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ON COMPLETE LATTICE ORDERED GROUPS WITH STRONG UNITS

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A theorem of Cantor-Bernstein type has been proved for Boolean σ-algebras by Sikorski [5] and Tarski [2] (cf. also Sikorski [6], pp. 90 and 193).

Next, a theorem of such type was proved by the author [3] for lattice ordered groups which are complete and orthogonally complete.

For a lattice ordered group G we denote by $\ell(G)$ the underlying lattice.

In the present paper we prove the following result.

(A) Let G_1 and G_2 be complete divisible lattice ordered groups having strong units u_1 and u_2, respectively. Suppose that
  - (i) there exists an isomorphism φ_1 of $\ell(G_1)$ into $\ell(G_2)$ such that $\varphi_1(\ell(G_1))$ is a convex sublattice of $\ell(G_2)$;
  - (ii) there exists an isomorphism φ_2 of $\ell(G_2)$ into $\ell(G_1)$ such that $\varphi_2(\ell(G_2))$ is a convex sublattice of $\ell(G_1)$.

Then there exists one isomorhpism φ of the lattice ordered group G_1 onto G_2 such that $\varphi(u_1) = u_2$.

If G_1 and G_2 are lattice ordered groups satisfying the conditions from (A) then they need not be orthogonally complete. Hence the consideration of the present paper cannot be subsummed under that of [3].

By means of examples we show that neither the assumption of completeness nor the assumption of the existence of strong unit can be omitted in (A).

On the other hand, the question whether (A) remains valid without assuming that G_1 and G_2 are divisible is open.

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1. Preliminaries and auxiliary results

Let L be a lattice with the least element 0. For $\emptyset \neq X \subseteq L$ we put

$$X^{\perp} = \{y \in L \colon y \wedge x = 0 \quad \text{for each} \quad x \in X\}.$$

The system $\{X^{\perp} \colon \emptyset \neq X \subseteq L\}$ will be denoted by $\mathscr{P}_1(L)$; it is partially ordered by the set theoretical inclusion.

Now, let e be an element of L such that $e \wedge a > 0$ whenever $0 < a \in L$. For $\emptyset \neq X \subseteq [0, e]$ let X^* be defined analogously as $X^{\perp}$ above with the distinction that L is replaced by $[0, e]$. For each $P \in \mathscr{P}_1(L)$ we put

$$\varphi(P) = P \cap [0, e].$$

1.1. Lemma. *Let $P \in \mathscr{P}_1(L)$. Then $\varphi(P) \in \mathscr{P}_1([0, e])$.*

Proof. Put $X = P^{\perp}$. Then $P = X^{\perp}$. Denote $X_1 = X \cap [0, e]$. We have $\emptyset \neq X_1$. If $p \in \varphi(P)$ then $p \wedge x_1 = 0$ for each $x_1 \in X_1$, whence $\varphi(P) \subseteq X_1^*$. Assume that $\varphi(P) \neq X_1^*$. Hence there is $x_1^* \in X_1^* \setminus \varphi(P)$. Thus $x_1^* \notin P$. This yields that there exists $x \in X$ such that $x_1^* \wedge x > 0$. Put $x_1^* \wedge x \wedge e = y$. Then $0 < y \in X_1$ and, at the same time, $y \leqslant x_1^*$, which is a contradiction. Therefore $\varphi(P) \in \mathscr{P}_1([0, e])$. □

1.2. Lemma. *φ is an isomorphism of $\mathscr{P}_1(L)$ onto $\mathscr{P}_1([0, e])$.*

Proof. According to 1.1, φ is a mapping of $\mathscr{P}_1(L)$ into $\mathscr{P}_1([0, e])$.

a) Let P, X and X_1 be as above. First we shall show that

$$X_1^{\perp} = X^{\perp}. \tag{1}$$

In fact, the relation $X_1 \subseteq X$ gives $X_1^{\perp} \supseteq X^{\perp}$. By way of contradiction, suppose that (1) does not hold. Hence there exists $y \in X_1^{\perp} \setminus X^{\perp}$. Then there is $x \in X$ with $y \wedge x > 0$. Thus $y \wedge x \wedge e > 0$. But $y \wedge e \in X^{\perp}$, $x \wedge a \in X_1$ and so $y \wedge x \wedge e = 0$, which is a contradiction.

The relation (1) yields that

$$(\varphi(X))^{\perp} = X^{\perp} \quad \text{for each} \quad X \in \mathscr{P}_1(L).$$

Hence

$$(\varphi(X))^{\perp\perp} = X \quad \text{for each} \quad X \in \mathscr{P}_1(L). \tag{2}$$

b) Let $Z \in \mathscr{P}_1([0,e])$. Denote

$$Z^{\perp} = P, \quad P^{\perp} = X.$$

Hence $X \supseteq Z$ and so $\varphi(X) \supseteq Z$. Assume that $\varphi(X) \neq Z$. Thus there is $x \in \varphi(X)\setminus Z$. Clearly $Z^* \subseteq P$, yielding that $x \wedge z^* = 0$ for each $z^* \in Z^*$. We obtain $x \in (Z^*)^* = Z$, which is a contradiction. Therefore φ is a bijection.

c) Let $X, Y \in \mathscr{P}_1(L)$. If $X \subseteq Y$, then clearly $\varphi(X) \subseteq \varphi(Y)$. Conversely, if $\varphi(X) \subseteq \varphi(Y)$, then (2) implies that $X \subseteq Y$. Therefore in view of b), φ is an isomorphism of $\mathscr{P}_1(L)$ onto $\mathscr{P}_1([0,e])$. □

For lattice ordered groups we apply the standard notation; the group operation will be written additively. Let G be a lattice ordered group. For $\emptyset \neq X \subseteq G$ put

$$X^{\delta} = \{g \in G \colon |g| \wedge |x| = 0 \quad \text{for each} \quad x \in X\}.$$

The set X^{δ} is a polar of G; the system of all polars of G will be denoted by $\mathscr{P}(G)$. This system is considered to be partially ordered by the set theoretical inclusion.

For each $Y \in \mathscr{P}(G)$ we put $\varphi_1(Y) = Y \cap G^+$.

1.3. Lemma. *φ_1 is an isomorphism of $\mathscr{P}(G)$ onto $\mathscr{P}_1(G^+)$.*

The proof is a routine, it will be omitted.

If G is a lattice ordered group, then to simplify the expression we often say "the lattice G" instead of "the lattice $\ell(G)$". Also, the meaning of the expression "the lattice G^+" is clear.

1.4. Lemma. *Let G_1 and G_2 be lattice ordered groups. Suppose that φ is an isomorphism of the lattice G_1^+ into the lattice G_2^+ such that $\varphi(0) = 0$ and $\varphi(G_1^+)$ is a convex sublattice of the lattice G_2^+. Then there exists an isomorphism φ_1 of the lattice G_1 into the lattice G_2 such that $\varphi_1(x) = \varphi(x)$ for each $x \in G_1^+$ and $\varphi_1(G_1)$ is a convex sublattice of the lattice G_2.*

Proof. Let $x \in G$. Put

$$\varphi_1(x) = \varphi(x^+) - \varphi(x^-), \tag{3}$$

where, as usual, $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$. Hence $\varphi_1(x) = \varphi(x)$ for each $x \in G^+$.

a) Since $x^+ \wedge x^- = 0$, we have

$$\varphi(x^+) \wedge \varphi(x^-) = 0. \tag{4}$$

From this we immediately obtain

$$(5) \qquad (\varphi_1(x))^+ = \varphi(x^+), \quad (\varphi_1(x))^- = \varphi(x^-).$$

If y_1, y_2, z_1, z_2 are elements of G_2 such that $y_1 \wedge y_2 = 0 = z_1 \wedge z_2$ and $y_1 - y_2 = z_1 - z_2$, then clearly $y_1 = z_1$ and $y_2 = z_2$. Hence (3), (4) and (5) yield that φ_1 is injective.

b) Let $x_1, x_2 \in G_1$, $y_1 = \varphi_1(x_1)$, $y_2 = \varphi_1(x_2)$, $y \in G_2$, $y_1 \leqslant y \leqslant y_2$. Hence

$$0 \leqslant y^+ \leqslant \varphi(y_2^+), \quad 0 \leqslant y^- \leqslant \varphi(y_1^-).$$

Thus there are elements $x_1', x_2' \in G_1^+$ such that

$$y^+ = \varphi(x_1'), \quad y^- = \varphi(x_2').$$

Since $y^+ \wedge y^- = 0$, we have $x_1' \wedge x_2' = 0$. Put $x_3 = x_1' - x_2'$. Then $x_3^+ = x_1'$ and $x_3^- = x_2'$. Thus $\varphi_1(x_3) = y$. Therefore $\varphi(G_1)$ is a convex subset of G_2.

c) Let x_i and y_i $(i = 1, 2)$ be as in b) with the distinction that we do not suppose the validity of the relation $y_1 \leqslant y_2$. Put

$$z_1 = x_1^+ \vee x_2^+, \quad z_2 = x_1^- \vee x_2^-.$$

Then $y_1, y_2 \in [-\varphi(z_2), \varphi(z_1)]$. This yields that both $y_1 \vee y_2$ and $y_1 \wedge y_2$ belong to the interval $[-\varphi(z_2), \varphi(z_1)]$. Hence according to b), $\varphi_1(G_1)$ is a sublattice of G_2.

d) Again, let x_i and y_i $(i = 1, 2)$ be as in c). If $x_1 \leqslant x_2$, then clearly $y_1 \leqslant y_2$. Assume that $y_1 \leqslant y_2$. Hence

$$y_1^+ \leqslant y_2^+, \quad -y_1^- \leqslant -y_2^-.$$

Since φ is an isomorphism of G_1^+ into G_2^+, from (5) we infer that

$$x_1^+ \leqslant x_2^+, \quad -x_1^- \leqslant -x_2^-.$$

Hence according to a) and c), φ_1 is an isomorphism of $\ell(G_1)$ into $\ell(G_2)$. $\square$

Let $0 < e \in G$. The element e is said to be a weak unit of G if, whenever $0 < g \in G$, then $e \wedge g > 0$. Next, e is called a strong unit in G, if for each $g \in G$ there exists a positive integer n such that $|g| \leqslant ne$. Each strong unit is a weak unit.

A nonempty set X of strictly positive elements of G is called orthogonal if $x_1 \wedge x_2 = 0$ whenever x_1 and x_2 are distinct elements of X. The lattice ordered group G is said to be orthogonally complete if each its orthogonal subset has a supremum.

2. Proof of Theorem (A)

2.1. Lemma. *Let G be a complete lattice ordered group. There exists a complete lattice ordered group G' such that*

(i) *G is a convex ℓ-subgroup of G';*
(ii) *G' is orthogonally complete;*
(iii) *if $0 < x \in G'$, then there exists an orthogonal subset $\{x_i\}_{i \in I}$ of G such that $x = \bigvee_{i \in I} x_i$ is valid in G'.*

Proof. Cf. [2], 2.20. □

2.2. Lemma. *Let G and G' be as in 2.1. If G is divisible, then G' is divisible as well.*

Proof. Assume that G is divisible. Let n be a positive integer. It suffices to verify that for each $0 < x \in G'$ there exists $y \in G'$ such that $ny = x$. Let $\{x_i\}_{i \in I}$ be as in 2.1. For each $i \in I$ there exists $y_i \in G$ with $ny_i = x_i$. The system $\{y_i\}_{i \in I}$ is orthogonal, hence there is $y \in G'$ with $y = \bigvee_{i \in I} y_i$. Thus

$$ny = n \bigvee_{i \in I} y_i = \bigvee_{i \in I} ny_i = x.$$

□

2.3. Lemma. *Let G be a lattice ordered group which is complete and divisible. Then we can define a multiplication of elements of G with reals such that G turns out to be a vector lattice.*

Proof. Cf., e.g., [1], Theorem 4.9, Corollary 2. □

2.4. Lemma. *Let G, G' be as in 2.1 and let H be an orthogonally complete lattice ordered group. Let φ be an isomorphism of the lattice G^+ into the lattice H^+ such that $\varphi(0) = 0$ and $\varphi(G^+)$ is a convex sublattice of the lattice H^+. Let $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$ be orthogonal subsets of G such that $\bigvee_{i \in I} x_i = \bigvee_{j \in J} y_j$ is valid in G'. Then $\bigvee_{i \in I} \varphi(x_i) = \bigvee_{j \in J} \varphi(y_j)$ holds in H.*

Proof. Both the sets $\{\varphi(x_i)\}_{i \in I}$ and $\{\varphi(y_j)\}_{j \in J}$ are orthogonal in H. Hence there exist y_1 and y_2 in H such that

$$y_1 = \bigvee_{i \in I} \varphi(x_i), \quad y_2 = \bigvee_{j \in J} \varphi(y_j).$$

Also, there is $0 < x \in G'$ with

$$x = \bigvee_{i \in I} x_i = \bigvee_{j \in J} y_j.$$

This yields that

$$x = \bigvee_{(i,j) \in I \times J} (x_i \wedge y_j).$$

Put $K = \{(i,j) \in I \times J : x_i \wedge y_j \neq 0\}$. Hence $K \neq \emptyset$ and the set $\{x_i \wedge y_j\}_{(i,j) \in K}$ is orthogonal. Therefore the set $\{\varphi(x_i) \wedge \varphi(y_j)\}_{(i,j) \in K}$ is orthogonal as well. Hence there exists $y_3 \in H$ such that

$$y_3 = \bigvee_{(i,j) \in K} (\varphi(x_i) \wedge \varphi(y_j)).$$

Clearly $y_3 \leqslant y_1$. Let $i \in I$. We have

$$x_i = x_i \wedge x = x_i \wedge \Big(\bigvee_{j \in J} y_j \Big) = \bigvee_{j \in J} (x_i \wedge y_j). \tag{1}$$

According to the assumption, $\varphi([0, x_i]) = [0, \varphi(x_i)]$. Hence from (1) we obtain

$$\varphi(x_i) = \bigvee_{j \in J} \varphi(x_i \wedge y_j) = \bigvee_{j \in J} (\varphi(x_i) \wedge \varphi(y_j)). \tag{1'}$$

This yields that $y_1 \leqslant y_3$. Summarizing, we obtain $y_1 = y_3$. Similarly, $y_2 = y_3$ and hence $y_1 = y_2$, completing the proof. □

Let us apply the same assumptions as in 2.4. If $x \in G'$, $x = \bigvee_{i \in I} x_i$, where $\{x_i\}_{i \in I}$ is an orthogonal subset of G, then we put

$$\varphi_1(x) = \bigvee_{i \in I} \varphi(x_i).$$

Next, let $\varphi_1(0) = 0$.

2.5. Lemma. *φ_1 is a convex isomorphism of the lattice ${G'}^+$ into the lattice H^+. For each $x \in G^+$, $\varphi_1(x) = \varphi(x)$.*

Proof. In view of 2.4, the mapping φ_1 is correctly defined. Clearly $\varphi_1(x) = \varphi(x)$ for each $x \in G^+$.

a) Let $0 < z \in H, x \in {G'}^+, z \leqslant \varphi_1(x)$. Under the notation as above we have

$$z = z \wedge \varphi_1(x) = z \wedge \Big(\bigvee_{i \in I} \varphi(x_i) \Big) = \bigvee_{i \in I} (z \wedge \varphi(x_i)).$$

Let $i \in I$. The element $z \wedge \varphi(x_i)$ belongs to the interval

$$[0, \varphi(x_i)] = \varphi([0, x_i]). \tag{2}$$

Hence there is $t_i \in [0, x_i]$ such that $\varphi(t_i) = z \wedge \varphi(x_i)$. The system of nonzero elements of the set $\{t_i\}_{i \in I}$ is orthogonal and

$$z = \bigvee_{i \in I} \varphi(t_i).$$

Thus $z \in \varphi_1(G'^+)$. Therefore $\varphi_1(G'^+)$ is a convex subset of H^+.

b) Suppose that $x, y \in G'$ amd that $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$ are orthogonal subsets of G such that the relations

$$x = \bigvee_{i \in I} x_i, \quad y = \bigvee_{j \in J} y_j$$

are valid in G'. Next, suppose that $\varphi_1(x) = \varphi_1(y)$, i.e.,

$$\bigvee_{i \in I} \varphi(x_i) = \bigvee_{j \in J} \varphi(y_j). \tag{3}$$

Let $i \in I$. Then the relation

$$\varphi(x_i) = \bigvee_{j \in J} (\varphi(x_i) \wedge \varphi(y_j) \tag{4}$$

is valid and thus, according to (2),

$$x_i = \bigvee_{j \in J} (x_i \wedge y_j). \tag{5}$$

This yields that $x \leqslant y$. Similarly we obtain that $y \leqslant x$. Therefore φ_1 is injective.

c) Let x, y be as in b) with the distinction that we do not assume that the validity of (3). If $x \leqslant y$, then clearly $\varphi_1(x) \leqslant \varphi_1(y)$. Conversely, let $\varphi_1(x) \leqslant \varphi_1(y)$. Then for each $i \in I$ the relation (4) holds; by applying (2) we obtain that (5) is valid. Thus $x \leqslant y$.

d) Again, let $x, y \in G'$. If $G \neq \{0\}$, then there is $z \in G'$ with $x < z, y < z$. Hence $\varphi_1(x) < \varphi_1(z), \varphi_1(y) < \varphi_1(z)$. Thus both the elements $\varphi_1(x) \vee \varphi_1(y), \varphi_1(x) \wedge \varphi_1(y)$ belong to the interval $[0, \varphi_1(z)]$. Therefore according to a), $\varphi_1(G'^+)$ is a sublattice of H^+. This completes the proof. $\square$

Now let G_i and φ_i $(i = 1, 2)$ satisfy the assumptions of (A). Denote

$$\varphi_{10}(x) = \varphi_1(x) - \varphi_1(0), \quad \varphi_{20}(y) = \varphi_2(y) - \varphi_2(0)$$

for each $x \in G_1$ and each $y \in G_2$. Hence φ_{10} is an isomorphism of $\ell(G_1)$ into $\ell(G_2)$ and $\varphi_{10}(0) = 0$; also, $\varphi_{10}(G_1)$ is a convex sublattice of $\ell(G_2)$. The situation for φ_{20} is analogous. Thus without loss of generality we can suppose that $\varphi_1(0) = 0$ and $\varphi_2(0) = 0$.

For $i \in \{1, 2\}$ let G'_i be a lattice ordered group such that the relation between G_i and G'_i is the same as the relation between G and G' in 2.1. Further, let $\varphi_i^+ = \varphi_i|G_i^+$. We define mappings

$$\varphi_{11}^+ : {G'_1}^+ \longrightarrow {G'_2}^+ \quad \textit{and} \quad \varphi_{21}^+ : {G'_2}^+ \longrightarrow {G'_1}^+$$

analogously as we did above for G and H.

If ψ is an isomorphism of a lattice L_1 into a lattice L_2 such that $\psi(L_1)$ is a convex sublattice of L_2, then ψ is said to be a convex lattice isomorphism.

From 2.5 we obtain

2.6. Lemma. *Both φ_{11}^+ and φ_{21}^+ are convex lattice isomorphisms. If $x \in G_1^+$ and $y \in G_2^+$, then $\varphi_{11}^+(x) = \varphi_1^+(x)$ and $\varphi_{21}^+(y) = \varphi_2(y)$.*

2.7. Lemma. *Let us apply the notation as above. There exists a convex isomorphism φ_{12} of $\ell(G'_1)$ into $\ell(G'_2)$ such that $\varphi_{12}(x) = \varphi_{11}^+(x)$ for each $x \in G_1^+$. Analogously, there exists a convex isomorphism φ_{22} of $\ell(G'_2)$ into $\ell(G'_1)$ such that $\varphi_{22}(y) = \varphi_{21}^+(y)$ for each $y \in {G'_2}^+$.*

P r o o f. This is a consequence of 2.6 and 1.4. □

2.8. Lemma. *There exists an isomorphism φ of the lattice ordered group G'_1 onto the lattice ordered group G'_2.*

P r o o f. This follows from 2.7 and [3]. □

Denote $u'_2 = \varphi(u_1)$. Since u_1 is a strong unit in G_1, it is a weak unit in G'_1. Hence u'_2 is a weak unit in G'_2. Let G'_{21} be the ℓ-subgroup of G'_2 consisting of all elements of G'_2 which are bounded with respect to u'_2; i.e., G'_{21} is the set of all $g \in G'_2$ such that there exists a positive integer n with $-nu'_2 \leqslant g \leqslant nu'_2$. Thus u'_2 is a strong unit of G'_{21}.

2.9. Lemma. *There exists an isomorphism φ^1 of G_1 onto G'_{21} such that $\varphi^1(u_1) = u'_2$.*

Proof. We have $\varphi([0,u_1]) = [0,u_2']$. Next, u_1 is a weak unit in G_1' and u_2' is a weak unit in G_2'. Thus the assertion follows from 1.3, 2.3 and from [4], Chap. XIII, Section 3.12. $\square$

2.10. Lemma. *There exists an isomorphism φ^2 of G_2 onto G_{21}' such that $\varphi^2(u_2) = u_2'$.*

Proof. Since u_2 and u_2' are weak units in G_2', we infer from 1.2 that the partially ordered sets $\mathscr{P}_1([0,u_2])$ and $\mathscr{P}_1([0,u_2'])$ are isomorphic. Hence by applying 2.3 and [4], loc. cit., we obtain the desired result. $\square$

The validity of (A) follows from 2.9 and 2.10. $\square$

2.11. Corollary. *Let G_1 and G_2 be complete divisible lattice ordered groups having strong units. If $\ell(G_1)$ and $\ell(G_2)$ are isomorphic, then G_1 and G_2 are isomorphic.*

2.12.1. Example. Let A be the additive group of all real functions defined on the set of all reals with the partial order defined componentwise. Next, let B be the ℓ-subgroup of A consisting of all functions which are bounded. Put $G_1 = B$ and $G_2 = A \times B$. Both G_1 and G_2 are divisible and complete. The conditions (i) and (ii) from (A) are satisfied. G_1 has a strong unit, but G_2 has no strong unit, hence G_1 is not isomorphic to G_2. Therefore the condition of existence of strong units cannot be omitted in (A).

2.12.2. Example. Let R be the additive group of all reals with the natural linear order and let G_1 be the subgroup of R consisting of all rationals. Next, let G_2 be the ℓ-subgroup of R consisting of all $x \in R$ which have the form $x = r_1 + r_2y$, where y is a fixed irrational number and r_1, r_2 run over G_1. Then $\ell(G_1)$ and $\ell(G_2)$ are isomorphic, but G_1 and G_2 fail to be isomorphic. Both G_1 and G_2 are divisible and have strong units. Neither G_1 nor G_2 is complete.

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