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MULTIPLICATION GROUPS OF FREE LOOPS II

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This paper is a sequel to author's work [1]. It is concerned with the structure of the multiplication group of a free loop. First we show that every non-identity permutation from the multiplication group of a free loop W fixes at most two elements, and then we explicitly describe a set of permutations generating the point-wise stabilizer $\mathrm{Mlt}(W)_{a,b}$ for arbitrary $a, b \in W$, $a \neq b$.

In [4] Kepka and Niemenmaa asked whether there exists any loop Q such that $\mathrm{Mlt}(Q)$ contains a permutation fixing exactly two elements of Q, but no $\varphi \in \mathrm{Mlt}(Q)$, $\mathrm{id}_Q \neq \varphi$ fixes three or more elements. Our result answers this question affirmatively. However, their problem seems to remain open for finite loops.

The notation and terminology of [1] will be used without explanation or apology in this paper. Our numbering here begins with Section 6; references to material in Sections 1 through 5 concern the relevant parts of [1].

6. LEFT-RIGHT SYMMETRY AND CANCELLATION

First we augment the set of permutations determined by an element a of a quasigroup Q by the (right) division $D_a\colon b \to a/b$. The division D_a can be defined for every $a \in Q$, and $D_a^{-1}(b) = b\backslash a$. The permutation group $\mathrm{Tot}(Q) = \langle L_a, R_a, D_a;\ a \in Q\rangle$ is known as the *total multiplication group*. If $Q = Q(\cdot, /, \backslash, 1)$ is a loop, then the *opposite loop* $Q^{op} = Q(\cdot^{op}, /^{op}, \backslash^{op}, 1)$ is defined by $a \cdot^{op} b = b \cdot a$, $a /^{op} b = b\backslash a$, $a \backslash^{op} b = b/a$.

If Q_1 and Q_2 are two loops, then $\varphi\colon Q_1 \to Q_2$ is called an *antihomomorphism* if it is a homomorphism of Q_1 to Q_2^{op}.

If w is a loop word over the basis X, then w^{op} denotes the loop word defined by

(i) $x^{op} = x$ for every $x \in X \cup \{1\}$,

(ii) $(u \cdot v)^{op} = v^{op} \cdot u^{op}$, $(u/v)^{op} = v^{op} \backslash u^{op}$ and $(u \backslash v)^{op} = v^{op}/u^{op}$ for any loop words u, v.

Clearly, w^{op} is a reduced word iff w is a reduced word. Thus for each $a \in W$ there exists a unique $a^{op} \in W$ with $\varrho_X(a^{op}) = (\varrho_X(a))^{op}$. The loop W^{op} is again free and the mapping $a \to a^{op}$ obviously establishes an antiisomorphism of W onto W^{op}. Moreover, $|a| = |a^{op}|$ for every $a \in W$.

For $e \in W$ we denote the set $\{L_e, L_e^{-1}, R_e, R_e^{-1}\}$ by $T(e)$ and the set $\{L_e, L_e^{-1}, R_e, R_e^{-1}, D_e, D_e^{-1}\}$ by $O(e)$. For $\varepsilon \in \{-1, +1\}$ we put $(L_e^{\varepsilon})^{op} = R_{e^{op}}^{\varepsilon}$, $(R_e^{\varepsilon})^{op} = L_{e^{op}}^{\varepsilon}$, $(D_e)^{op} = D_{e^{op}}^{-1}$ and $(D_e^{-1})^{op} = D_{e^{op}}$. Clearly we have

6.1 Lemma. *Let $a_i \in W$, $0 \leqslant i \leqslant k$ be such that $a_i = \varphi_i(a_{i-1})$, $\varphi_i \in O(e_i)$, $1 \leqslant i \leqslant k$. Then $a_i^{op} = \varphi_i^{op}(a_{i-1}^{op})$ for all $1 \leqslant i \leqslant k$ and $|a_i^{op}| = |a_i|$ for all $0 \leqslant i \leqslant k$.*

The property of the free loop expressed in this lemma will be known as the *left-right symmetry.*

If a and e are elements of W, then there exist unique reduced loop words b, f over X such that $b = \varrho_X(a)$ and $f = \varrho_X(e)$. Let $\varphi \in O(e)$, say $\varphi = L_e$ (or $\varphi = L_e^{-1}$ or $\varphi = R_e$ or ...). We will say that φ *does not cancel at a* if $f \cdot b$ (or $f \backslash b$ or $b \cdot f$ or ...) is also a reduced loop word.

The mappings $\varphi_i \in O(e_i)$, $1 \neq e_i \in W$, $i = 1, 2$ are said to have *complementary types*, if — after a possible exchange of φ_1 and φ_2 — we have either $\varphi_1 = L_{e_1}$ and $\varphi_2 = D_{e_2}$, or $\varphi_1 = R_{e_1}$ and $\varphi_2 = D_{e_2}^{-1}$, or $\varphi_1 = L_{e_1}^{-1}$ and $\varphi_2 = R_{e_2}^{-1}$.

6.2 Lemma. *Let $a, e \in W$ and $\varphi \in O(e)$ be such that φ does not cancel at a. Then $a < \varphi(a)$, $|a| + |e| = |\varphi(a)| \geqslant |a|$ and $|\varphi(a)| = |a|$ iff $\varphi = D_1^{\pm}$.*

6.3 Lemma. *Let $a, b, e, f \in W$ and $\varphi \in O(e)$, $\psi \in O(f)$ be such that $\varphi(a) = \psi(b)$, $\varphi \neq \psi$, φ does not cancel at a and ψ does not cancel at b. Then $a = f$, $b = e$, and φ^{-1} and ψ^{-1} have complementary types.*

6.4 Lemma. *Let $a, e \in W$ be such that $\varphi \in O(e)$ does not cancel at a. Then φ does not cancel at $\varphi^i(a)$ for each $i \geqslant 0$.*

6.5 Lemma. *Let $a, e \in W$ be such that $\varphi \in O(e)$ does not cancel at a. Then φ^{-1} cancels at $\varphi(a)$.*

Note that $\varphi \in O(e)$ coincides with id_W only when $e = 1$ and $\varphi \in T(e)$.

6.6 Lemma. *Let $a, e \in W$ be such that $\mathrm{id}_W \neq \varphi \in O(e)$ cancels at a. Then one of the following possibilities holds:*

(i) *$a = 1$ and $\varphi \in \{L_e, R_e, D_e, D_e^{-1}\}$, or*

(ii) $a = e$ *and* $\varphi \in \{L_e^{-1}, R_e^{-1}, D_e, D_e^{-1}\}$, *or*

(iii) φ^{-1} *does not cancel at* $\varphi(a)$, *or*

(iv) $e = \kappa(a)$, *where* $\kappa \in O(\varphi(a))$, κ *does not cancel at* a, *and* κ *and* φ *have complementary types.*

Proof. Let b, f and w be the loop words over X such that $a = \varrho_X(b)$, $e = \varrho_X(f)$ and w is the composition of b and f induced by an action of φ upon a. If w is of the form $u \cdot 1$ or $1 \cdot u$ or $u/1$ or $1\backslash u$, then clearly $b = 1 = a$, $f = u$, and (i) applies. If w is of the form u/u or $u\backslash u$, then $b = u = f$, $a = e$, and (ii) can be used. As w is not a reduced word, the only other possibility is that w is equal to one of $u \cdot (u\backslash v)$, $(v/u) \cdot u$, $u\backslash(u \cdot v)$, $(v \cdot u)/u$, $u/(v\backslash u)$ and $(u/v)\backslash u$. Then $v = \varrho_X(\varphi(a))$, and the case $u = f$ is covered by (iii). The remaining case $u = b$ corresponds to (iv) — for example $w = b \cdot (b\backslash v)$ implies $e = a\backslash\backslash\varphi(a)$, $\varphi = R_e$ and $\kappa = D_{\varphi(a)}^{-1}$. □

6.7 Lemma. *Let* $a, b, c, d, e \in W$ *and* $\varphi \in O(e)$ *be such that* $c = \varphi(a)$, $d = \varphi(b)$, $1 \notin \{a, b, c, d, e\}$ *and* $|a| + |b| = |c| + |d|$. *Then* $|c| \neq |a|$ *and the inequality* $|c| > |a|$ *yields that* φ *does not cancel at* a, φ *cancels at* b, φ^{-1} *does not cancel at* d *and* φ^{-1} *cancels at* c. *In particular,* $|c| = |a| + |e|$ *and* $|d| = |b| - |e|$.

Proof. Let $|c| \geqslant |a|$. It follows from 6.2 that φ^{-1} cancels at c. If φ did not cancel at b, we would have $|c| + |d| \geqslant |a| + |d| = |a| + |e| + |b| > |a| + |b|$. This contradicts our hypothesis, and hence φ cancels at b. To prove that φ does not cancel at a, we shall start from the opposite. Assuming that φ cancels at a, we obtain from 6.6(iv) that $e = \kappa_1(a)$, $\kappa_1 \in O(c)$, κ_1 does not cancel at a and κ_1 and φ have complementary types. If φ^{-1} cancels at d, then for similar reasons $e = \kappa_2(b)$, $\kappa_2 \in O(d)$ and κ_2 does not cancel at b and κ_2 and φ have complementary types. We have $c \neq d$, and hence $\kappa_1 \neq \kappa_2$. As $e = \kappa_1(a) = \kappa_2(b)$, by 6.3 κ_1^{-1} and κ_2^{-1} have also complementary types. However, this contradicts the fact that κ_i and φ are of complementary types for $i = 1,2$. Therefore φ^{-1} cannot cancel at d. But then by 6.2, $|a| + |b| = |a| + |e| + |d| = 2|a| + |c| + |d| \neq |c| + |d|$ — a contradiction again. We have proved that φ does not cancel at c. The rest is clear. □

7. Lifting the unit

The neutral element 1 vanishes in terms like $a \cdot 1$, $1 \cdot a$, $a \in W$, but it need not cancel in terms $a\backslash 1$, $1/a$, $(a\backslash 1)\backslash 1$, $1/(1/a)$ etc. This twofold rôle brings certain problems when dealing with the occurencies of 1 in reduced loop words. To deal with these difficulties, we construct for each element $a \in W$ the element $\overline{a}$ so that each occurence of 1 is substituted by y.

More formally, choose y with $y \notin W$ and put $\overline{X} = X \cup \{y\}$. Let $\overline{W}$ be the free loop with the basis $\overline{X}$. Define recursively a mapping $a \to \overline{a}$ of W to $\overline{W}$ by $\overline{1} = y$, $\overline{x} = x$ for $x \in X$, $\overline{a \circ b} = \overline{a} \cdot \overline{b}$, $\overline{a \backslash\backslash b} = \overline{a} \backslash \overline{b}$ and $\overline{a /\!/ b} = \overline{a}/\overline{b}$.

Clearly, $\overline{a \circ b} = \overline{a} \circ \overline{b}$, $\overline{a \backslash\backslash b} = \overline{a} \backslash\backslash \overline{b}$ and $\overline{a /\!/ b} = \overline{a} /\!/ \overline{b}$. Moreover, $|\overline{a}| > 0$ for any $a \in W$. Put also $\overline{L_e} = L_{\overline{e}}$, $\overline{L_e^{-1}} = L_{\overline{e}}^{-1}$, $\overline{R_e} = R_{\overline{e}}$ etc. Note that for any $a, b \in W$ the inequality $a < b$ implies $\overline{a} < \overline{b}$ and $|\overline{a}| < |\overline{b}|$.

7.1 Lemma. *Let $\varphi \in T(e)$, $1 \neq e \in W$, $c = \varphi(a)$. Then $\overline{c} \neq \overline{\varphi}(\overline{a})$ implies that either*

(i) *$a = 1$ and $\varphi \in \{L_e, R_e\}$, or*
(ii) *$e = a$ and $\varphi \in \{L_e^{-1}, R_e^{-1}\}$.*

P r o o f. Assume $\varphi = L_e^{\pm 1}$ and use 1.1 and 1.2. □

7.2 Corollary. *Let $\varphi \in T(e)$, $1 \neq e \in W$, $c = \varphi(a)$. Then $\overline{c} \neq \overline{\varphi}(\overline{a})$ iff $\{a, c\} = \{1, e\}$.*

7.3 Lemma. *Let $a, b, c, d, e \in W$ and $\varphi \in T(e)$ be such that $c = \varphi(a)$, $d = \varphi(b)$, $a \neq b$ and $e \neq 1$. Suppose that $\overline{c} = \overline{\varphi}(\overline{a})$ and $\overline{d} = \overline{\varphi}(\overline{b})$. Then $|c| + |d| < |a| + |b|$ implies $|\overline{c}| + |\overline{d}| < |\overline{a}| + |\overline{b}|$.*

P r o o f. We can assume that $\varphi = L_e^{\pm 1}$. It follows from 7.1, 2.1(c) and 2.2(c) that $|\overline{a}| + |\overline{b}| - |\overline{c}| - |\overline{d}| \in \{2|\overline{e}|, 2|\overline{a}|, 2|\overline{b}|\}$. □

7.4 Lemma. *Let $a, b, c, d, e \in W$ and $\varphi \in T(e)$ be such that $c = \varphi(a)$, $d = \varphi(b)$, $a \neq b$ and $e \neq 1$. If $|c| + |d| < |a| + |b|$, then $1 \notin \{a, b\}$.*

P r o o f. For $\varphi = L_e^{\pm 1}$ this is a corollary of 3.1. □

7.5 Lemma. *Let $a, e \in W$, $\varphi \in O(e)$ be such that $|\overline{\varphi(a)}| \leqslant |\overline{a}| \geqslant |\overline{\varphi^{-1}(a)}|$. Then $\varphi(a) = \varphi^{-1}(a)$ and for $\mathrm{id}_W \neq \varphi$ we have $\varphi = D_e^{\pm 1}$ and either $e = a \circ a$, or $e = a \neq 1$, or $e \in X \cup \{1\}$ and $a = 1$.*

P r o o f. Assume that $\mathrm{id}_W \neq \varphi$. Then $a = 1$ implies $\varphi = D_x^{\pm 1}$ with $x \in X \cup \{1\}$, and so we can assume $a \neq 1$ for the rest of the proof. Clearly, φ and φ^{-1} cancel at a. If φ^{-1} did not cancel at $\varphi(a)$, then φ^{-1} would not cancel at $\varphi^{-1}(\varphi(a)) = a$ by 6.4. Thus φ^{-1} cancels at $\varphi(a)$ and φ cancels at $\varphi^{-1}(a)$. Furthermore, $L_a^{\pm 1} \neq \varphi \neq R_a^{\pm 1}$, as L_a and R_a do not cancel at a. The case $\varphi = D_a^{\pm 1}$ is covered by the hypothesis, and so we can assume $e \neq a$. By 6.6 $e = \kappa_i(a)$, $i = 1, 2$, κ_i does not cancel at a. $\kappa_1 \in O(\varphi(a))$ and $\kappa_2 \in O(\varphi^{-1}(a))$. From 6.6 it also follows that κ_1 and φ are of complementary types. As the same holds for κ_2 and φ^{-1}, we see that $\kappa_1 \neq \kappa_2$. Now

6.3 applies to $\kappa_1(a) = e = \kappa_2(a)$, and we obtain $\varphi(a) = a = \varphi^{-1}(a)$. To compose irreducibly e from a and a, we cannot use a/a or $a\backslash a$. Thus $e = a \circ a$, and $\varphi = D^{\pm 1}_{a \circ a}$ follows. □

7.6 Corollary. *Let $a, e \in W$, $\mathrm{id}_W \neq \varphi \in O(e)$ be such that either $\varphi \in T(e)$, or $\varphi = D^{\pm 1}_e$ and $1 \neq a \neq e \neq a \circ a$. Then there exists $k \geqslant 0$ such that $|\overline{\varphi^{i+1}(a)}| < |\overline{\varphi^i(a)}|$ for every $0 \leqslant i \leqslant k-2$, $|\overline{\varphi^k(a)}| \leqslant |\overline{\varphi^{k-1}(a)}|$ if $k \geqslant 1$, and $|\overline{\varphi^{i+1}(a)}| > |\overline{\varphi^i(a)}|$ for every $i \geqslant k$. In particular, if the above conditions hold, then the set $\{|\overline{\varphi^i(a)}|;\, i \geqslant 0\}$ is never bounded.*

7.7 Proposition. *Let $a, e \in W$ and $\varphi \in O(e)$ be such that $\mathrm{id}_W \neq \varphi$. If $\varphi^k(a) = a$ for some $k \geqslant 1$, then $\varphi = D^{\pm 1}_e$ and either $a \in \{1, e\}$, or $e = a \circ a$. If $a \in \{1, e\}$ and $e \neq 1$, then $\varphi^k(a) = a$ iff k is even. In the other cases $\varphi^k(a) = a$ for any integer k.*

8. Fixed points

8.1 Lemma. *Let $a, e \in W$ be such that $\mathrm{id}_W \neq \varphi \in O(e)$ cancels at a. If $|\bar{a}| > |\bar{e}|$, then φ^{-1} does not cancel at $\varphi(a)$. In particular, $|\bar{a}| = |\overline{\varphi(a)}| + |\bar{e}|$.*

Proof. Consider the alternatives of 6.6. □

8.2 Lemma. *For $i = 1, 2$ let $a, e_i \in W$, $\mathrm{id}_W \neq \varphi_i \in O(e_i)$, $\varphi_1 \neq \varphi_2$, $|\bar{a}| > |\bar{e_i}|$, and suppose that φ_i cancels at a. Then φ_1 and φ_2 have complementary types and $a = \varphi_1^{-1}(e_2) = \varphi_2^{-1}(e_1)$.*

Proof. By 8.1, φ_i^{-1} does not cancel at $\varphi_i(a)$ for $i = 1,2$. The rest follows from 6.3. □

8.3 Corollary. *Let $a_j \in W$, $0 \leqslant j \leqslant 2$ and for $i = 1,2$ let $1 \neq e_i \in W$, $\varphi_i \in T(e_i)$, $\varphi_i(a_{i-1}) = a_i$, $\varphi_2 \neq \varphi_1^{-1}$. Suppose that φ_1^{-1} and φ_2 cancel at a_1 and $|\bar{a_1}| > |\bar{e_i}|$, $i = 1, 2$. Then $a_0 = e_2$, $a_2 = e_1$ and either $\varphi_1 = L_{e_1}$, $\varphi_2 = R^{-1}_{e_2}$ and $a_1 = e_1 \circ e_2$, or $\varphi_1 = R_{e_1}$, $\varphi_2 = L^{-1}_{e_2}$ and $a_2 = e_2 \circ e_1$.*

8.4 Lemma. *Let $a_j \in W$, $0 \leqslant j \leqslant 2$ and for $i = 1,2$ let $e_i \in W$, $\mathrm{id}_W \neq \varphi_i \in O(e_i)$, $\varphi_i(a_{i-1}) = a_i$, $\varphi_2 \neq \varphi_1^{-1}$. If φ_1 does not cancel at a_0 and $|\bar{a_0}| > |\bar{e_2}|$, then φ_2 does not cancel at a_1.*

Proof. Suppose that φ_2 cancels at a_1. By 6.5 we can then use 8.2 for $\varphi_1' = \varphi_1^{-1}$, $\varphi_2' = \varphi_2$. As 8.2 yields $a_0 = \varphi_1^{-1}(a_1) = e_2$, we get a contradiction. □

8.5 Lemma. *Let $a_i \in W$, $0 \leqslant i \leqslant k$ be such that for every $1 \leqslant i \leqslant k$ we have $\varphi_i(a_{i-1}) = a_i$, $\mathrm{id}_W \neq \varphi_i \in O(e_i)$, $e_i \in W$ and $|\overline{a_0}| > |\overline{e_i}|$. Further, let $\varphi_{i+1}^{-1} \neq \varphi_i$ for each $1 \leqslant i \leqslant k-1$ and suppose that φ_1 does not cancel at a_0. Then φ_i for all $1 \leqslant i \leqslant k$ does not cancel at a_{i-1} and $|\overline{a_i}| = (\sum_{1 \leqslant j \leqslant i} |\overline{e_j}|) + |\overline{a_0}|$.*

Proof. The lemma can be proved directly by induction — the inductive step is contained in 8.4. □

8.6 Lemma. *Let $a_i, b_i, c_i \in W$, $0 \leqslant i \leqslant k$ be such that $k \geqslant 2$ and for every $1 \leqslant i \leqslant k$ we have $\varphi_i(a_{i-1}) = a_i$, $\varphi_i(b_{i-1}) = b_i$, $\varphi_i(c_{i-1}) = c_i$, $\varphi_i \in T(e_i)$, $1 \neq e_i \in W$. Further, let $\varphi_{i+1}^{-1} \neq \varphi_i$ for every $1 \leqslant i \leqslant k-1$. If $a_k = a_0$, $b_k = b_0$ and $c_k = c_0$, then $a_0 = b_0$ or $a_0 = c_0$ or $b_0 = c_0$.*

Proof. Assume that $a_k = a_0$, $b_k = b_0$, $c_k = c_0$ and that a_0, b_0, c_0 are pairwise distinct. Clearly, it suffices to obtain a contradiction for the case $\varphi_k \neq \varphi_1^{-1}$. Assume that $\varphi_k \neq \varphi_1^{-1}$, and cyclically permute a_i, b_i, c_i and φ_i so that $|\overline{e_1}| = \max\{|\overline{e_i}|: 1 \leqslant i \leqslant k\}$. Denote $|\overline{e_1}|$ by m. The rest of the proof is divided into four steps. The indices of a_i, b_i, c_i, e_i and φ_i are computed modulo k.

(i) Let $1 \leqslant i \leqslant k$ be such that $|\overline{a_i}| > m$. If φ_{i+1} did not cancel at a_i, we could apply 8.5 and obtain $|\overline{a_i}| > |\overline{a_i}|$. This is not possible, and so φ_{i+1} cancels at a_i. For a similar reason φ_i^{-1} cancels at a_i, too. Now 8.3 can be used, and we see that $a_{i-1} = e_{i+1}$, $a_{i+1} = e_i$, and either $\varphi_i = L_{e_i}$, $\varphi_{i+1} = R_{e_{i+1}}^{-1}$ and $a_i = e_i \circ e_{i+1}$, or $\varphi_i = R_{e_i}$, $\varphi_{i+1} = L_{e_{i+1}}^{-1}$ and $a_i = e_{i+1} \circ e_i$.

(ii) Let $|\overline{a_1}| > m$. By (i) and the left-right symmetry we can assume that $a_0 = e_2$, $a_1 = e_1 \circ e_2$, $a_2 = e_1$, $\varphi_1 = L_{e_1}$ and $\varphi_2 = R_{e_2}^{-1}$. As $b_0 \neq a_0$, we get from (i) that $|\overline{b_1}| \leqslant m$. Because $\varphi_1 = L_{e_1}$, $|\overline{b_0}| \leqslant m$ is implied by (i), too. Thus φ_1 cancels at b_0 and φ_1^{-1} cancels at b_1. If $|\overline{b_1}| = m$ and $b_0 \neq 1$, then e_1 is by 6.6 equal to $\kappa(b_0)$, where $\kappa \in O(b_1)$ does not cancel at b_0. But then $|\overline{b_1}| \neq |\overline{b_0}| + |\overline{e_1}|$, and from 7.1 we obtain $b_0 = 1$. Hence $|\overline{b_1}| = m$ if and only if $b_0 = 1$. D_{b_1} is complementary to L_{e_1}, and so if $b_0 \neq 1$, then by 6.6 e_1 equals $b_1 /\!/ b_0$. Our argument can be repeated for c_0, c_1, and as $c_0 \neq b_0$ holds, $b_0 = 1$ and $b_1 = e_1 = c_1 /\!/ c_0$ can be assumed. If $\varphi_2 = R_{e_2}^{-1}$ does not cancel at b_1, then $|\overline{b_2}| > |\overline{b_1}| = m$. By (i) this is not possible, and therefore φ_2 cancels at b_1. But $|\overline{e_2}| \leqslant m$ implies that e_2 cannot be of the form $b_2 \backslash\!\backslash e_1$. We have $b_2 = (c_1 /\!/ c_0)/e_2$, and thus $e_2 = e_1 = c_1 /\!/ c_0$ is the only way how φ_2 can cancel at $b_1 = c_1 /\!/ c_0$. But then $|\overline{c_2}| > m$ as $e_2 = c_1 /\!/ (c_1 /\!/ c_0)$, and a contradiction follows from (i). We see that any of $|\overline{a_1}|$, $|\overline{b_1}|$ and $|\overline{c_1}|$ must be less than or equal to m.

(iii) Let $|\overline{a_0}| > m$. Then we can apply (ii) for $a'_1 = a_0$, $a'_0 = a_1$, $\varphi'_1 = \varphi_1^{-1}$, $\varphi'_2 = \varphi_k^{-1}$ etc. Thus $m \geqslant |\overline{a_0}|$, and hence also $m \geqslant |\overline{b_0}|$ and $m \geqslant |\overline{c_0}|$.

(iv) By (ii) and (iii) φ_1 cancels at a_0, b_0, c_0, and φ_1^{-1} cancels at a_1, b_1, c_1. Suppose that $a = a_0$, $\varphi = \varphi_1$ satisfy any of the conditions (i) and (ii) of 6.6. As $b_0 \neq a_0 \neq c_0$

and as $\varphi_1 \in T(e)$, none of these two conditions can be satisfied also by $a' = b_0$ or by $a' = c_0$. Thus 6.6 allows us to assume that $e_1 = \kappa_b(b_0) = \kappa_c(c_0)$, where $\kappa_b \in O(b_1)$, $\kappa_c \in O(c_1)$, κ_b does not cancel at b_0, κ_c does not cancel at c_0, κ_b and φ_1 have complementary types, and κ_c with φ_1 have complementary types, too. By the latest two statements κ_b^{-1} and κ_c^{-1} cannot have complementary types. As $b_1 \neq c_1$ implies $\kappa_b \neq \kappa_c$, we get a contradiction by applying 6.3 to $e_1 = \kappa_b(b_0) = \kappa_c(c_0)$. □

8.7 Corollary. *Each non-identity permutation contained in* Mlt(W) *fixes at most two elements of* W.

9. Common factors

For any $a, b \in W$, $a \neq 1 \neq b$, $a \neq b$ we will say that a and b have a *common factor* $w \in W$, if one of the following possibilities takes place.

(i) There exist $u, v \in W$ and $\psi \in O(w)$ such that ψ cancels neither at u nor at v, and $a = \psi(u)$, $b = \psi(v)$.

(ii) There exist $u \in W$ and $\psi \in T(u)$ such that $a = w$, $b = \psi(w)$ and ψ does not cancel at w.

(iii) There exist $u \in W$ and $\psi \in T(u)$ such that $b = w$, $a = \psi(w)$ and ψ does not cancel at w.

9.1 Lemma. *Let $a, b, c, d, e \in W$, $1 \notin \{a, b, c, d, e\}$, $\varphi \in O(e)$ be such that $\varphi(c) = a$, $\varphi(d) = b$ and $d < c$. If φ does not cancel at c and $|c| + |d| = |a| + |b|$, then $b < a$, and a and b have no common factor.*

Proof. By 6.7, φ^{-1} does not cancel at b, and hence by 6.2 and the hypothesis we have $b < d < c < a$. Suppose that w is a common factor of a and b. Following the definition, we shall distinguish three separate cases. Symbols u, v and ψ have the same meaning as in the above definition.

(i) If $a = \psi(u)$, $b = \psi(v)$, then $\varphi = \psi$ would imply that $d = v$, and φ would not cancel at d. This contradicts 6.7, and hence $\psi \neq \varphi$. But then 6.3 implies $e = w$, $u = e$, and 6.2 yields $w < b < c = w$.

(ii) If $b = \psi(a)$ and $\psi \in T(u)$ does not cancel at a, then $a < b < a$ by 6.2.

(iii) Let $a = \psi(b)$, $\psi \in T(u)$ and suppose that ψ does not cancel at b. Then $\varphi = \psi$ is not possible, as $c = b$ would follow. Hence $\varphi \neq \psi$, and by 6.3 $u = c$ and $b = e$. As $d = \varphi^{-1}(b) = \varphi^{-1}(e) \neq 1$, we obtain from $\varphi^{-1} \in O(e)$ that $\varphi^{-1} \in \{L_e, R_e\}$. As φ^{-1} and ψ^{-1} have complementary types, we see that $\psi^{-1} = D_c^{\pm 1}$. However, this contradicts $\psi \in T(u)$. □

9.2 Lemma. *Let $a, b, c, d, e \in W$ be such that $1 \notin \{a, b, e\}$, $a \neq b$ and for $\varphi = L_e^{\pm 1}$, $c = \varphi(a)$, $d = \varphi(b)$ let $|a| + |b| > |c| + |d|$. If a and b have no common factor and $|b| \leqslant |a|$, then $a = e \backslash\backslash c$ and $e = d /\!/ b$ if $\varphi = L_e$, and $a = e \circ c$ and $e = b /\!/ d$ if $\varphi = L_e^{-1}$.*

P r o o f. This follows directly from 2.1(c) and 2.2(c). □

Applying the left-right symmetry to 9.2, we obtain

9.3 Lemma. *Let $a, b, c, d, e \in W$ be such that $1 \notin \{a, b, e\}$, $a \neq b$ and for $\varphi = R_e^{\pm 1}$, $c = \varphi(a)$, $d = \varphi(b)$ let $|a| + |b| > |c| + |d|$. If a and b have no common factor and $|b| \leqslant |a|$, then $a = c /\!/ e$ and $e = b \backslash\backslash d$ if $\varphi = R_e$, and $a = c \circ e$ and $e = d \backslash\backslash b$ if $\varphi = R_e^{-1}$.*

In the rest of this section we state some further auxiliary assertions.

The correspondence between $a \in W$ and a^{op} can be used to dualize the results obtained in [1] for the left translations of W. We will do so explicitly for Lemmas 2.1 and 2.2. We obtain

9.4 Lemma. *Let $a, b, c, d, e \in W$ be such that $c = R_e(a)$, $d = R_e(b)$, $a \neq b$, and $e \neq 1$. Then exactly one of the following possibilities takes place.*

(a) $|c| + |d| > |a| + |b|$. *Then either*

(1) $c = a \circ e$, $d = b \circ e$ *or* $c = e$, $a = 1$, $d = b \circ e$ *or* $c = a \circ e$, $b = 1$, $d = e$, *or*

(2) $e = b \backslash\backslash d$, $1 \neq d$, *and* $c = a \circ e$ *or* $c = e$, $a = 1$, *or*

(3) $e = a \backslash\backslash c$, $1 \neq c$, *and* $d = b \circ e$ *or* $d = e$, $b = 1$.

(b) $|c| + |d| = |a| + |b|$. *Then either*

(1) $b = d /\!/ e$, *and* $c = a \circ e$ *or* $c = e$, $a = 1$, *or*

(2) $a = c /\!/ e$, *and* $d = b \circ e$ *or* $d = e$, $b = 1$, *or*

(3) $d = 1$, $e = b \backslash\backslash 1$, *and* $c = a \circ e$ *or* $c = e$, $a = 1$, *or*

(4) $c = 1$, $e = a \backslash\backslash 1$, *and* $d = b \circ e$ *or* $d = e$, $b = 1$.

(c) $|c| + |d| < |a| + |b|$. *Then either*

(1) $a = c /\!/ e$ *and* $b = d /\!/ e$, *or*

(2) $e = b \backslash\backslash d$ *and* $a = c /\!/ e$, *or*

(3) $e = a \backslash\backslash c$ *and* $b = d /\!/ e$.

9.5 Lemma. *Let $a, b, c, d, e \in W$ be such that $c = R_e^{-1}(a)$, $d = R_e^{-1}(b)$, $a \neq b$, and $e \neq 1$. Then exactly one of the following possibilities takes place.*

(a) $|c| + |d| > |a| + |b|$. *Then either*

(1) $c = a /\!/ e$ *and* $d = b /\!/ e$, *or*

(2) $e = d\backslash\backslash b$ *and* $c = a/\!/e$, *or*

(3) $e = c\backslash\backslash a$ *and* $d = b/\!/e$.

(b) $|c| + |d| = |a| + |b|$. *Then either*

(1) $d = b/\!/e$, *and* $a = c \circ e$ *or* $a = e$, $c = 1$, *or*

(2) $c = a/\!/e$, *and* $b = d \circ e$ *or* $b = e$, $d = 1$, *or*

(3) $b = 1$, $e = d\backslash\backslash 1$, *and* $a = c \circ e$ *or* $a = e$, $c = 1$, *or*

(4) $a = 1$, $e = c\backslash\backslash 1$, *and* $b = d \circ e$ *or* $b = e$, $d = 1$.

(c) $|c| + |d| < |a| + |b|$. *Then either*

(1) $a = c \circ e$, $b = d \circ e$ *or* $a = e$, $c = 1$, $b = d \circ e$ *or* $a = c \circ e$, $d = 1$, $b = e$, *or*

(2) $e = d\backslash\backslash b$, $1 \neq b$, *and* $a = c \circ e$ *or* $a = e$, $c = 1$, *or*

(3) $e = c\backslash\backslash a$, $1 \neq a$, *and* $b = d \circ e$ *or* $b = e$, $d = 1$.

9.6 Lemma. *Let* $a, b, u, v, c, d, e \in W$, $a \neq b$, $1 \neq e$, $\varphi \in T(e)$ *be such that* $b = u \circ v$, $a \in \{u, v\}$, $c = \varphi(a)$, $d = \varphi(b)$ *and* $|c| + |d| < |a| + |b|$. *Then* $\varphi = L_u^{-1}$ *or* $\varphi = R_v^{-1}$.

Proof. The left-right symmetry allows us to assume $\varphi = L_e^{\pm 1}$. If $\varphi = L_e$, then by 2.1(c) $e = d/\!/b$ and $a = e\backslash\backslash c = (d/\!/(u \circ v))\backslash\backslash c > a$. If $\varphi = L_e^{-1}$, $u \neq e$, then by 2.2(c) we have either $a = b \circ c$, or $e = b/\!/d$ with $a = e \circ c$ or $a = e$, $c = 1$. If $a = b \circ c = (u \circ v) \circ c$, then we get again $a < a$, which is a contradiction. If $e = b/\!/d$, then $a < b < e \leqslant a$ also implies $a < a$. □

9.7 Lemma. *Let* $a_i \in W$, $0 \leqslant i \leqslant k$ *be such that for every* $1 \leqslant i \leqslant k$ *we have* $\varphi_i(a_{i-1}) = a_i$, $\mathrm{id}_W \neq \varphi_i \in O(e_i)$, $e_i \in W$. *Suppose that* φ_i *cancels at* a_{i-1} *for no* $1 \leqslant i \leqslant k$ *and that* $1 \leqslant k$. *Then* $(\varphi_k \dots \varphi_1)^2(a_0) \neq a_0$.

Proof. If $\varphi_i = \varphi_{k+1-i}^{-1}$ for every $1 \leqslant i \leqslant k$, the define s as $k+1$. Otherwise denote by s the least $i \geqslant 1$ with $\varphi_i \neq \varphi_{k+1-i}^{-1}$. Then $(\varphi_k \dots \varphi_1)^2 = \varphi_k \dots \varphi_s \varphi_{k+1-s} \dots \varphi_1$. Suppose that $s-1 \geqslant k/2$. If $k = 2t$ is even, then $\varphi_t = \varphi_{t+1}^{-1}$. If $k = 2t+1$ is odd, then $\varphi_{t+1} = \varphi_{t+1}^{-1}$. As $(\varphi_{t+1})^2 \neq \mathrm{id}_W$ by 7.7, we see that $s-1 < k/2$. Hence $k+1 \geqslant 2s$ and $(\varphi_k \dots \varphi_1)^2 = \varphi_1^{-1} \dots \varphi_{s-1}^{-1}(\varphi_{k+1-s} \dots \varphi_s)^2 \varphi_{s-1} \dots \varphi_1$. If $k+1 = 2s$, then $\varphi_{k+1-s} = \varphi_s$ does not cancel at $a_{k+1-s} = \varphi_{k+1-s}(a_{k-s})$ by 6.4. If $k \geqslant 2s$, then $e_s < a_{k-s}$ and φ_s does not cancel at $a_{k+1-s} = \varphi_{k+1-s}(a_{k-s})$ by 8.4. Let i be the least integer such that $s+1 \leqslant i \leqslant k$ and φ_i cancels at $\varphi_{i-1} \dots \varphi_s(a_{k+1-s})$. If $s+1 \leqslant i \leqslant k+1-s$, then obviously $e_i < a_{k+1-s}$. For $k+2-s \leqslant i \leqslant k$ we have $e_i < a_{k+1-s}$ by $e_i = e_{k+1-i}$. Thus $e_i < a_{k+1-s} \leqslant \varphi_{i-2} \dots \varphi_s(a_{k+1-s})$ and 8.4 can be applied to $a_i = \varphi_i \varphi_{i-1}(\varphi_{i-2} \dots \varphi_s(a_{k+1-s}))$. Therefore φ_i cancels at $\varphi_{i-1} \dots \varphi_s(a_{k+1-s})$ for no $s \leqslant i \leqslant k$, and hence $a_0 < (\varphi_k \dots \varphi_1)^2(a_0)$. □

For any $a, b, c \in W$ define $\mu(a,b,c) = R_{b\backslash c}^{-1} L_a L_b^{-1} R_{a\backslash c}$ and $\nu(a,b,c) = L_{c/b}^{-1} R_a R_b^{-1} L_{c/a}$. If $\varphi \in \mathrm{Mlt}(W)$, we denote by $\mu_\varphi(a,b,c)$ and $\nu_\varphi(a,b,c)$ the permutations $\varphi^{-1}\mu(\varphi(a),\varphi(b),\varphi(c))\varphi$ and $\varphi^{-1}\nu(\varphi(a),\varphi(b),\varphi(c))\varphi$, respectively.

10.1 Lemma. *Let $a, b, c \in W$, $a \neq b$, $\varphi, \pi \in \mathrm{Mlt}(W)$. Then*

(i) $\mu_\varphi(a,b,c)^{-1} = \mu_\varphi(b,a,c)$ *and* $\nu_\varphi(a,b,c)^{-1} = \nu_\varphi(b,a,c)$;

(ii) $\nu_\varphi(a,b,c)^{op} = \mu_\varphi(a,b,c)$ *and* $\nu_\varphi(a,b,c)^{op} = \mu_\varphi(a,b,c)$;

(iii) $\pi\mu_\varphi(a,b,c)\pi^{-1} = \mu_{\varphi\pi^{-1}}(\pi(a),\pi(b),\pi(c))$ *and* $\pi\nu_\varphi(a,b,c)\pi^{-1} = \nu_{\varphi\pi^{-1}}(\pi(a), \pi(b),\pi(c))$.

10.2 Corollary. *Let $\psi \in \mathrm{Mlt}(W)_{a,b}$ and $\pi \in \mathrm{Mlt}(W)$. Then $\pi\psi\pi^{-1} \in \langle \mu_\varphi(\pi(a), \pi(b), c),\ \nu_\varphi(\pi(a),\pi(b),c);\ c \in W, \varphi \in \mathrm{Mlt}(W)\rangle$ iff $\psi \in \langle \mu_\varphi(a,b,c), \nu_\varphi(a,b,c);\ c \in W, \varphi \in \mathrm{Mlt}(W)\rangle$.*

10.3 Lemma. *Let $a, b, c \in W$, $a \neq b$, $\varphi \in \mathrm{Mlt}(W)$ and $\kappa \in \{\mu_\varphi(a,b,c), \nu_\varphi(a,b,c)\}$. Then $\kappa(a) = a$, $\kappa(b) = b$ and $\kappa(d) \neq d$ for every $d \in W$, $a \neq d \neq b$.*

P r o o f. By direct computation we obtain $\kappa(a) = a$, $\kappa(b) = b$. By 8.6 it remains to prove that $\kappa \neq \mathrm{id}_W$. It follows from 10.1(iii) that we can assume $\kappa \in \{\mu(a,b,c), \nu(a,b,c)\}$. We have $\nu(a,b,c) = \mu(a,b,c)^{op}$ for any $a, b, c \in W$. Thus we need to verify that $\mu(a,b,c) \neq \mathrm{id}_W$. Put $\varphi_1 = R_{b\backslash c}^{-1}$, $\varphi_2 = L_a$, $\varphi_3 = L_b^{-1}$, $\varphi_4 = R_{a\backslash c}$. We have $\varphi_i \neq \varphi_j^{-1}$ whenever $\varphi_i \neq \mathrm{id}_W$, $1 \leqslant i,j \leqslant 4$, and hence $\mu(a,b,c) = \mathrm{id}_W$ iff $\varphi_i = \mathrm{id}_W$ for all $1 \leqslant i \leqslant 4$ (use 1.5). However, this is not possible, as $a \neq b$. □

Let $1 \neq e_i \in W$, $\varphi_i \in T(e_i)$, $1 \leqslant i \leqslant k$ and let $a_0, b_0 \in W$, $a_0 \neq b_0$. By a_i, b_i, $1 \leqslant i \leqslant k$ denote the elements $a_i = \varphi_i(a_{i-1})$, $b_i = \varphi_i(b_{i-1})$. We say that the sequence $\varphi_i, 1 \leqslant i \leqslant k$ *reduces at* $\{a_0, b_0\}$ whenever for some $t \geqslant 0$ we can find a sequence $\psi_j \in T(f_j)$, $1 \neq f_j \in W$, $1 \leqslant j \leqslant t$ such that for $c_0 = a_0$, $d_0 = b_0$, $c_j = \psi_j(c_{j-1})$ and $d_j = \psi_j(d_{j-1})$ we have

(i) $c_t = a_k$ and $d_t = b_k$,

(ii) $\sum_{1 \leqslant j < t} (|\overline{c_j}| + |\overline{d_j}|) < \sum_{1 \leqslant i < k} (|\overline{a_i}| + |\overline{b_i}|)$,

(iii) either there exist such $w \in W$, $\pi \in \mathrm{Mlt}(W)$ and $\kappa \in \{\mu_\pi(a_0, b_0, w)^{\pm 1}, \nu_\pi(a_0, b_0, w)^{\pm 1}\}$ that $\varphi_k \dots \varphi_1 = \psi_t \dots \psi_1 \kappa$, or there exist $w \in W$, $\pi \in \mathrm{Mlt}(W)$ and $\kappa \in \{\mu_\pi(a_k, b_k, w)^{\pm 1}, \nu_\pi(a_k, b_k, w)^{\pm 1}\}$ such that $\varphi_k \dots \varphi_1 = \kappa\psi_t \dots \psi_1$.

Note that we have not excluded the case $t = 0$.

Let now $\varphi_1, \dots, \varphi_k$ be a sequence of permutations such that $\varphi_k \dots \varphi_1$ fixes exactly two elements of W and $\varphi_i \in T(e_i)$, $1 \neq e_i \in W$ for each $1 \leqslant i \leqslant k$. Let $a_0 \neq b_0$ be the points fixed by the permutation and let $a_i = \varphi_i(a_{i-1})$, $b_i = \varphi_i(b_{i-1})$ for $1 \leqslant i \leqslant k$ (we have $a_k = a_0$ and $b_k = b_0$). We say that the sequence $\varphi_1, \dots, \varphi_k$ *reduces at its fixed points* if there exist $1 \leqslant r, s \leqslant k$ such that the sequence $\varphi_r, \varphi_{r+1}, \dots, \varphi_{s-1}, \varphi_s$ reduces at $\{a_{r-1}, b_{r-1}\}$ (the indices being computed modulo k).

Assume now that the sequence $\varphi_1, \dots, \varphi_k$ satisfies $\varphi_i \neq \varphi_{i+1}^{-1}$ for all $1 \leqslant i \leqslant k-1$ and put $\psi = \varphi_k \dots \varphi_1$ and $n(\psi) = \sum\limits_{1 \leqslant i \leqslant k} |\overline{a}_i| + |\overline{b}_i|$. As $\varphi_k, \dots, \varphi_1$ are determined by ψ uniquely, $n(\psi)$ is well defined for any $\psi \in \mathrm{Mlt}(W)$ that fixes exactly two elements of W. Suppose that $n(\psi)$ is the least possible with respect to the property $\psi \notin \langle \mu_\pi(a_0, b_0, w), \nu_\pi(a_0, b_0, w)\,;\, w \in W, \pi \in \mathrm{Mlt}(W)\rangle$ (running over all possible choices of a_0 and b_0). The minimality condition posed on $n(\psi)$ together with 10.2 yield $\varphi_1 \neq \varphi_k^{-1}$. Put $\psi_j = \varphi_{j-1} \dots \varphi_1 \varphi_k \dots \varphi_j$ for any $1 \leqslant j \leqslant k$. Clearly, $\psi_1 = \psi$ and $n(\psi_j) = n(\psi)$ for all $1 \leqslant j \leqslant k$. Moreover, $\psi_j \notin \langle \mu_\pi(a_{j-1}, b_{j-1}, w), \nu_\pi(a_{j-1}, b_{j-1}, w)\,;\, w \in W, \pi \in \mathrm{Mlt}(W)\rangle$ — this follows again from 10.2. If the sequence $\varphi_1, \dots, \varphi_k$ can be reduced at its fixed points, we can find $1 \leqslant j \leqslant k$ and $\psi' \in \mathrm{Mlt}(W)$ such that $\psi'(a_{j-1}) = a_{j-1}$, $\psi'(b_{j-1}) = b_{j-1}$, $n(\psi') < n(\psi)$, and for some $\kappa \in \{\mu_\pi(a_{j-1}, b_{j-1}, w), \nu_\pi(a_{j-1}, b_{j-1}, w)\,;\, \pi \in \mathrm{Mlt}(W), w \in W\}$ we have $\psi_j = \psi'\kappa$ or $\psi_j = \kappa\psi'$. Thus we have proved

10.4 Lemma. *Suppose that every sequence $\varphi_i \in T(e_i), 1 \neq e_i \in W$, $1 \leqslant i \leqslant k$ such that $\varphi_k \dots \varphi_1$ fixes exactly two elements of W, $\varphi_i \neq \varphi_{i+1}^{-1}$ for $1 \leqslant i \leqslant k-1$ and $\varphi_k \neq \varphi_1^{-1}$ reduces at its fixed points. Then for any $a, b \in W, a \neq b$ we have $\mathrm{Mlt}(W)_{a,b} = \langle \mu_\varphi(a, b, c), \nu_\varphi(a, b, c)\,;\, \varphi \in \mathrm{Mlt}(W)$ and $c \in W\rangle$.*

The rest of this paper is devoted to the proof that each sequence φ_i satisfying the hypothesis of 10.4 can be reduced at its fixed points. Like in Sections 2, 3 and 4 we proceed by considering the norm sums $|\varphi_j \dots \varphi_1(a)| + |\varphi_j \dots \varphi_1(b)|$.

10.5 Lemma. *Let $a_i, b_i, e, f \in W$, $a_i \neq b_i$, $0 \leqslant i \leqslant 2$, $e \neq 1 \neq f$ be such that $|a_0| + |b_0| < |a_1| + |b_1| > |a_2| + |b_2|$, $a_2 = a_1/f$, $b_2 = b_1/f$, $a_1 = e \cdot a_0$ and $b_1 = e \circ b_0$. If $b_0 = f$ or $b_2 = e$, then the sequence L_e. R_f^{-1} can be reduced at $\{a_0, b_0\}$.*

P r o o f. From $b_0 = f$ it follows that $b_2 = b_1/f = (e \cdot f)/f = e$. As $b_2 = e$ similarly implies $b_0 = f$, we see that the assumptions $b_0 = f$ and $b_2 = e$ are equivalent. Under these assumptions we have $|a_0| + |f| < |a_1| + |e| + |f| > |a_2| + |e|$, $a_2 = a_1/f$, $a_1 = e \cdot a_0$ and $b_1 = e \circ f$. We put $\psi = R_{a_0}^{-1} L_{a_2}$ — clearly $\psi(a_0) = a_2$ and $\psi(f) = e$. Further, $R_f^{-1} L_e = \psi L_{a_2}^{-1} R_{a_0} R_f^{-1} L_e = \psi\nu(a_0, f, e \cdot a_0)$. This proves the lemma whenever $1 \in \{a_0, a_2\}$. Suppose that $a_0 \neq 1 \neq a_2$. Then we have to show that $|\overline{L_{a_2}(f)}| + |\overline{L_{a_2}(a_0)}| < |\overline{a_1}| + |\overline{b_1}|$. However, $L_{a_2}(f) = a_1 = e \cdot a_0$, and hence it suffices to prove that $|\overline{a_2}|+|\overline{a_0}|<|\overline{b_1}|=|\overline{e}|+|\overline{f}|$. We have $e\neq 1\neq a_2$ and $f\neq e\backslash\backslash b_1$, and thus 9.5(c) implies that either $a_1=a_2\circ f$, or $f=a_2\backslash\backslash a_1$. Similarly, by 2.1(a) either $a_1=e\circ a_0$, or $e=a_1/\!/a_0$. Nevertheless, $a_1=a_2\circ f$ and $a_1=e\circ a_0$ cannot hold simultaneously, as $a_0=f=b_0$ would follow. Thus $a_1=e\circ a_0$ implies $f=a_2\backslash\backslash a_1$, and we get $|\overline{a_2}|+|\overline{a_0}|<|\overline{a_2}|+|\overline{a_1}|=|\overline{f}|$. If $e=a_1/\!/a_0$, then for $a_1=a_2\circ f$ we obtain $|\overline{a_0}|+|\overline{a_2}|<|\overline{a_0}|+|\overline{a_1}|=|\overline{e}|$, while for $f=a_2\backslash\backslash a_1$ we have $|\overline{a_0}|<|\overline{e}|$ and $|\overline{a_2}|<|\overline{f}|$. □

10.6 Lemma. *Let $a_i,b_i,e,f\in W$, $a_i\neq b_i$, $0\leqslant i\leqslant 2$ be such that for $\varphi_1\in T(e)$, $\varphi_2\in T(f)$, $e\neq 1\neq f$, $\varphi_1\neq\varphi_2^{-1}$ it holds $a_j=\varphi_j(a_{j-1})$, $b_j=\varphi_j(b_{j-1})$, $j=1,2$. If $|b_2|+|a_2|<|b_1|+|a_1|>|b_0|+|a_0|$, then the sequence φ_1, φ_2 can be reduced at $\{a_0,b_0\}$.*

P r o o f. Taking into account the left-right symmetry, we can assume $\varphi_1=L_e^{\pm1}$. It follows from 2.3 that then $\varphi_2=R_f^{\pm1}$ can be assumed as well. Thus there are four different cases to be investigated. However, the case $\varphi_1=L_e^{-1}$, $\varphi_2=R_f^{-1}$ can be reduced to the case $\varphi_1=L_e$, $\varphi_2=R_f$, as $(R_f^{-1}L_e^{-1})^{-1}=L_eR_f$ and $(L_eR_f)^{op}=R_{e^{op}}L_{f^{op}}$.

If $\varphi_1=L_e^{-1}$ and $\varphi_2=R_f$, then by 2.2(a) we can assume $b_1=e\backslash\backslash b_0$. Then $b_1\neq b_2/\!/f$, and hence by 9.4(c) $a_1=a_2/\!/f$ and $f=b_1\backslash\backslash b_2$. Then $e=a_1/\!/a_2$ by 2.2(a) and $b_1=(a_1/\!/a_2)\backslash\backslash b_0=((a_2/\!/f)/\!/a_2)\backslash\backslash b_0=((a_2/\!/(b_1\backslash\backslash b_2))/\!/a_2)\backslash\backslash b_0$ leads to a contradiction.

Therefore we can have $\varphi_1=L_e$ in the rest of the proof. By 2.1(a) we can assume that either $b_1=e\circ b_0$, or $a_0=1$ and $a_1=e=b_1/\!/b_0$.

Let now $\varphi_2=R_f$ and suppose first that $a_1=e=b_1/\!/b_0$ and $a_0=1$. If $a_1=a_2/\!/f$, then $a_2=b_1$, $f=b_0$ and $b_2=b_1\cdot f=b_1\cdot b_0$, $a_2=b_1=b_1\cdot a_0$. This together with $\varphi_2\varphi_1=L_{b_1}L_{b_1}^{-1}R_{b_0}L_{b_1/\!/b_0}=L_{b_1}\nu(b_0,1,b_1)$ yields a reduction. If $a_1\neq a_2/\!/f$, then $f=a_1\backslash\backslash a_2$ and $b_1=b_2/\!/f$ by 9.4(c). However, $f=a_1\backslash\backslash a_2=(b_1/\!/b_0)\backslash\backslash a_2=((b_2/\!/f)/\!/b_0)\backslash\backslash a_2$ leads to a contradiction. Thus we can suppose that $b_1=e\circ b_0$. Then $b_1\neq b_2/\!/f$ and 9.4(c) implies $f=b_1\backslash\backslash b_2$ and $a_1=a_2/\!/f$. Therefore we have $a_1=a_2/\!/(b_1\backslash\backslash b_2)=a_2/\!/((e\circ b_0)\backslash\backslash b_2)$. By 2.1(a) either $e=a_1$, or $e=a_1/\!/a_0$ — but clearly none of that can hold.

Assuming $\varphi_2=R_f^{-1}$ let us again suppose first that $a_1=e=b_1/\!/b_0$ and $a_0=1$. By 9.5(c) then either $a_1=f$ and $b_1=b_2\circ f$, or $f=a_2\backslash\backslash a_1$. If $a_1=f=e$ and $b_1=b_2\circ f$, then $e=b_1/\!/b_0=(b_2\circ e)/\!/b_0$. If $f=a_2\backslash\backslash a_1=a_2\backslash\backslash(b_1/\!/b_0)$, then by 9.5(c) either $b_1=f$, or $b_1=b_2\circ f$, none of which is possible. Thus we can assume that $b_1=e\circ b_0$. If $b_1=b_2\circ f$, then $f=b_0$ and we can use 10.5. If $a_1=a_2\circ f$, then by 10.5 we can omit the case $a_1=e\circ a_0$. Thus for $a_1=a_2\circ f$ we get from 2.1(a) that either $a_1=e$, or $e=a_1/\!/a_0$. Then b_1 is equal to one of $(a_2\circ f)\circ b_0$ and $((a_2\circ f)/\!/a_0)\circ b_0$. By 9.5(c) $a_1=a_2\circ f$, $b_1\neq b_2\circ f$ imply $f\in\{b_1,b_2\backslash\backslash b_1\}$, and thus we always obtain a contradiction in such a case. If $a_1 \neq a_2 \circ f$ and $e \circ b_0 = b_1 \neq b_2 \circ f$, then it follows from 9.5(c) that $f = a_1 = b_2 \backslash\backslash b_1$, and hence $a_1 = b_2 \backslash\backslash (e \circ b_0)$. By 2.1(a) in such a case either $e = a_1$, or $e = a_1 /\!/ a_0$ — a contradiction again. □

11. Sequences containing the unit element

11.1 Lemma. *Let $c, d, e, g \in W$ and $\varphi \in T(e)$ be such that $1 \neq e$, $d = \varphi(1)$, $c = \varphi(g \backslash\backslash 1)$ and $|d| + |c| \leqslant |g|$. Then $c = 1$ and either*

(i) *$d = g$ and $\varphi \in \{L_g, R^{-1}_{g\backslash\backslash 1}\}$, or*
(ii) *$d = (g\backslash\backslash 1)\backslash\backslash 1$ and $\varphi \in \{L^{-1}_{g\backslash\backslash 1}, R_{(g\backslash\backslash 1)\backslash\backslash 1}\}$.*

Proof. If $\varphi = L_e^{\pm 1}$, then we can use 3.1. If $\varphi = R_c^{\pm 1}$, then $\varphi^{op} = L^{\pm 1}_{e^{op}}$, $(g\backslash\backslash 1)^{op} = 1 /\!/ g^{op}$, and we obtain the result again from 3.1. □

11.2 Lemma. *Let $c, d, e, g, h \in W$ and $\varphi \in T(e)$ be such that $e \neq 1 \neq h$, $d = \varphi(1)$, $c = \varphi(g\backslash\backslash h)$ and $|d| + |c| \leqslant |g| + |h|$. If $c \neq 1$, then $c = h$, $d = g$ and $\varphi = L_g$. If $c = 1$, then either*

(i) *$d = 1/\!/(g\backslash\backslash h)$ and $\varphi \in \{R^{-1}_{g\backslash\backslash h}, L_{1/\!/(g\backslash\backslash h)}\}$, or*
(ii) *$d = (g\backslash\backslash h)\backslash\backslash 1$ and $\varphi \in \{L^{-1}_{g\backslash\backslash h}, R_{(g\backslash\backslash h)\backslash\backslash 1}\}$.*

Proof. Like above, employ 3.1. □

11.3 Lemma. *Let $a_i, b_i \in W$, $a_i \neq b_i$, $0 \leqslant i \leqslant k$ be such that $k \geqslant 2$, $b_0 = a_1 = 1$, $b_1 = a_0\backslash\backslash 1$, and for $1 \leqslant i \leqslant k$ let $|a_i| + |b_i| = |a_0| + |b_0| = |a_0|$, $\varphi_i(a_{i-1}) = a_i$, $\varphi_i(b_{i-1}) = b_i$, $\varphi_i \in T(e_i)$, with $1 \neq e_i \in W$. Further, let $\varphi^{-1}_{i+1} \neq \varphi_i$ for each $1 \leqslant i \leqslant k-1$. Then either there exists $1 \leqslant j \leqslant k-1$ such that the sequence $\varphi_j, \dots, \varphi_k$ can be reduced at $\{a_{j-1}, b_{j-1}\}$, or for i odd $b_i = a_{i-1}\backslash\backslash 1$, $a_i = 1$ and for i even $a_i = b_{i-1}\backslash\backslash 1$, $b_i = 1$, $1 \leqslant i \leqslant k$.*

Proof. Suppose that $b_i = a_{i-1}\backslash\backslash 1$, $a_i = b_{i-1} = 1$ and $1 \leqslant i \leqslant k-1$. Then 11.1 can be applied for $g = a_{i-1}$, $e = e_{i+1}$, $c = b_{i+1}$ and $d = a_{i+1}$. If $d = (a_{i-1}\backslash\backslash 1)\backslash\backslash 1$, then $a_{i+1} = b_i\backslash\backslash 1$ and $b_{i+1} = 1$. If $d = g$, then $a_{i-1} = a_{i+1} = g$, $b_{i-1} = b_{i+1} = 1$ and by 11.1, $\varphi_{i+1} \in \{R^{-1}_{g\backslash\backslash 1}, L_g\}$. Applying 11.1 to φ_i^{-1} we obtain $\varphi_i^{-1} \in \{R^{-1}_{g\backslash\backslash 1}, L_g\}$, too, and so $\varphi_{i+1}\varphi_i = (L_g R_{g\backslash\backslash 1})^{\pm 1}$. As $L_g R_{g\backslash\backslash 1} = \mu(g, 1, 1)$, we see that the lemma can be proved by induction. □

11.4 Lemma. *Let $a_i, b_i \in W$, $a_i \neq b_i$, $0 \leqslant i \leqslant k$ be such that $k \geqslant 2$, and for $1 \leqslant i \leqslant k$ let $|a_i| + |b_i| = |a_0| + |b_0|$, $\varphi_i(a_{i-1}) = a_i$, $\varphi_i(b_{i-1}) = b_i$, $\varphi_i \in T(e_i)$, with $1 \neq e_i \in W$. Furthermore, let $\varphi^{-1}_{i+1} \neq \varphi_i$ for each $1 \leqslant i \leqslant k-1$. Suppose that there* *exists $2 \leqslant j \leqslant k-1$ with $b_j = 1$ so that $b_{j-1} = e_j$ or $b_{j+1} = e_{j+1}$. Then there exists $1 \leqslant r \leqslant k$ such that at least one of the following possibilities is true:*

(i) $r \leqslant k-1$, $b_r = a_{r+1} = 1$ *and either* $b_{r+1} = a_r \backslash\!\!\backslash 1$, *or* $b_{r+1} = 1 /\!\!/ a_r$;
(ii) $r \geqslant 1$, $b_r = a_{r-1} = 1$ *and either* $b_{r-1} = a_r \backslash\!\!\backslash 1$, *or* $b_{r-1} = 1 /\!\!/ a_r$;
(iii) $r \leqslant k-1$, $a_r = b_{r+1} = 1$ *and either* $a_{r+1} = b_r \backslash\!\!\backslash 1$, *or* $a_{r+1} = 1 /\!\!/ b_r$;
(iv) $r \geqslant 1$, $a_r = b_{r-1} = 1$ *and either* $a_{r-1} = b_r \backslash\!\!\backslash 1$, *or* $a_{r-1} = 1 /\!\!/ b_r$.

Proof. As we can consider $\varphi_k^{-1}, \dots, \varphi_1^{-1}$ in place of $\varphi_1, \dots, \varphi_k$, we can assume that $e_j = b_{j-1}$. Then clearly $\varphi_j \in \{L_{b_{j-1}}^{-1}, R_{b_{j-1}}^{-1}\}$, and by the left-right symmetry we can choose the case $\varphi_j = L_{b_{j-1}}^{-1}$. By 3.1 either $a_j = b_{j-1} \backslash\!\!\backslash a_{j-1}$, or $a_{j-1} = 1$ and $b_{j-1} = 1 /\!\!/ a_j$. In the latter case (ii) applies for $r = j$, and so $a_j = b_{j-1} \backslash\!\!\backslash a_{j-1}$ can be assumed. If $a_{j-1} = 1$, then (iii) holds for $r = j-1$. Finally, for $a_{j-1} \neq 1$ use 11.2 with $\varphi = \varphi_{j+1}$, $e = e_{j+1}$, $g = b_{j-1}$, $h = a_{j-1}$. As $\varphi_{j+1} \neq L_g = \varphi_j^{-1}$, we see that (i) takes place for $r = j$. □

11.5 Lemma. *Let $a_i, b_i \in W$, $a_i \neq b_i$, $0 \leqslant i \leqslant k$ be such that $k \geqslant 2$, and for $1 \leqslant i \leqslant k$ let $|a_i| + |b_i| = |a_0| + |b_0|$, $\varphi_i(a_{i-1}) = a_i$, $\varphi_i(b_{i-1}) = b_i$, $\varphi_i \in T(e_i)$, with $1 \neq e_i \in W$. Further, let $\varphi_{i+1}^{-1} \neq \varphi_i$ for each $1 \leqslant i \leqslant k-1$, $\varphi_k \neq \varphi_1^{-1}$ and let $a_k = a_0$, $b_k = b_0$. Then there exists no $0 \leqslant j \leqslant k$ with $1 = b_j$ so that $b_{j+1} = e_{j+1}$ or $b_{j-1} = e_j$ (the indices being computed modulo k), or the sequence $\varphi_1, \dots, \varphi_k$ can be reduced at its fixed points.*

Proof. Assume the contrary. First, cyclically permute a_i, b_i, e_i, φ_i so that the hypothesis of 11.4 is satisfied. Considering the four possibilities of 11.4, we see that by exchanging a_i and b_i we can reduce (iii) and (iv) to (i) and (ii). As the inverse mappings φ_i^{-1} can be used in place of φ_i, it is enough to consider just the case (i). Because of the left–right symmetry, we can choose the case $b_{r+1} = a_r \backslash\!\!\backslash 1$. Finally, using cyclic permutation we can assume $r = 0$. Then the hypothesis of 11.3 gets satisfied, and we obtain $|\overline{a_0}| + |\overline{b_0}| < |\overline{a_1}| + |\overline{b_1}| < \dots < |\overline{a_k}| + |\overline{b_k}|$ — a contradiction. □

11.6 Lemma. *Let $a_i, b_i \in W$, $a_i \neq b_i$, $0 \leqslant i \leqslant k$ be such that $k \geqslant 3$, and for $1 \leqslant i \leqslant k$ let $\varphi_i(a_{i-1}) = a_i$, $\varphi_i(b_{i-1}) = b_i$, $\varphi_i \in T(e_i)$, with $1 \neq e_i \in W$. Further, for each $1 \leqslant i \leqslant k-1$ let $\varphi_{i+1}^{-1} \neq \varphi_i$ and $|a_0| + |b_0| < |a_1| + |b_1| = |a_i| + |b_i| > |a_k| + |b_k|$. Then the sequence $\varphi_1, \dots, \varphi_{k-1}$ can be reduced at $\{a_j, b_j\}$ for some $2 \leqslant j \leqslant k-3$, or there exists no $1 \leqslant j \leqslant k-1$ with $b_j = 1$ such that $b_{j+1} = e_{j+1}$ or $b_{j-1} = e_j$.*

Proof. Start from the contrary and suppose that there exist $1 \leqslant j \leqslant k-1$ with $b_j = 1$ such that $b_{j+1} = e_{j+1}$ or $b_{j-1} = e_j$. Then 7.4 implies that $2 \leqslant j \leqslant k-2$. Proceeding similarly as in the preceding proof, we see that we can assume existence of such $2 \leqslant r \leqslant k-2$ that $b_r = a_{r+1} = 1$, $b_{r+1} = a_r \backslash\backslash 1$. But then $1 \in \{a_{k-1}, b_{k-1}\}$ by 11.3, and a contradiction follows from 7.4. □

12. Sequences without the unit element and with equal norm sums

Let $a_i, b_i \in W$, $1 \neq a_i \neq b_i \neq 1$, $0 \leqslant i \leqslant k$, $k > 1$ be such that for $1 \leqslant i \leqslant k$ we have $|a_i|+|b_i| = |a_0|+|b_0|$, $\varphi_i(a_{i-1}) = a_i$, $\varphi_i(b_{i-1}) = b_i$, $\varphi_i \in T(e_i)$, with $1 \neq e_i \in W$. Moreover, let $\varphi_{i-1}^{-1} \neq \varphi_i$ for each $1 \leqslant i \leqslant k$.

We will deal with such situation throughout this section. First, inductively define $c_i, d_i \in \{a_i, b_i\}$, $0 \leqslant i \leqslant k$ so that $\{c_i, d_i\} = \{a_i, b_i\}$ and

(i) $c_0 = a_0$ if $|a_1| > |a_0|$, and $c_0 = b_0$ if $|b_1| > |b_0|$;

(ii) if $1 \leqslant i \leqslant k$, then $c_i = a_i$ if $|a_i| > |a_{i-1}|$, and $c_i = b_i$ if $|b_i| > |b_{i-1}|$.

We denote by J the set $\{0 \leqslant i \leqslant k-1\,;\ c_i = b_i$ and $c_{i+1} = a_{i+1}\} \cup \{0 \leqslant i \leqslant k-1\,;\ c_i = a_i$ and $c_{i+1} = b_{i+1}\}$. Clearly, we have

12.1 Lemma. *$0 \leqslant i \leqslant k-1$ belongs to J iff $\varphi_{i+1}(d_i) \neq d_{i+1}$.*

12.2 Lemma. *If $i \in J$, then $i \geqslant 1$ and $i+1 \notin J$, $i-1 \notin J$. For any $i \in J$ we have $c_{i-1} = e_{i+1}$, $d_{i+1} = e_i$ and either $\varphi_i = L_{e_i}$, $\varphi_{i+1} = R_{e_{i+1}}^{-1}$, $c_i = e_i \circ e_{i+1}$, $d_{i-1} = e_i \backslash\backslash d_i$ and $c_{i+1} = d_i /\!/ e_{i+1}$, or $\varphi_i = R_{e_i}$, $\varphi_{i+1} = L_{e_{i+1}}^{-1}$, $c_i = e_{i+1} \circ e_i$, $d_{i-1} = d_i /\!/ e_i$ and $c_{i+1} = e_{i+1} \backslash\backslash d_i$.*

P r o o f. Consider an arbitrary $i \in J$. By the definition, $c_1 = a_1$ iff $c_0 = a_0$, and hence $i \geqslant 1$. By 6.7, φ_i^{-1} and φ_{i+1} cancel at c_i, and hence 8.3 can be applied. It follows that c_i must be of the form $e_i \circ e_{i+1}$ or $e_{i+1} \circ e_i$. By the left-right symmetry we can restrict ourselves to the case $c_i = e_i \circ e_{i+1}$. By 8.3 $\varphi_i = L_{e_i}$, $\varphi_{i+1} = R_{e_{i+1}}^{-1}$ and by 6.7 $\varphi_i^{-1}(d_i) = e_i \backslash\backslash d_i$ and $\varphi_{i+1}(d_i) = d_i /\!/ e_{i+1}$. If $\varphi_i^{-1}(d_i) = c_{i-1}$, then $i-1 \in J$ by 12.1, and $c_{i-1} \in \{e_{i-1} \circ e_i, e_i \circ e_{i-1}\}$ by the preceding part of the proof. While this is not the case, we have $\varphi_i^{-1}(d_i) = d_{i-1}$, and for the very same reason $\varphi_{i+1}(d_i) = d_{i+1}$ as well. Thus $i-1 \notin J$ and $i+1 \notin J$. □

For every $i \in J$ we define $\pi_i = D_{d_i}$ if $\varphi_i = L_{e_i}$, and $\pi_i = D_{d_i}^{-1}$ if $\varphi_i = R_{e_i}$. If $i \in J$, then 12.2 implies that π_i does not cancel at c_{i-1}, $\pi_i(c_{i-1}) = c_{i+1}$ and $\pi_i(d_{i-1}) = d_{i+1}$.

We now put $K = \{0 \leqslant i \leqslant k\,;\ i \notin J\}$ and $r = \mathrm{card}(K) - 1$. Clearly $K \cap J = \emptyset$, $K \cup J = \{i\,;\ 0 \leqslant i \leqslant k\}$ and by 12.2 $r \geqslant k/2$. For each $i \in K$, $i < k$ we define $\beta(i) = i+1$ if $i+1 \in K$, and $\beta(i) = i+2$ if $i+1 \in J$. Thus $\beta(i) = \min\{j\,;\ j \in K$ and $j > i\}$ is always the "successor" of i in K.

For every $i \in K$, $i > 0$ put $\psi_i = \pi_{i-1}$ if $i-1 \in J$, and $\psi_i = \varphi_i$ if $i-1 \in K$.

12.3 Lemma. *Let $i \in K$, $i < k$. Then $\psi_{\beta(i)}(c_i) = c_{\beta(i)}$, $\psi_{\beta(i)}(d_i) = d_{\beta(i)}$, $|c_i| + |d_i| = |c_{\beta(i)}| + |d_{\beta(i)}|$ and $\varphi_{\beta(i)}$ does not cancel at c_i. Moreover, $\psi_i \neq \psi^{-1}_{\beta(i)}$ whenever $i > 0$.*

Proof. If $\beta(i) - 1 \in K$, then $\psi_{\beta(i)} = \varphi_{i+1}$ does not cancel at $c_i = \varphi^{-1}_{i+1}(c_{i+1})$ by 6.7. For $\beta(i) - 1 \in J$ we have $\psi_{\beta(i)} = \pi_{i+1}$, and thus by the observations preceding the lemma it remains to prove only that $\psi_i \neq \psi^{-1}_{\beta(i)}$. To do that we need to consider just the case $i - 1 \in J$ and $\beta(i) - 1 \in J$. By the left-right symmetry we can choose the case $\pi_{i-1} = D_{d_{i-1}}$ and $\pi_{i+1} = D^{-1}_{d_{i+1}}$. Then by 12.2 we have $c_i = d_{i-1} /\!\!/ e_i = e_{i+2}$ and $c_{i+2} = e_{i+2} \backslash\!\!\backslash d_{i+1}$. Therefore $c_{i+2} = (d_{i-1} /\!\!/ e_i) \backslash\!\!\backslash d_{i+1}$, and hence $d_{i-1} \neq d_{i+1}$. □

12.4 Lemma. *$a_k \neq a_0$ or $b_k \neq b_0$.*

Proof. Put $\kappa_i = \psi_{\beta^i(0)}$ for each $1 \leqslant i \leqslant r$ and assume $a_0 = a_k$ and $b_0 = b_k$. Then $c_k = \kappa_r \dots \kappa_1(c_0) \neq d_k = \kappa_r \dots \kappa_1(d_0)$, $\{c_k, d_k\} = \{c_0, d_0\}$, and hence $c_0 = (\kappa_r \dots \kappa_1)^2(c_0)$. However, by 12.3 and 9.7 this is not possible. □

12.5 **Remark.** For $2 \leqslant k' \leqslant k$ the sequences $a'_i = a_i$, $b'_i = b_i$, $0 \leqslant i \leqslant k'$ satisfy the conditions introduced at the beginning of this section. Clearly, $c'_i = c_i$ and $d'_i = d_i$ for all $0 \leqslant i \leqslant k'$.

12.6 Lemma. *If there exists $j \in K$ with $d_j < c_j$ and $j \neq k$, then for all $k \geqslant i \geqslant j + 1$ the element d_i has no common factor with c_i and $d_i < c_i$.*

Proof. By 12.5 we can assume $i = k$. The lemma then follows from 12.3 and 9.1. □

12.7 Lemma. *Let $0 \leqslant j \leqslant k - 1$, $j \in K$ be such that $d_j < c_j$ and suppose that there exists $\varphi \in T(e)$, $1 \neq e \in W$ with $\varphi \neq \varphi^{-1}_k$ and $|\varphi(a_k)| + |\varphi(b_k)| < |a_k| + |b_k|$. Then $j = k - 1$ and there exist $g, h \in W$ such that either $c_{k-1} = d_k /\!\!/ h$ and $d_{k-1} = d_k /\!\!/ g$, or $c_{k-1} = h \backslash\!\!\backslash d_k$ and $d_{k-1} = g \backslash\!\!\backslash d_k$.*

Proof. By 12.6, d_i and c_i have no common factor for any $i \in K$, $j + 1 \leqslant i \leqslant k$. By the left-right symmetry $\varphi = L^{\pm 1}_e$ can be assumed, and so 9.2 can be used. Put $g = \varphi(c_k)$ and $h = \varphi(d_k)$. As $d_k < c_k$, we obtain from 9.2 that either $\varphi = L_e$ and $c_k = e \backslash\!\!\backslash g$, or $\varphi = L^{-1}_e$, $e = d_k /\!\!/ h$ and $c_k = e \circ g$. However, $c_k = e \backslash\!\!\backslash g$ implies $\varphi_k = L^{-1}_e = \varphi^{-1}$, and so we have $\varphi = L^{-1}_e$. Then $\varphi_k = R_g$, $\varphi^{-1}_k(c_k) = d_k /\!\!/ h$, $\varphi^{-1}_k(d_k) = d_k /\!\!/ g$ and by 12.1, $k - 1 \in K$. As c_{k-1} and d_{k-1} have a common factor d_k, we see that $k - 1 = j$. □

12.8 Lemma. *For each $0 \leqslant i \leqslant k$ put $a'_i = a_{k-i}$, $b'_i = b_{k-i}$ and then generically define K', J', c'_i and d'_i, $0 \leqslant i \leqslant k$. Then $i \in K'$ iff $k - i \in K$, $i \in J'$ iff $k - i \in J$ and $c'_i = d_{k-i}$ for $i \in K$, while $c'_i = c_{k-i}$ for $i \in J$.*

Proof. This is easy. □

13. Sequences with a plateau

Let $a_i, b_i \in W$, $a_i \neq b_i$, $0 \leqslant i \leqslant k+1$, $k \geqslant 2$ be such that for $1 \leqslant i \leqslant k$ we have $|a_0| + |b_0| < |a_1| + |b_1| = |a_i| + |b_i| > |a_{k+1}| + |b_{k+1}|$, $a_i \neq 1 \neq b_i$ and for $1 \leqslant i \leqslant k+1$ we have $\varphi_i(a_{i-1}) = a_i$, $\varphi_i(b_{i-1}) = b_i$, $\varphi_i \in T(e_i)$, where $1 \neq e_i \in W$. Further, let $\varphi_{i+1}^{-1} \neq \varphi_i$ for each $1 \leqslant i \leqslant k$.

The results of the preceding section can be used for a_i, b_i, $1 \leqslant i \leqslant k$. To facilitate applications of these results, we define in a corresponding way the elements $c_i, d_i \in \{a_i, b_i\}$; *i.e.* $\{c_i, d_i\} = \{a_i, b_i\}$ for $1 \leqslant i \leqslant k$ and

(i) $c_1 = a_1$ if $|a_2| > |a_1|$, and $c_1 = b_1$ if $|b_2| > |b_1|$;

(ii) if $2 \leqslant i \leqslant k$, then $c_i = a_i$ if $|a_i| > |a_{i-1}|$, and $c_i = b_i$ if $|b_i| > |b_{i-1}|$.

We also put $K = \{k\} \cup \{1 \leqslant i \leqslant k-1;\ c_i = a_i \text{ and } c_{i+1} = a_{i+1}\} \cup \{1 \leqslant i \leqslant k-1;\ c_i = b_i \text{ and } c_{i+1} = b_{i+1}\}$.

For $d_i < c_i$, $k \neq i \in K$ we can use 12.7. However, 12.7 can be used also when $c_i < d_i$ and $1 \neq i \in K$. In that case we follow 12.8 and consider the sequence $\varphi_{k+1}^{-1}, \dots, \varphi_1^{-1}$. Thus 12.7 and 12.8 together yield

13.1 Lemma. *Let $j \in K$ be such that either $j \neq k$ and $d_j < c_j$, or $j \neq 1$ and $c_j < d_j$. Then there exist $f, g, h \in W$ such that either $a_j = f /\!/ g$ and $b_j = f /\!/ h$, or $a_j = g \backslash\!\backslash f$ and $b_j = h \backslash\!\backslash f$. Moreover, $j = k-1$ if $d_j < c_j$, and $j = 2$ if $c_j < d_j$.*

13.2 Lemma. *Suppose that there exist $f, g, h \in W$ and $1 \leqslant j \leqslant k$ such that either $a_j = f /\!/ g$ and $b_j = f /\!/ h$, or $a_j = g \backslash\!\backslash f$ and $b_j = h \backslash\!\backslash f$. Then $k \leqslant 3$ and the sequence $\varphi_1, \dots, \varphi_{k+1}$ can be reduced at $\{a_0, b_0\}$.*

Proof. The left-right symmetry allows us to choose the case $a_j = g \backslash\!\backslash f$, $b_j = h \backslash\!\backslash f$. As the inverse sequence $\varphi_{k+1}^{-1}, \dots, \varphi_1^{-1}$ could be considered in place of $\varphi_1, \dots, \varphi_{k+1}$, we can omit the case $j = k$. Thus $\varphi_{j+1} \in \{L_g, L_h\}$ and we can choose the case $\varphi_{j+1} = L_g$. Then $d_{j+1} = a_{j+1} = f$, $c_{j+1} = b_{j+1} = g \circ (h \backslash\!\backslash f)$ and thus $d_{j+1} < c_{j+1}$. If $j+1 < k$, then $j+1 = k-1$ by 13.1. Moreover, 13.1 also implies that b_{j+1} cannot equal $g \circ (h \backslash\!\backslash f)$ if $j+1 < k$. Therefore $k = j+1$ is true. As f and $g \circ (h \backslash\!\backslash f)$ have no common factor, $\varphi_{k+1} = R_{h \backslash\!\backslash f}^{-1}$ by 9.2 and 9.3.

Assume first $k \geqslant 3$. Then 12.2 yields $2 \leqslant k-1 \in K$, and so $c_{k-1} = b_{k-1} = h\backslash\backslash f$ implies $\varphi_{k-1} = L_h^{-1}$. Thus $b_{k-2} = f < a_{k-2} = h \circ (g\backslash\backslash f)$ and $k-2 \notin K$ gives $k \geqslant 4$, $\varphi_{k-2} = R_{g\backslash\backslash f}$, $a_{k-3} = h$ and $b_{k-3} = g$. Then $|a_{k-3}| + |b_{k-3}| < |a_k| + |b_k|$ and we see that $k-2 \in K$. But by 13.1 from $c_{k-2} = b_{k-2} < a_{k-2} = d_{k-2} = h \circ (g\backslash\backslash f)$ we get $k-2 = 1$. Furthermore, from 9.2 and 9.3 we obtain $\varphi_1 = R_{g\backslash\backslash f}$. As $a_0 = h$, $b_0 = g$ and $\varphi_4\varphi_3\varphi_2\varphi_1 = R^{-1}_{h\backslash\backslash f} L_g L_h^{-1} R_{g\backslash\backslash f} = \mu(g,h,f)$, we can proceed to the case $k = 2$.

If $k = 2$, then $a_1 = g\backslash\backslash f$, $b_1 = h\backslash\backslash f$ and $\varphi_2 = L_g$. For $\varphi_1 = R_{e_1}^{\pm 1}$ 9.4(a) and 9.5(a) show that only the cases $a_0 = 1$, $a_1 = b_0\backslash\backslash b_1$, $\varphi_1 = R_{a_1}$ and $b_0 = 1$, $b_1 = a_0\backslash\backslash a_1$, $\varphi_1 = R_{b_1}$ need to be considered. But $a_1 = b_0\backslash\backslash b_1$ implies $b_1 = f = h\backslash\backslash f$ and $b_1 = a_0\backslash\backslash a_1$ implies $a_1 = f = g\backslash\backslash f$. Thus $\varphi_1 = L_{e_1}^{\pm 1}$ and from 2.1(a) we get $\varphi_1 = L_{e_1}^{-1}$. By 2.2(a) we have to consider two cases. First, let $e_1 = b_0/\!/b_1$ and $a_1 = e_1\backslash\backslash a_0$. Then $g = e_1 = b_0/\!/(h\backslash\backslash f)$, but this contradicts $b_{j+1} = g \circ (h\backslash\backslash f)$. Hence $e_1 = a_0/\!/a_1$, $b_1 = e_1\backslash\backslash b_0$, and therefore $h = e_1$, $f = b_0$, $\varphi_1 = L_h^{-1}$, $\psi = \varphi_3\varphi_2\varphi_1 = R^{-1}_{h\backslash\backslash f} L_g L_h^{-1}$ and $\psi(a_0) = h = a_0/\!/(g\backslash\backslash f) = R^{-1}_{g\backslash\backslash f}(a_0)$, $\psi(f) = g = R^{-1}_{g\backslash\backslash f}(f)$. It now remains to observe that $R_{g\backslash\backslash f}\psi = R_{g\backslash\backslash f}\mu(g,h,f)R^{-1}_{g\backslash\backslash f}$ and $\psi = R^{-1}_{g\backslash\backslash f} R_{g\backslash\backslash f}\psi$. □

13.3 Lemma. *$k \leqslant 3$ and the sequence $\varphi_1, \dots, \varphi_{k+1}$ can be reduced at $\{a_0, b_0\}$.*

P r o o f. Because of the left-right symmetry $\varphi_1 = L_{e_1}^{\pm 1}$ can be assumed. We can also assume that $|a_2| < |a_1|$, *i.e.* $d_1 = a_1$. If $\varphi_2 = L_{e_2}^{\pm 1}$, then $d_1 < c_1$ by 4.2, and 13.1 together with 13.2 apply. Thus we can assume $\varphi_2 = R_{e_2}^{\pm 1}$. For $d_1 < c_1$ we can again use 13.1 and 13.2, and hence we need only to consider the cases when $d_1 < c_1$ does not hold. Let first $\varphi_2 = R_{e_2}$. Then $a_1 = a_2/\!/e_2$ and $b_2 = b_1 \circ e_2$ by 9.5(b). If $\varphi_1 = L_{e_1}$, then by 2.1(a) either $b_1 = a_1 \circ b_0$, or $a_1 = b_1/\!/b_0 = e_1$, or $b_1 = (a_1/\!/a_0) \circ b_0$, or $b_1 = a_1/\!/a_0$. Neglecting the cases with $d_1 = a_1 < b_1 = c_1$, we thus have $a_0 = 1$, $a_1 = a_2/\!/e_2$, $b_0 = e_2$, $b_1 = a_2$ and $b_2 = a_2 \circ e_2$. Then $d_2 = a_2 < c_2 = a_2 \circ e_2$ and 13.1 can be applied if $k \neq 2$. For $k = 2$ use 9.6. We get $\varphi_3 = L_{a_2}^{-1}$, and so $\psi = \varphi_3\varphi_2\varphi_1 = \nu(e_2, 1, a_2)$, $\psi(1) = 1$ and $\psi(e_2) = e_2$. If $\varphi_1 = L_{e_1}^{-1}$, then $b_1 = (a_0/\!/a_1)\backslash\backslash b_0$ by 2.2(a), and hence $d_1 < c_1$.

Let now $\varphi_2 = R_{e_2}^{-1}$. Then $a_1 = a_2 \circ e_2$ and $b_2 = b_1/\!/e_2$ by 9.5(b). If $\varphi_1 = L_{e_1}$, then we shall distinguish several cases according to 2.1(a). If $e_1 = b_1/\!/b_0$, then $a_1 \neq b_1/\!/b_0$, and hence $a_1 = e_1 \circ a_0 = (b_1/\!/b_0) \circ a_0$. But then $a_2 = b_1/\!/b_0$ and 13.2 can be applied. If $e_1 = a_1/\!/a_0$ and $b_1 = (a_1/\!/a_0) \circ b_0$ or $b_1 = a_1/\!/a_0$, then we get $d_1 = a_1 < b_1 = c_1$. This is also true if $a_0 = 1$, $a_1 = e_1$ and $b_1 = a_1 \circ b_0$. In the remaining cases $a_1 = e_1 \circ a_0$ and either $b_1 = e_1 \circ b_0$, or $b_1 = e_1$. Therefore $a_2 = e_1$, $a_0 = e_2$ and either $b_2 = (a_2 \circ b_0)/\!/a_0$, or $b_2 = a_2/\!/a_0$. Thus $c_2 = b_2 > a_2 = d_2$, and for $2 \neq k$ we can use 13.1. Hence $k = 2$ will be assumed. By 2.1(c) $\varphi_3 \neq L_{e_3}$, suppose first that $\varphi_3 = L_{e_3}^{-1}$. Then $b_2 \neq e_3 \circ b_3$ and $a_2 \neq \{b_2 \circ a_3, (b_2/\!/b_3) \circ a_3, b_2/\!/b_3\}$. By 2.2(c) $b_2 = a_2/\!/a_3$, $b_0 = 1 = b_3$, $a_0 = a_3$ and we obtain $\varphi_3\varphi_2\varphi_1 = L^{-1}_{a_2/\!/a_0} R^{-1}_{a_0} L_{a_2} = \nu(1, a_0, a_2)$. Further, $\varphi_3 = R_{e_3}$ is not possible, as 9.4(c) implies $a_2 = a_3/\!/a_0$ or $a_0 = a_2\backslash\!\backslash a_3$ — both of which are contradictory to $a_1 = a_2 \circ a_0$. Finally, from 9.5(c) we also obtain that $\varphi_3 \neq R_{e_3}^{-1}$. This settles all the cases induced by $\varphi_1 = L_{e_1}$ and we can assume $\varphi_1 = L_{e_1}^{-1}$. As $a_1 \neq e_1\backslash\!\backslash a_0$, we obtain from 2.2(a) that $b_1 = (a_1/\!/a_0)\backslash\!\backslash b_0$, and hence $d_1 = a_1 < b_1 = c_1$. □

14. Two-point stabilizers

Recall that by $\overline{W}$ we denote the free loop with the basis $\overline{X} = X \cup \{y\}$ (see Section 7). Denote further by π the epimorphism $\overline{W} \to W$ defined by $\pi(x) = x$ for $x \in X$, and $\pi(y) = 1$. Clearly $\pi(\overline{a}) = a$ for all $a \in W$.

The epimorphism π induces an epimorphism Π: $\mathrm{Mlt}(\overline{W}) \to \mathrm{Mlt}(W)$, $\Pi(L_a) = L_{\pi(a)}$, $\Pi(R_a) = R_{\pi(a)}$ for all $a \in \overline{W}$. We easily get

14.1 Lemma. $\Pi(\varphi)(\pi(a)) = \pi(\varphi(a))$ *for any* $a \in \overline{W}$ *and* $\varphi \in \mathrm{Mlt}(\overline{W})$.

14.2 Corollary. $\Pi(\mu_\varphi(a,b,c)) = \mu_{\Pi(\varphi)}(\pi(a),\pi(b),\pi(c))$ *and* $\Pi(\nu_\varphi(a,b,c)) = \nu_{\Pi(\varphi)}(\pi(a),\pi(b),\pi(c))$, *for any* $a,b,c \in \overline{W}$ *and* $\varphi \in \mathrm{Mlt}(\overline{W})$.

14.3 Lemma. *Let* $a_i, b_i \in W$, $a_i \neq b_i$, $0 \leqslant i \leqslant k+1$ *be such that* $k \geqslant 1$, *and for every* $1 \leqslant i \leqslant k+1$ *we have* $\varphi_i(a_{i-1}) = a_i$, $\varphi_i(b_{i-1}) = b_i$, $\varphi_i \in T(e_i)$ *with* $1 \neq e_i \in W$. *Furthermore, let* $\varphi_{i+1}^{-1} \neq \varphi_i$ *and* $|a_0| + |b_0| < |a_1| + |b_1| = |a_i| + |b_i| > |a_{i+1}| + |b_{i+1}|$ *for each* $1 \leqslant i \leqslant k$. *Then* $\varphi_1, \dots, \varphi_{k+1}$ *can be reduced at* $\{a_j, b_j\}$ *for some* $0 \leqslant j \leqslant k$.

Proof. By 10.6 we have $k \geqslant 2$. Suppose that $\{b_i, b_{i+1}\} = \{1, e_{i+1}\}$ for some $0 \leqslant i \leqslant k$. Then $e_{i+1} = b_i$ or $b_{i+1} = e_{i+1}$, and 11.6 yields a reduction. If $\{b_i, b_{i+1}\} \neq \{1, e_{i+1}\}$ for all $0 \leqslant i \leqslant k$, then by 7.2 we have $\overline{\varphi_i}(\overline{a_{i-1}}) = \overline{a_i}$ and $\overline{\varphi_i}(\overline{b_{i-1}}) = \overline{b_i}$ for all $1 \leqslant i \leqslant k+1$. By 7.3 we can therefore find integers s and r such that $0 \leqslant s \leqslant s+r \leqslant k$ and $|\overline{a_s}| + |\overline{b_s}| < |\overline{a_{s+i}}| + |\overline{b_{s+i}}| > |\overline{a_{s+r+1}}| + |\overline{b_{s+r+1}}|$ for every $1 \leqslant i \leqslant r$. By 13.3 the sequence $\overline{\varphi_{s+1}}, \dots, \overline{\varphi_{s+r+1}}$ can be reduced at $\{\overline{a_s}, \overline{b_s}\}$. Examining the proofs of 13.2 and 13.3 we see that $\overline{\varphi_{s+r+1}} \dots \overline{\varphi_{s+1}}$ is equal either to κ or to $\kappa\varphi$ or to $\varphi\kappa$, where κ is a permutation that can be expressed in a respective μ– or ν–form, and $\varphi \in T(e)$ for some $e \in \overline{W}$. Hence $\varphi_{s+r+1} \dots \varphi_{s+1}$ equals $\Pi(\kappa)$ or $\Pi(\kappa)\Pi(\varphi)$ or $\Pi(\varphi)\Pi(\kappa)$ and with respect to 14.2 we obtain that $\varphi_{s+1}, \dots, \varphi_{s+r+1}$ reduces at $\{a_s, b_s\}$, too. □

14.4 Theorem. *Let* W *be a free loop with a basis* $X \neq \emptyset$. *For any* $a,b,c \in W$ *and* $\varphi \in \mathrm{Mlt}(W)$ *put* $\mu_\varphi(a,b,c) = \varphi^{-1} R_{\varphi(b)\backslash\varphi(c)}^{-1} L_{\varphi(a)} L_{\varphi(b)}^{-1} R_{\varphi(a)\backslash\varphi(c)} \varphi$ *and* $\nu_\varphi(a,b,c) = \varphi^{-1} L_{\varphi(c)/\varphi(b)}^{-1} R_{\varphi(a)} R_{\varphi(b)}^{-1} L_{\varphi(c)/\varphi(a)} \varphi$. *If* $a,b \in W$ *and* $a \neq b$, *then* $\mathrm{Mlt}(W)_{a,b} = \langle \mu_\varphi(a,b,c), \nu_\varphi(a,b,c);\ \varphi \in \mathrm{Mlt}(W)$ *and* $c \in W\rangle$. *Moreover, for any* $\mathrm{id}_W \neq \psi \in \mathrm{Mlt}(W)_{a,b}$ *and any* $c \in W$ *we have* $\psi(c) = c$ *iff* $c \in \{a,b\}$.

Proof. By Lemma 10.4 it is enough to prove that whenever $\varphi_i \in T(e_i)$ for $1 \neq e_i \in W$, $1 \leqslant i \leqslant k$ satisfy $\varphi_1 \neq \varphi_k^{-1}$, $\varphi_i \neq \varphi_{i+1}^{-1}$ and $\psi_1 = \varphi_k \ldots \varphi_1$ fixes exactly two elements of W, then the sequence $\varphi_1, \ldots, \varphi_k$ reduces at its fixed points. For $2 \leqslant j \leqslant k$ put $\psi_j = \varphi_{j-1} \ldots \varphi_1 \varphi_k \ldots \varphi_j$. Let $a_0, b_0 \in W$ be such that $\psi_1(a_0) = a_0$, $\psi_1(b_0) = b_0$, $a_0 \neq b_0$ and for $1 \leqslant i \leqslant k$ put $a_i = \varphi_i \ldots \varphi_1(a_0)$, $b_i = \varphi_i \ldots \varphi_1(b_0)$. Then clearly $\psi_j(a_{j-1}) = a_{j-1}$, $\psi_j(b_{j-1}) = b_{j-1}$, $a_0 = a_k$ and $b_0 = b_k$. Let us assume that the sequence $\varphi_1, \ldots, \varphi_k$ cannot be reduced at its fixed points, and suppose first that there exist $1 \leqslant i_1 < i_2 \leqslant k$ such that $|a_{i_1}| + |b_{i_1}| \neq |a_{i_2}| + |b_{i_2}|$. This implies that there exist $0 \leqslant j \leqslant k-1$ and $r < k$ such that for $m = \max\{|a_i| + |b_i|;\ 1 \leqslant i \leqslant k\}$ and any $1 \leqslant i \leqslant r$ we have $|a_j| + |b_j| < |a_{j+i}| + |b_{j+i}| = m > |a_{j+r+1}| + |b_{j+r+1}|$ (the indices are computed modulo k). However, in such a case a contradiction follows from 14.3. Hence $|a_i| + |b_i| = |a_0| + |b_0|$ holds for all $1 \leqslant i \leqslant k$. By 11.5 and 7.2 we can also assume $\overline{\varphi_i}(\overline{a_{i-1}}) = \overline{a_i}$ and $\overline{\varphi_i}(\overline{b_{i-1}}) = \overline{b_i}$ for all $1 \leqslant i \leqslant k$. If $|\overline{a_{i_1}}| + |\overline{b_{i_1}}| \neq |\overline{a_{i_2}}| + |\overline{b_{i_2}}|$ for some $1 \leqslant i_1 < i_2 \leqslant k$, we get a contradiction by the preceding part of the proof. However, $|\overline{a_i}| + |\overline{b_i}| = |\overline{a_0}| + |\overline{b_0}|$ for all $1 \leqslant i \leqslant k$ is not possible by 12.4. $\square$

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