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A WEAK MAXIMUM PRINCIPLE AND ESTIMATES OF $\operatorname{ess\,sup}_{\Omega} u$ FOR NONLINEAR DEGENERATE ELLIPTIC EQUATIONS

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1. Introduction

Maximum principles for elliptic equations, in the linear case, have been studied extensively for many years, see e.g. [4], [7], [8], and their importance for the problem of uniqueness and existence of solutions of boundary value problems is now well understood. In this paper we investigate estimates of $\operatorname{ess\,sup}_{\Omega} u(x)$ for a weak subsolution of nonlinear equations of the form

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) - f_1(x, u, \nabla u) = f_2(x, u, \nabla u) \tag{1.1}$$

in a bounded open set $\Omega \subset \mathbb{R}^m$, where functions $f_1(x, \xi, p)$, $f_2(x, \xi, p)$ satisfy different hypotheses and different conditions of growth on ξ and p, namely:

$$f_1(x, \xi, p) \leqslant [\bar{f}(x) + c_1|\xi|^{1+\sigma} + c_2(\sqrt{\nu}|p|)^{1+\mu}] \quad \text{a.e. } x \in \Omega,$$

for any real numbers $\xi, p_1, p_2, \ldots, p_m$ and

$$|f_2(x, \xi, p)| \leqslant \tilde{c}[f^*(x) + \xi^{r-1} + (\sqrt{\nu}|p|)^{2(r-1)/r}] \quad \text{a.e. } x \in \Omega,$$

for any real numbers $p_1, p_2, \ldots, p_m$ and $\xi \in \mathbb{R}_0^+$, while the coefficients of the principal part of the operator are supposed to satisfy the following elliptic degenerate condition:

$$\sum_{i=1}^{m} a_i(x, \xi, p) p_i \geqslant \nu(x)|p|^2, \tag{1.2}$$

p being the vector $(p_1, p_2, \dots, p_m)$, $|p|$ its module, and with $\nu(x)$ satisfying sufficiently general hypotheses. The estimate for $\operatorname{ess\,sup}_\Omega u(x)$ depending only on the boundary, initial data and on the structure of the equation implies, in a special case (linear growth of $f_2(x, \xi, p)$ with respect to ξ and p with $f^*(x) \equiv 0$, see remark (4.1)), the maximum principle for a weak subsolution $u(x)$, that is, the nonnegative maximum of $u(x)$ is attained on the boundary $\partial\Omega$. It is perhaps worth mentioning that similar results, in the classical case, have been obtained in [5] and [3], the latter with regular coefficients, and, in non-degenerate case, in [7] and [13].

This paper may be regarded as a continuation and completion of the preceding papers [9] and [2].

2. Functional spaces, definitions and hypotheses

Let $\mathbb{R}^m$ $(m \geqslant 2)$ be the Euclidean m-dimensional space having the generical point $x = (x_1, \dots, x_m)$, let Ω be an open and bounded set of $\mathbb{R}^m$.

Hypothesis (2.1). Let $\nu(x)$ be a positive function defined in Ω such that

$$\nu(x) \in L^1(\Omega), \quad \frac{1}{\nu(x)} \in L^1_{\mathrm{loc}}\Omega.$$

$H^1(\nu, \Omega)$ denotes the completion of $C^1(\bar{\Omega})$ with respect to

$$\|u\|_1 = \left(\int_\Omega |u|^2 + \nu(x) |\nabla u|^2 \, dx \right)^{1/2}.$$

$H_0^1(\nu, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\nu, \Omega)$.[1]

Definition 1. Any function $u(x) \in H^1(\nu, \Omega)$ such that

$$\int_\Omega \left\{ \sum_{i=1}^m a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + f_1(x, u, \nabla u)\varphi + f_2(x, u, \nabla u)\varphi \right\} dx \leqslant 0 \tag{2.1}$$

for any $\varphi \in H_0^1(\nu, \Omega)$, $\varphi \geqslant 0$ almost everywhere x in Ω, will be called a subsolution of the equation (1.1).

Definition 2. Given a real number h, if $u(x) \in H^1(\nu, \Omega)$, we will say that $u(x) \leqslant h$ on $\partial\Omega$ if there exists a sequence $\{u_n\}$ of functions belonging to $C^1(\bar{\Omega})$ such that $u_n \leqslant h$ on $\partial\Omega$ and

$$\lim_{n \to \infty} \|u_n - u\|_1 = 0.$$

[1] See also [11] for another definition of the space $H^1(\nu, \Omega)$. We remark that, in the last case, for having $C_0^\infty(\Omega)$ as a subset of $H^1(\nu, \Omega)$ it is sufficient to suppose $\nu(x) \in L^1_{\mathrm{loc}}(\Omega)$.

See also [9] and [8]. If h is such that $u(x) \leqslant h$ on $\partial\Omega$, we will say that $u(x)$ is bounded from above on $\partial\Omega$. In this case the symbol $\sup_{\partial\Omega} u$ stands for the greatest lower bound of the real numbers h such that $u(x) \leqslant h$ on $\partial\Omega$.

Hypothesis (2.2). There exist r and β ($2 < r < +\infty$, $0 < \beta < +\infty$) such that

$$|u|_r \leqslant \beta \|u\|_1$$

for any $u \in H^1(\nu, \Omega)$.[2]

For more details see also [10] and [6].

Hypothesis (2.3). Functions $f_j(x, \xi, p)$ ($j = 1, 2$), $a_i(x, \xi, p)$ ($i = 1, 2, \dots, m$) are Carathéodory's functions in $\Omega \times \mathbb{R} \times \mathbb{R}^m$, i.e. measurable with respect to x for any $(\xi, p) \in \mathbb{R} \times \mathbb{R}^m$, continuous with respect to (ξ, p) for a.e. x in Ω.

Hypothesis (2.4). There exists a positive constant f_0 such that, for a.e. x in Ω, we have

$$f_1(x, \xi, p) - f_0 \xi \geqslant 0$$

for any real numbers $p_1, p_2, \dots, p_m$ and for any positive real number ξ.

Hypothesis (2.5). There exist two nonnegative real numbers c_1 and c_2, the former greater than or equal to f_0, a function $\bar{f}(x)$ of $L^{r/(r-1)}(\Omega)$, and two positive real numbers σ and μ, both less than $\frac{r-2}{r}$, such that, for a.e. x in Ω, we have

(i) $\bar{f}(x) \geqslant f_0$,

(ii) $f_1(x, \xi, p) \leqslant [\bar{f}(x) + c_1 |\xi|^{1+\sigma} + c_2 (\sqrt{\nu}|p|)^{1+\mu}]$

for any real numbers $\xi, p_1, p_2, \dots, p_m$.[3]

Hypothesis (2.6). There exists a positive constant $\tilde{c}$ and a function $f^*(x) \in L^g(\Omega)$ with $g > \frac{r}{r-2}$ such that, for x a.e. in Ω, we have

$$|f_2(x, \xi, p)| \leqslant \tilde{c}[f^*(x) + \xi^{r-1} + (\sqrt{\nu}|p|)^{2(r-1)/r}]$$

for any real numbers $p_1, p_2, \dots, p_m$ and for any nonnegative real number ξ.

Hypothesis (2.7). The function $f_2(x, \xi, p)$ is monotone nondecreasing in $\mathbb{R}^+$ for a.e. x in Ω and for any $p_1, p_2, \dots, p_m \in \mathbb{R}$, that is:

$$f_2(x, \xi, p) \leqslant f_2(x, \eta, p) \quad \text{if } 0 < \xi < \eta.$$[4]

[2] If $1 \leqslant s \leqslant +\infty$, the symbol $|u|_s$ denotes the norm in $L^s(\Omega)$.

[3] Hypotheses (2.5), (2.6) and (2.8) ensure (2.1).

[4] Hypothesis (2.7), e.g., is true for

$$f_2(x, \xi, p) = f^*(x) + \xi^{r-1} + (\sqrt{\nu}|p|)^{2(r-1)/r}.$$

Hypothesis (2.8). There exist a function $a_i^*(x) \in L^2(\Omega)$ $(i = 1, 2, \dots, m)$ and a constant $\alpha_i > 0$ such that, for a.e. x in Ω, we have

$$\frac{|a_i(x, \xi, p)|}{\sqrt{\nu}} \leqslant \alpha_i[a_i(x) + |\xi| + \sqrt{\nu}|p|]$$

for any real numbers $\xi, p_1, p_2, \dots, p_m$.

Hypothesis (2.9). Let us assume that (1.2) holds for a.e. x in Ω and for any real numbers $\xi, p_1, p_2, \dots, p_m$.

In Sec. 4 we will prove

Theorem (2.1). *Let us assume hypotheses* (2.1)–(2.9) *hold and let* $u(x)$ *be a subsolution of the equation* (1.1) *bounded from above on* $\partial\Omega$. *Then* $u(x)$ *is bounded from above in* Ω*; moreover,*

$$\operatorname*{ess\,sup}_{\Omega} u \leqslant M.[5] \tag{2.2}$$

In Sec. 5 we will extend the result cited above to the case when the hypothesis (2.7) does not hold, but it will be necessary to suppose $f^*(x) \in L^\infty(\Omega)$. Then we will get

Theorem (2.2). *Let us assume hypotheses* (2.1)–(2.6), (2.8), (2.9) *hold with* $f^*(x) \in L^\infty(\Omega)$ *and let* $u(x)$ *be a subsolution of equation* (1.1) *bounded from above on* $\partial\Omega$. *Then* $u(x)$ *is bounded from above in* Ω *and* (2.2) *holds.*

Finally, in Sec. 6 we will extend the results cited above to the case when the assumptions (2.4) and (2.5) are replaced by $f_1(x, \xi, p) \geqslant 0$ for a.e. x in Ω and for any real numbers $\xi, p_1, p_2, \dots, p_m$.

However, it will be necessary to substitute hypothesis (2.1) with another one slightly more restrictive:

Hypothesis (2.10). Let $\nu(x)$ be a positive function defined in Ω such that

$$\nu(x) \in L^1(\Omega), \quad \frac{1}{\nu(x)} \in L^\kappa(\Omega),$$

where $\frac{m}{2} < \kappa < +\infty$ $(1 < \kappa < +\infty)$ if $m \geqslant 3$ $(m = 2)$.

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[5] M stands for a constant dependent on $\max(0, \sup_{\partial\Omega} u)$, β, r, $\tilde{c}$, $\operatorname{meas}\Omega$, $|f^*|_g$, f_0.

3. Preliminary lemmas

Lemma (3.1). *Let $u(x) \in H^1(\nu, \Omega)$ be bounded from above on $\partial\Omega$ and $k \geqslant \sup_{\partial\Omega} u$, then the function $v = u - \min(u, k)$ belongs to $H_0^1(\nu, \Omega)$.*

See [8], Corollary (2.10).

Lemma (3.2). *If the hypothesis* (2.10) *is satisfied, we get*

$$|u|_{2^\sharp} \leqslant L|\sqrt{\nu}|\nabla u||_2 \quad \text{for any } u \in H_0^1(\nu, \Omega), \tag{3.1}$$

where $2^\sharp = \frac{2m\kappa}{m\kappa + m - 2\kappa}$.[6]

The proof is based on Sobolev's imbedding theorem (see e.g. [1]).

Remark (3.3). If the hypothesis (2.10) holds, then $\left|\sqrt{\nu}|\nabla u|\right|_2$ constitutes an equivalent norm in $H_0^1(\nu, \Omega)$. We will denote this norm by $\|u\|_{1,0}$.

4. Proof of Theorem (2.1)

Let us fix k: $k \geqslant \max(0, \sup_{\partial\Omega} u)$, then from (2.1) for $w = v$ (see Lemma (3.1)) we get

$$\int_\Omega \left\{ \sum_{i=1}^m a_i(x, u, \nabla u)\frac{\partial v}{\partial x_i} + f_1(x, u, \nabla u)v + f_2(x, u, \nabla u)v \right\} \mathrm{d}x \leqslant 0. \tag{4.1}$$

Hypotheses (2.4), (2.7) and (2.8) imply

$$\int_\Omega \sum_{i=1}^m a_i(x, u, \nabla u)\frac{\partial v}{\partial x_i}\,\mathrm{d}x \geqslant \int_\Omega \nu(x)|\nabla v|^2\,\mathrm{d}x; \tag{4.2}$$

$$\int_\Omega f_2(x, u, \nabla u)v\,\mathrm{d}x = \int_{\Omega(u>k)} f_2(x, u, \nabla v)v\,\mathrm{d}x \geqslant \int_\Omega f_2(x, v, \nabla v)v\,\mathrm{d}x; \tag{4.3}$$

$$\int_\Omega f_1(x, u, \nabla u)v\,\mathrm{d}x \geqslant \int_{\Omega(u>k)} f_0 uv\,\mathrm{d}x \geqslant \int_{\Omega(u>k)} f_0(u-k)v\,\mathrm{d}x = f_0\int_\Omega v^2\,\mathrm{d}x. \tag{4.4}$$

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[6] We note that $2^\sharp$ is greater than 2; moreover, if Ω satisfies cone property, then the hypothesis (2.2) is true with $r = 2^\sharp$.

Therefore from (4.1) according to (4.2), (4.3) and (4.4) we get

$$f_0 \int_\Omega v^2 \, dx + \int_\Omega \nu(x)|\nabla v|^2 \, dx + \int_\Omega f_2(x, v, \nabla v)v \, dx \leqslant 0. \tag{4.5}$$

On the other hand, one has

$$\begin{aligned} -\int_\Omega f_2(x, v, \nabla v)v \, dx &\leqslant \int_\Omega |f_2(x, v, \nabla v)|v \, dx \\ &\leqslant \tilde{c}\Big\{ \int_{\Omega(u>k)} f^*(x)v \, dx + \int_\Omega v^r \, dx + \int_\Omega (\sqrt{\nu}|\nabla v|)^{2(r-1)/r} v \, dx \Big\}. \end{aligned}$$

Applying now the Hölder-Riesz inequality we get

$$\begin{aligned} &-\int_\Omega f_2(x, v, \nabla v)v \, dx \\ &\leqslant \tilde{c}\left\{|f^*(x)|_g[\operatorname{meas}\Omega(u>k)]^{1/\lambda}|v|_r + |v|_r^r + \|v\|_1^{2(r-1)/r}|v|_r\right\} \\ &\leqslant \tilde{c}\left\{\beta|f^*(x)|_g[\operatorname{meas}\Omega(u>k)]^{1/\lambda}\|v\|_1 + \beta^2|v|_r^{r-2}\|v\|_1^2 + \beta^{2/r}|v|_r^{1-\frac{2}{r}}\|v\|_1^2\right\} \\ &\leqslant \tilde{c}\left\{\beta|f^*(x)|_g[\operatorname{meas}\Omega(u>k)]^{1/\lambda}\|v\|_1 + \beta^2\left(\int_{\Omega(u>k)} u^r \, dx\right)^{(r-2)/r}\|v\|_1^2\right. \\ &\left.+\beta^{2/r}\left(\int_{\Omega(u>k)} u^r \, dx\right)^{(r-2)/(r^2)}\|v\|_1^2\right\}. \end{aligned}$$

Accordingly, (4.5) yields

$$\begin{aligned} \min(1, f_0)\|v\|_1 &\leqslant \tilde{c}\beta|f^*|_g[\operatorname{meas}\Omega(u>k)]^{1/\lambda} \\ &+ \tilde{c}\left[\beta^2\left(\int_{\Omega(u>k)} u^r \, dx\right)^{(r-2)/r} + \beta^{2/r}\left(\int_{\Omega(u>k)} u^r \, dx\right)^{(r-2)/(r^2)}\right]\|v\|_1, \end{aligned}$$

and, moreover,

(4.6)

$$\begin{aligned} &\left\{\min(1, f_0) - \tilde{c}\beta^2\left(\int_{\Omega(u>k)} u^r \, dx\right)^{(r-2)/r} - \tilde{c}\beta^{2/r}\left(\int_{\Omega(u>k)} u^r \, dx\right)^{(r-2)/(r^2)}\right\}\|v\|_1 \\ &\leqslant \tilde{c}\beta|f^*|_g[\operatorname{meas}\Omega(u>k)]^{1/\lambda}. \end{aligned}$$

Recalling that

$$\lim_{k\to\infty} \operatorname{meas}\Omega(u>k) = 0$$

and that the integral function of $|u|^r$ is absolutely continuous, we can certainly choose $\tilde{k} \geqslant \max(0, \sup_{\partial\Omega} u)$ such that for any $k \geqslant \tilde{k}$ we have

$$\tilde{c}\left\{\beta^2\left(\int_{\Omega(u>k)} u^r \,\mathrm{d}x\right)^{(r-2)/r} + \beta^{2/r}\left(\int_{\Omega(u>k)} u^r \,\mathrm{d}x\right)^{(r-2)/(r^2)}\right\} \leqslant \frac{1}{2}\min(1, f_0).$$

We apply this inequality to (4.6), obtaining

$$\|v\|_1 \leqslant \frac{2\tilde{c}\beta|f^*|_g}{\min(1, f_0)}[\operatorname{meas}\Omega(u > k)]^{1/\lambda} \quad \text{for any } k \geqslant \tilde{k}. \tag{4.7}$$

Let h, k be real numbers: $h > k \geqslant \tilde{k}$. Then one has

$$|v|_r = \left[\int_{\Omega(u>k)} |u - k|^r \,\mathrm{d}x\right]^{1/r} \geqslant (h - k)[\operatorname{meas}\Omega(u > h)]^{1/r}; \tag{4.8}$$

furthermore, (4.7), (4.8) and hypothesis (2.2) yield

$$[\operatorname{meas}\Omega(u > h)]^{1/r} \leqslant \frac{1}{(h - k)}\frac{2\tilde{c}\beta^2|f^*|_g}{\min(1, f_0)}[\operatorname{meas}\Omega(u > k)]^{1/\lambda}. \tag{4.9}$$

Noticing that $r > \lambda$ (see hypothesis (2.6)) we get

$$[\operatorname{meas}\Omega(u > k)]^{1/\lambda} \leqslant [\operatorname{meas}\Omega(u > k)]^{\tilde{\beta}/r}(\operatorname{meas}\Omega)^{(r-\lambda)/2r\lambda}$$

where $\tilde{\beta} = \frac{1}{2}\left(1 + \frac{r}{\lambda}\right)$.

So from (4.9) we obtain

$$[\operatorname{meas}\Omega(u > h)]^{1/r} \leqslant \frac{1}{(h - k)}\left\{\frac{2\tilde{c}\beta^2|f^*|_g}{\min(1, f)}(\operatorname{meas}\Omega)^{\frac{(r-\lambda)}{2r\lambda}}\right\}[\operatorname{meas}\Omega(u > k)]^{\tilde{\beta}/r}$$

for any $h, k \in \mathbb{R}$ with $h > k \geqslant \tilde{k}$.

If we assume $\varphi(h) = [\operatorname{meas}\Omega(u > h)]^{1/r}$ for any $h \geqslant \tilde{k}$, we get

$$\varphi(h) \leqslant \frac{M}{(h - k)}\varphi(k)^{\tilde{\beta}} \quad \text{for any } h > k \geqslant \tilde{k},$$

and from Stampacchia's lemma (see [12] p. 212) we deduce

$$\varphi(\tilde{k} + d) = 0,$$

where $d = \frac{2\tilde{c}\beta^2|f^*|_g}{\min(1,f)}(\operatorname{meas}\Omega)^{\frac{(r-\lambda)}{2r\lambda}}2^{\tilde{\beta}/(\tilde{\beta}-1)}[\operatorname{meas}\Omega(u > \tilde{k})]^{(\tilde{\beta}-1)/r}$. The proof of theorem (2.1) now follows easily.

Remark (4.1) (Maximum principle). We can find the exact value of the constant M in some cases; if, e.g., $f_2(x,\xi,p)$ has linear growth with respect to ξ and p, and if $\tilde{c} < \min(\frac{2}{3}c_0, 2)$, we deduce by the same argument:

$$\operatorname*{ess\,sup}_{\Omega} u \leqslant \max(0, \sup_{\partial\Omega} u) + \gamma |f^*|_g.$$ [7]

5. Proof of theorem (2.2)

Let us fix k and v as in Theorem (2.1); from (4.1), (4.2) and (4.4) we get

$$f_0 \int_\Omega v^2\,dx + \int_\Omega \nu(x)|\nabla v|^2\,dx \leqslant -\int_\Omega f_2(x,u,\nabla u)v\,dx. \tag{5.1}$$

On the other hand, one has

$$-\int_\Omega f_2(x,u,\nabla u)v\,dx \leqslant \hat{c}\int_\Omega [1+(v+k)^{r-1}+(\sqrt{\nu}|\nabla u|)^{2(r-1)/r}]v\,dx$$

$$\leqslant \tilde{c}\{(1+2^{r-1}k^{r-1})\beta[\operatorname{meas}\Omega(u>k)]^{(r-1)/r}\|v\|_1 + 2^{r-2}\beta^2|v|_r^{r-2}\|v\|_1^2 + \beta^{2/r}|v|_r^{1-\frac{2}{r}}\|v\|_1^2\}.$$

Then, similarly to Theorem (2.1), we can immediately deduce that

$$\|v\|_1 \leqslant \frac{2\tilde{c}\beta(1+2^{r-2}k^{r-1})}{\min(1,f_0)}[\operatorname{meas}\Omega(u>k)]^{(r-1)/r},$$

$$|v|_r \leqslant \frac{2\tilde{c}\beta^2(1+2^{r-2}k^{r-1})}{\min(1,f_0)}[\operatorname{meas}\Omega(u>k)]^{(r-1)/r} \quad \text{for any } k \geqslant \tilde{k}.$$

Consequently, if $h > k \geqslant \tilde{k}$, we obtain

$$[\operatorname{meas}\Omega(u>h)]^{1/r} \leqslant \frac{2\tilde{c}\beta^2(1+2^{r-2}k^{r-1}}{\min(1,f_0)(h-k)}[\operatorname{meas}\Omega(u>k)]^{\frac{(r-1)}{r}}.\tag{5.2}$$ [8]

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[7] $\gamma = \frac{2^{\beta/(\beta-1)}\beta c}{\min\{(c_0-\frac{3}{2}\tilde{c}),(1-(\tilde{c}/2))\}}(\operatorname{meas}\Omega)^{\frac{(r-\lambda)(r+1)}{2r\lambda}}$.

[8] Observe that we could not apply directly Stampacchia's lemma because $\frac{2\tilde{c}\beta^2(1+2^{r-2}k^{r-1})}{\min(1,f_0)}$ depends on k.

Next, if $k > 0$, we get

$$\operatorname{meas}\Omega(u > k) \leqslant \frac{1}{k^r}\int_\Omega u^r \, dx,$$

$$\frac{2\tilde{c}\beta^2(1 + 2^{2r-3}k^{r-1})}{k\min(1, f_0)} 2^{(r-1)/(r-2)}[\operatorname{meas}\Omega(u > k)]^{\frac{(r-1)}{r}}$$

$$\leqslant \frac{2\tilde{c}\beta^2(1 + 2^{2r-3}k^{r-1})}{k^{r-1}\min(1, f_0)} 2^{(r-1)/(r-2)}\left(\int_{\Omega(u>k)} u^r \, dx\right)^{(r-2)/r}.$$

Now, the first term of the above inequality converges to zero as k goes to $+\infty$, therefore we can fix k_1 ($\geqslant \tilde{k}$) such that

$$\frac{2\tilde{c}\beta^2(1 + 2^{2r-3}k_1^{r-1})}{\min(1, f_0)} 2^{(r-1)/(r-2)}[\operatorname{meas}\Omega(u > k_1)]^{\frac{(r-1)}{r}} \leqslant k_1. \tag{5.3}$$

Moreover, one has

$$\frac{2\tilde{c}\beta^2(1 + 2^{r-2}k^{r-1})}{\min(1, f_0)} \leqslant \frac{2\tilde{c}\beta^2(1 + 2^{2r-3}k_1^{r-1})}{\min(1, f_0)} \quad \text{if } 0 \leqslant k \leqslant 2k_1. \tag{5.4}$$

Combining (5.2) and (5.4) we obtain

$$[\operatorname{meas}\Omega(u > h)]^{1/r} \leqslant \frac{2\tilde{c}\beta^2(1 + 2^{2r-3}k_1^{r-1})}{(h-k)\min(1, f_0)}[\operatorname{meas}\Omega(u > k)]^{(r-1)/r}$$

for any $h, k \in \mathbb{R}$ such that $h > k \geqslant k_1, k \leqslant 2k_1$.

Assuming in $[k_1, +\infty[$ that

$$\varphi(k) = \begin{cases} [\operatorname{meas}\Omega(u > k)]^{1/r} & \text{if } k_1 \leqslant k \leqslant 2k_1, \\ 0 & \text{if } k > 2k_1 \end{cases}$$

we can complete the proof as in Theorem (2.1).[9]

[9] We remark that, in this case, d is the first term of (5.3).

6. A generalisation of Theorems (2.1) and (2.2)

We suppose that (2.10) holds. Moreover, let $f_1(x,\xi,p)$ be greater than or equal to zero for a.e. x in Ω and for any real numbers $\xi, p_1, \dots, p_m$.

If $u(x)$ is a subsolution of (1.1) we get

$$\int_\Omega \left\{ \sum_{i=1}^{m} a_i(x,u,\nabla u)\frac{\partial v}{\partial x_i} + f_2(x,u,\nabla u)v \right\} \mathrm{d}x \leqslant 0 \tag{6.1}$$

where $v = u - \min(u,k)$ and $k \geqslant \max(0, \sup_{\partial\Omega} u)$.

Observing that $v \in H_0^1(\nu,\Omega)$ and that $\|v\|_{1,0}$ is an equivalent norm in $H_0^1(\nu,\Omega)$ (see remark (3.3)), one concludes

$$\|v\|_{1,0}^2 \leqslant -\int_\Omega f_2(x,u,\nabla u)v\,\mathrm{d}x$$

which, as in Theorems (2.1) and (2.2), implies

$$\operatorname*{ess\,sup}_{\Omega} u \leqslant M.$$

7. Open question

It is an open question if it is possible to obtain similar results in nonlinear degenerate parabolic case.

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