(\varepsilon(d)) = g(z) = b = \varepsilon(a),$$

thus $f(d) = a$, which is a contradiction, since $f^{-1}(a) = \emptyset$. Therefore $g^{-1}(b) = \emptyset$. Then $f^{-1}(b) = \emptyset$, since $b \in L - I$ by 5.1. We have two possibilities:

(1.1) $$b \in L - J,$$

(1.2) $$b \in J - I, \text{ i.e., } b = \overline{v} \text{ for some } v \in V.$$

If (1.1) is valid, then $k(b) > m$ and the definition of $k(b)$ implies

$$\operatorname{card} f^{-1}(f^m(b)) = 1,$$

a contradiction, since $\operatorname{card} f^{-1}(f^m(a)) > 1$. Hence (1.2) holds. In this case $g^{-1}(g^m(\overline{v})) = \{g^{m-1}(\overline{v})\}$, which is a contradiction to $\operatorname{card} f^{-1}(f^m(a)) > 1$ as well. □

5.3. Lemma. *If $a \in I$, then there exists an endomorphism φ_a of (A, f) such that $\varphi_a(x) \neq x$ iff $x \in A_a$ and $\varphi_a(A_a) \subseteq B_0$.*

Proof. Put $\varphi_a(x) = x$ for each $x \in A - A_a$.

First suppose that $f^{-m}(f^m(a)) \subseteq J$. Denote $v = f^m(a)$ and put

$$\varphi_a(a) = \overline{v}, \varphi_a(f(a)) = f(\overline{v}), \dots, \varphi_a(f^{m-1}(a)) = f^{m-1}(\overline{v}). \tag{1}$$

Then $a \neq v$ and we obtain

$$\varphi_a(x) \neq x \quad \text{iff} \quad x \in A_a, \tag{2}$$

$$\varphi_a(A_a) \subseteq B_0. \tag{3}$$

If $x \in A_a - \{f^{m-1}(a)\}$, then (1) implies

$$\varphi_a(f(x)) = f(\varphi_a(x)). \tag{4}$$

If $x \in A - A_a$, then (4) is valid, too. Let $x = f^{m-1}(a)$. Then $f(x) \in A - A_a$, thus we have

$$\varphi_a(f(x)) = f(x) = f^m(a) = v = f^m(\overline{v}) = f(\varphi_a(x)). \tag{5}$$

Therefore

$$\varphi_a \text{ is an endomorphism of } (A, f). \tag{6}$$

According to (2), (3) and (6), φ_a has the desired properties.

Now suppose that $f^{-m}(f^m(a)) \nsubseteq J$. Then there exists $y \in f^{-m}(f^m(a)) \cap B_0$ and we can proceed analogously, only with y instead of $\overline{v}$. □

5.4. Notation. Denote

$$T_0 = \{b \in B \colon b(0) \in B_0,\ b(a) = c \quad \text{for each } a \in I\}$$

and if $a \in I$, then

$$T_a = \{b \in B \colon b(a) \in A_a,\ b(i) = c \quad \text{for each } i \in I - \{a\},\ b(0) = \varphi_a(b(a))\}.$$

Let

$$T = \bigcup_{a \in I \cup \{0\}} T_a.$$

Consider $b \in T$, i.e., $b \in T_a$, $a \in I \cup \{0\}$.

a) If $a = 0$, then $b(0) \in B_0$, $b(i) = c$ for each $i \in I$, thus $(g(b))(0) = g(b(0)) = f(b(0)) \in B_0$ and $(g(b))(i) = g(b(i)) = g(c) = c$, which implies $g(b) \in T_0$.

b) Let $a \in I$. We obtain that $f(a) \in A_a$, $b(i) = c$ for each $i \in I - \{a\}, b(0) = \varphi_a(b(a))$. This yields that $(g(b))(a) = g(b(a)) \in A_a \cup \{c\}$. Further, if $i \in I - \{a\}$, then $(g(b))(i) = g(b(i)) = g(c) = c$ and, by 5.3, $(g(b))(0) = g(b(0)) = g(\varphi_a(b(a))) = \varphi_a(g(b(a))) = \varphi_a((g(b))(a))$. Thus either $g(b(a)) \in A_a$, which implies $g(b) \in T_a$, or $g(b(a)) = c$, which implies $g(b) \in T_0$.

Therefore the set T is closed under g.

Define a mapping $\nu \colon T \to A$ as follows: if $t \in T_a$, $a \in I \cup \{0\}$, then $\nu(t) = t(a)$.

5.5. Lemma. *(T, g) is a monounary algebra and ν is an isomorphism of (T, g) onto (A, f).*

Proof. Suppose that $b, t \in T$, $\nu(b) = \nu(t) = x$. If $x \in B_0$, then $\{b, t\} \subseteq T_0$ and

$$b(0) = \nu(b) = x = \nu(t) = t(0),$$
$$b(a) = c = t(a) \quad \text{for each } a \in I.$$

If $x \in B_a$, $a \in I$, then $\{b, t\} \subseteq T_a$ and we have

$$b(a) = \nu(b) = x = \nu(t) = t(a),$$
$$b(i) = c = t(i) \quad \text{for each } i \in I - \{a\},$$
$$b(0) = \varphi_a(b(a)) = \varphi_a(x) = \varphi_a(t(a)) = t(0).$$

Thus ν is an injective mapping.

Let $x \in A$. If $x \in B_0$, then $x = \nu(b)$, where $b(0) = x$, $b(a) = c$ for each $a \in I$, $b \in T_0$. Let $x \in A - B_0$. Then there is $a \in I$ such that $x \in A_a$ and then $x = \nu(b)$, where $b \in T_a$,

$$b(i) = \begin{cases} x & \text{if } i = a, \\ c & \text{if } i \in I - \{a\}, \\ \varphi_a(x) & \text{if } i = 0. \end{cases}$$

Hence the mapping ν is surjective.

Let $b \in T$. If $b \in T_0$, then $g(b) \in T_0$ and

$$\nu(g(b)) = (g(b))(0) = g(b(0)) = f(b(0)) = f(\nu(b)).$$

If $b \in T_a$, $a \in I$ and $g(b) \in T_a$, then $b(a) \neq c$ and

$$\nu(g(b)) = (g(b))(a) = g(b(a)) = f(b(a)) = f(\nu(b)).$$

If $b \in T_a$, $a \in I$ and $g(b) \notin T_a$, then $(g(b))(a) = c$, $g(b) \in T_0$ and by 5.3 we obtain

$$\nu(g(b)) = (g(b))(0) = g(b(0)) = g(\varphi_a(b(a))) = f(\varphi_a(b(a))) = \varphi_a(f(b(a))).$$

Since $g(b(a)) = c$, we get $f(b(a)) \in B_0$, thus $\varphi_a(f(b(a))) = f(b(a)) = f(\nu(b))$.
Therefore ν is an isomorphism and $(T, g) \cong (A, f)$. □

5.6. Lemma. *T is a retract of (B, g).*

Proof. We will prove the assertion by means of 1.2. Let $y \in g^{-1}(T)$. Then there is $b \in T$ with $g(y) = b$. If $b \in T_0$, then $b(0) \in B_0$ and $b(a) = c$ for each $a \in I$. Put

$$z(i) = \begin{cases} y(0) & \text{if } i = 0, \\ c & \text{if } i \in I. \end{cases}$$

Then $z \in T_0$, $g(z) = g(y)$. Further, $s_g(z(0)) = s_g(y(0))$ and $s_g(z(i)) = \infty \geqslant s_g(y(i))$ for each $i \in I$, thus $s_g(z) \geqslant s_g(y)$.
Suppose that $b \in T_a$, $a \in I$. Then

$$b(i) = \begin{cases} g(y(a)) & \text{if } i = a, \\ c & \text{if } i \in I - \{a\}, \\ \varphi_a(g(y(a))) & \text{if } i = 0. \end{cases}$$

Take $z \in T_a$ such that

$$z(i) = \begin{cases} y(a) & \text{if } i = a, \\ c & \text{if } i \in I - \{a\}, \\ \varphi_a(y(a)) & \text{if } i = 0. \end{cases}$$

Then $g(z) = g(y)$ by 5.3. Further, since φ_a is a homomorphism,

$$\begin{gathered} s_g(z(a)) = s_g(y(a)), \\ s_g(z(i)) = \infty \geqslant s_g(y(i)) \quad \text{for each } i \in I - \{a\}, \\ s_g(z(0)) = s_g(\varphi_a(y(a))) \geqslant s_g(y(a)), \end{gathered}$$

hence $s_g(z) \geqslant s_g(y)$.
Therefore we have proved that T is a retract of (B, g). □

5.7. Proposition. *If (A, f) satisfies (C5), then (A, f) is retract reducible.*

Proof. It is a consequence of 5.1, 5.2, 5.5 and 5.6. □

6. Proof of (R)

We conclude by proving Theorem (R) above.

Suppose that (A, f) is a connected monounary algebra with a cycle $\{c\}$. Then (A, f) satisfies one of the following conditions:

(1) If $a, b \in A$, $f(a) = f(b)$, then either $a = b$ or $c \in \{a, b\}$.

(2) There are $a, b \in A - \{c\}$ such that $a \neq b$ and $f(a) = f(b) = c$.

(3) The condition (2) is not fulfilled, there are $a, b \in A$ with $a \neq b$, $f(a) = f(b) \neq c$ and $s_f(x) = \infty$ for each $x \in A$.

(4) The conditions (2) and (3) are not fulfilled and there are $a, b \in A$ such that $a \neq b$, $f(a) = f(b) \neq c$.

If (1) is satisfied, then (A, f) is retract irreducible by 2.4. If (2), (3) or (4) holds, then 3.7, 4.8 and 5.7 imply that (A, f) is retract reducible.

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