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ON THE SET OF ALL SHORTEST PATHS OF A GIVEN LENGTH IN A CONNECTED GRAPH

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Let G be a connected graph (in the sense of the book [1], for example). Let V, E and d denote its vertex set, its edge set and its distance function, respectively. We denote by Σ_N the set of all finite nonempty sequences

$$u_0, \dots, u_i \tag{0}$$

where $i \geqslant 0$ and $u_0, \dots, u_i \in V$. Similarly as in [2], instead of (0) we will write

$$u_0 \dots u_i.$$

If $\alpha = v_0 \dots v_j$ and $\beta = w_0 \dots w_k$, where $j, k \geqslant 0$ and $v_0, \dots, v_j$, $w_0, \dots, w_k \in V$, then we write

$$\alpha\beta = v_0 \dots v_j w_0 \dots w_k.$$

Let $\gamma = x_0 \dots x_m$, where $m \geqslant 0$ and $x_0, \dots, x_m \in V$. We write

$$\bar{\gamma} = x_m \dots x_0, \quad \|\gamma\| = m, \quad A\gamma = x_0 \quad \text{and} \quad Z\gamma = x_m.$$

If $\mathscr{A} \subseteq \Sigma_N$, we define

$$\mathscr{A}(n) = \{\alpha \in \mathscr{A};\ d(A\alpha, Z\alpha)\} = n$$

for every integer $n \geqslant 0$. Put $\Sigma = \Sigma_N \cup \{*\}$, where $*$ denotes the empty sequence in the sense that $\delta * = \delta = * \delta$ for every $\delta \in \Sigma_N$, $** = *$ and $\bar{*} = *$.

As usual, by a walk in G we mean a finite nonempty sequence $u_0 \dots u_i$ such that $i \geqslant 0$, $u_0, \dots, u_i \in V$ and $\{u_j, u_{j+1}\} \in E$ for each integer j, $0 \leqslant j < i$. Let $\mathscr{W}$ denote the set of all walks in G. Obviously, $\mathscr{W} \subseteq \Sigma_N$.

Let $\alpha, \beta \in \Sigma_N$, $\|\alpha\|, \|\beta\| \geqslant 2$, and let $A\alpha = A\beta$ and $Z\alpha = Z\beta$. Then there exist $u, v, w, z \in V$ and $\varphi, \psi \in \Sigma$ such that $\alpha = uv\varphi z$ and $\beta = u\psi wz$. We define

$$\alpha \downarrow \beta = v\varphi zw \quad \text{and} \quad \alpha \uparrow \beta = vu\psi w.$$

It is clear that if $\alpha, \beta \in \mathscr{W}$, then $\alpha \downarrow \beta, \alpha \uparrow \beta \in \mathscr{W}$.

As usual, by a path in G we mean a finite nonempty sequence $v_0 \dots v_j$ such that $j \geqslant 0$, $v_0, \dots, v_j \in V$, the vertices $v_0, \dots$, and v_j are mutually distinct and $v_0 \dots v_j$ is a walk in G. Let $\mathscr{P}$ denote the set of all paths in G. If $\alpha \in \mathscr{P}$, then $\|\alpha\|$ is called the *length* of α. Obviously,

$$\begin{aligned} d(u,v) &= \min(\|\beta\|; \beta \in \mathscr{P},\ A\beta = u,\ Z\beta = v) \\ &= \min(\|\gamma\|; \gamma \in \mathscr{W},\ A\gamma = u,\ Z\gamma = v) \end{aligned}$$

for every pair of vertices u and v of G.

Let $\alpha \in \mathscr{W}$. Then α is called a *shortest path* in G, if

$$\|\alpha\| = d(A\alpha, Z\alpha).$$

Let $\mathscr{S}$ denote the set of all shortest paths in G. Obviously, $\mathscr{S} \subseteq \mathscr{P}$.

The next theorem gives a characterization of $\mathscr{S}$.

Theorem 0. *Let $\mathscr{R} \subseteq \mathscr{P}$. Then $\mathscr{R} = \mathscr{S}$ if and only if the following conditions* **A** – **G** *are fulfilled (for arbitrary $u, v, w, z \in V$ and $\alpha, \beta \in \Sigma$):*

A *If $uv\alpha w \in \mathscr{R}$, then $\{u, w\} \notin E$.*

B *If $uv\alpha w \in \mathscr{R}$, then $w\bar{\alpha}vu \in \mathscr{R}$.*

C *If $uv\alpha w \in \mathscr{R}$, then $v\alpha w \in \mathscr{R}$.*

D *If $uv\alpha w, v\beta w \in \mathscr{R}$, then $uv\beta w \in \mathscr{R}$.*

E *If $uv\alpha w, vu\beta z \in \mathscr{R}$ and $\{w, z\} \in E$, then $v\alpha wz \in \mathscr{R}$.*

F *If $uv\alpha w \in \mathscr{R}$, $\{w, z\} \in E$, $u\varphi zw \notin \mathscr{R}$ for any $\varphi \in \Sigma$ and $uv\psi z \notin \mathscr{R}$ for any $\psi \in \Sigma$, then $v\alpha wz \in \mathscr{R}$.*

G *There exists $\varphi \in \mathscr{R}$ such that $A\varphi = u$ and $Z\varphi = v$.*

The characterization of $\mathscr{S}$ given in Theorem 0 is "almost non-metric" in the sense that the lengths of paths greater than one are neither considered nor compared in the conditions **A** – **G**. Note that Theorem 0 is a modification of Theorem 1 in [2].

Let $n \geqslant 2$. As follows from the definition, $\mathscr{S}(n)$ is the set of all shortest paths of length n in G. The proof of Theorem 1 in [2] contains an implicit characterization of $\mathscr{S}(n)$ under the assumption that each of the sets $\mathscr{S}(0), \mathscr{S}(1), \dots, \mathscr{S}(n-1)$ is known. The next theorem gives a characterization of $\mathscr{S}(n)$ under the assumption that only $\mathscr{S}(n-1)$ is known. Note that the lengths of paths greater than $n-1$ are neither considered nor compared in the next theorem. Nonetheless, the knowledge of the distance function is assumed.

Theorem 1. *Let $n \geqslant 2$ be an integer, and let $\mathscr{R} \subseteq \mathscr{W}$. Assume that*

(1) $$\mathscr{R}(n-1) = \mathscr{S}(n-1).$$

Then $\mathscr{R}(n) = \mathscr{S}(n)$ if and only if the following conditions $\mathbf{B}_n - \mathbf{H}_n$ are fulfilled (for arbitrary $u, v, w, z \in V$ and $\alpha, \beta, \gamma \in \Sigma$):

$\mathbf{B}_n$ *If $uv\alpha w \in \mathscr{R}(n)$, then $w\bar{\alpha}vu \in \mathscr{R}$.*

$\mathbf{C}_n$ *If $uv\alpha w \in \mathscr{R}(n)$, then $v\alpha w \in \mathscr{R}$.*

$\mathbf{D}_n$ *If $uv\alpha w \in \mathscr{R}(n)$, $v\beta w \in \mathscr{R}$, then $uv\beta w \in \mathscr{R}$.*

$\mathbf{E}_n$ *If $uv\alpha w$, $vu\beta z \in \mathscr{R}(n)$ and $\{w, z\} \in E$, then $v\alpha wz \in \mathscr{R}$.*

$\mathbf{F}_n$ *If $uv\alpha w \in \mathscr{R}(n)$, $\{w, z\} \in E$, $u\varphi zw \notin \mathscr{R}$ for any $\varphi \in \Sigma$ and $uv\psi z \notin \mathscr{R}$ for any $\psi \in \Sigma$, then $v\alpha wz \in \mathscr{R}$.*

$\mathbf{G}_n$ *If $d(u, v) = n$, then there exists $\varphi \in \Sigma$ such that $A\varphi = u$ and $Z\varphi = v$.*

$\mathbf{H}_n$ *If $u\alpha v\beta w \in \mathscr{R}(n)$, then $w\gamma u\alpha v \notin \mathscr{R}(n)$.*

Proof. I. Let $\mathscr{R}(n) = \mathscr{S}(n)$. Then $\mathbf{B}_n - \mathbf{E}_n$, $\mathbf{G}_n$ and $\mathbf{H}_n$ can be verified easily.

Consider arbitrary $u, v, w, z \in V$ and $\alpha \in \Sigma$ such that $uv\alpha w \in \mathscr{R}(n)$, $\{w, z\} \in E$, $u\varphi zw \notin \mathscr{R}$ for any $\varphi \in \Sigma$ and $uv\psi z \notin \mathscr{R}$ for any $\psi \in \Sigma$. Since $\mathscr{R}(n) = \mathscr{S}(n)$, we see that $u \neq z$, $v\alpha w \in \mathscr{S}(n-1)$, $d(u, w) = n$, $d(v, z) \leqslant n$, $u\varphi zw \notin \mathscr{S}(n)$ for any $\psi \in \Sigma$ and $uv\psi z \notin \mathscr{S}(n)$ for any $\psi \in \Sigma$. We get $v \neq z$. (Otherwise, $uz\alpha w \in \mathscr{S}(n)$ and thus $uzw \in \mathscr{S}(n)$; a contradiction).

If $d(u, z) = n + 1$, then $d(v, z) = n$. Let $d(u, z) \neq n + 1$. Since $d(u, w) = n$, we get $d(u, z) = n$. Hence, $d(v, z) = n$ again. This implies that $v\alpha wz \in \mathscr{S}(n) \subseteq \mathscr{R}$. Thus $\mathbf{F}_n$ is verified, too.

II. Conversely, let $\mathbf{B}_n - \mathbf{H}_n$ be fulfilled (for arbitrary $u, v, w, z \in V$ and $\alpha, \beta, \gamma \in \Sigma$). This part of the proof will be divided into two steps. In Step 1 we will prove that $\mathscr{S}(n) \subseteq \mathscr{R}$. This result will be used in Step 2. We will prove there that $\mathscr{R}(n) \subseteq \mathscr{S}$.

Step 1. If $\mathscr{S}(n) = \emptyset$, then $\mathscr{S}(n) \subseteq \mathscr{R}$. Let $\mathscr{S}(n) \neq \emptyset$. Consider an arbitrary $\xi_0 \in \mathscr{S}(n)$. According to $\mathbf{G}_n$, there exists $\zeta_0 \in \mathscr{R}$ such that $A\xi_0 = A\zeta_0$ and $Z\xi_0 = Z\zeta_0$.

(2) Put $m = \|\zeta_0\|$. Obviously, $m \geqslant n$. We define $\zeta_{i+1} = \zeta_i \downarrow \xi_i$ and $\xi_{i+1} = \zeta_i \uparrow \xi_i$ for each $i \in \{0, \dots, m-1\}$. Clearly, $\|\zeta_j\| = m$ and $\|\xi_j\| = n$ for each $j \in \{0, \dots, m\}$.

We want to prove that $\xi_0 \in \mathscr{R}$. To the contrary, let $\xi_0 \notin \mathscr{R}$.

Recall that $\zeta_0 \in \mathscr{R}$ and $\xi_0 \in \mathscr{S} - \mathscr{R}$. There exists $k \in \{0, \dots, m-1\}$ such that

$$\zeta_0, \dots, \zeta_k \in \mathscr{R}, \quad \xi_0, \dots, \xi_k \in \mathscr{S} - \mathscr{R}$$

and

(3) $\qquad$ either $\zeta_{k+1} \notin \mathscr{R}$ or $\xi_{k+1} \notin \mathscr{S} - \mathscr{R}$ or $k = m - 1$.

(4) There exist $r, s, x, y \in V$ and $\varrho, \sigma \in \Sigma$ such that
$$\zeta_k = xr\varrho y \text{ and } \xi_k = x\sigma sy.$$
Then $\zeta_{k+1} = r\varrho ys$ and $\xi_{k+1} = rx\sigma s$. Since $\xi_k \in \mathscr{S}$, $d(x, y) = n$. We see that $x\sigma s \in \mathscr{S}(n-1)$ and therefore, $d(x, s) = n - 1$.

Assume that there exists $\tau \in \Sigma$ such that $x\tau sy \in \mathscr{R}$. Since $d(x, y) = n$, $x\tau sy \in \mathscr{R}(n)$. According to $\mathbf{B}_n, ys\bar{\tau}x \in \mathscr{R}(n)$. Obviously, $s\bar{\sigma}x \in \mathscr{S}(n-1)$. As follows from (1), $s\bar{\sigma}x \in \mathscr{R}$. Since $ys\bar{\tau}x \in \mathscr{R}(n)$, $\mathbf{D}_n$ implies that $ys\bar{\sigma}x \in \mathscr{R}(n)$. According to $\mathbf{B}_n$, $\xi_k = x\sigma sy \in \mathscr{R}$, which is a contradiction. Thus we see that

(5) $\qquad$ $x\varphi sy \notin \mathscr{R}$ for any $\varphi \in \Sigma$.

Assume that $d(r, s) < n - 1$. Since $d(x, y) = n$, we have $d(r, s) = n - 2$ and $d(r, y) = n - 1$. This implies that there exists $\pi \in \Sigma$ such that $r\pi sy \in \mathscr{S}(n-1)$. By virtue of (1), $r\pi sy \in \mathscr{R}(n-1)$. Since $\zeta_k \in \mathscr{R}(n)$, it follows from $\mathbf{D}_n$ that $xr\pi sy \in \mathscr{R}$, which contradicts (5). Thus

(6) $\qquad$ $n - 1 \leqslant d(r, s) \leqslant n$.

We distinguish two cases.

Case 1. Let $\zeta_{k+1} \in \mathscr{R}$. If $d(r, s) = n - 1$, then it follows from (1) that $\zeta_{k+1} \in \mathscr{S}(n-1)$, and therefore $m = n - 1$, which is a contradiction. Thus, by virtue of (6), $d(r, s) = n$. This means that $\xi_{k+1} \in \mathscr{S}(n)$.

Assume that $\xi_{k+1} \in \mathscr{R}$. Since $\xi_{k+1}, \zeta_k \in \mathscr{R}(n)$, $\mathbf{E}_n$ implies that $\xi_k \in \mathscr{R}$, which is a contradiction. Therefore, $\xi_{k+1} \notin \mathscr{R}$. This means that $\xi_{k+1} \in \mathscr{S} - \mathscr{R}$. Since $\zeta_{k+1} \in \mathscr{R}$, it follows from (3) that $k = m - 1$. Hence, $\zeta_m \in \mathscr{R}(n)$.

If $m = n$, then $\zeta_m = \bar{\xi}_0$ and therefore, according to $\mathbf{B}_n, \xi_0 \in \mathscr{R}$, which is a contradiction. Thus $m > n$.

(7) Clearly, there exist $t \in V$ and $\lambda, \mu, \nu \in \Sigma$ such that
$$\xi_0 = t\lambda r,\ \zeta_0 = t\mu s\nu r \text{ and } \zeta_m = r\bar{\lambda}t\mu s.$$
Since $\xi_0 \in \mathscr{S}(n)$, we have $\zeta_0 \in \mathscr{R}(n)$. Moreover, $\zeta_m \in \mathscr{R}(n)$, which contradicts $\mathbf{H}_n$.

Case 2. Let $\zeta_{k+1} \notin \mathscr{R}$. Combining the fact that $\zeta_k \in \mathscr{R}$ with (5) and $\mathbf{F}_n$, we see that

there exists $\psi \in \Sigma$ such that $xr\psi s \in \mathscr{R}$.

Since $d(x,s) = n-1$, it follows from (1) that $xr\psi s \in \mathscr{S}(n-1)$. Hence $d(r,s) = n-2$, which contradicts (6).

Thus $\xi_0 \in \mathscr{R}$. We have proved that

$$\mathscr{S}(n) \subseteq \mathscr{R}. \tag{8}$$

Step 2. If $\mathscr{R}(n) = \emptyset$, then $\mathscr{R}(n) \subseteq \mathscr{S}$. Let $\mathscr{R}(n) \neq \emptyset$. Consider an arbitrary $\zeta_0 \in \mathscr{R}(n)$. Since $\mathscr{R} \subseteq \mathscr{W}$, there exists $\xi_0 \in \mathscr{S}$ such that $A\zeta_0 = A\xi_0$ and $Z\zeta_0 = Z\xi_0$. We accept the convention given in (2).

We want to prove that $\zeta_0 \in \mathscr{S}$. To the contrary, let $\zeta_0 \notin \mathscr{S}$. Then $m > n$.

Clearly, there exists $k \in \{0, \dots, m-1\}$ such that

$$\zeta_0, \dots, \zeta_k \in \mathscr{R}, \quad \xi_0, \dots, \xi_k \in \mathscr{S}$$

and

$$\text{either } \zeta_{k+1} \notin \mathscr{R} \text{ or } \xi_{k+1} \notin \mathscr{S} \text{ or } k = m-1. \tag{9}$$

We accept the convention given in (4). Clearly, $n-2 \leqslant d(r,s) \leqslant n$ and $n-1 \leqslant d(r,y) \leqslant n+1$.

Assume that $d(r,y) = n-1$. Since $\zeta_k \in \mathscr{R}(n)$, $\mathbf{C}_n$ implies that $r\varrho y \in \mathscr{R}(n-1)$. By virtue of (1), $r\varrho y \in \mathscr{S}(n-1)$. Hence $m-1 = n-1$; a contradiction. Thus $d(r,y) \geqslant n$.

We get $d(r,s) \geqslant n-1$. Assume that $d(r,s) = n$. Then $\xi_{k+1} \in \mathscr{S}(n)$. Due to (8), $\xi_{k+1} \in \mathscr{R}$. Since $\zeta_k, \xi_{k+1} \in \mathscr{R}(n)$, it follows from $\mathbf{E}_n$ that $\zeta_{k+1} \in \mathscr{R}$. Due to (9), $k = m-1$. Hence $\zeta_m \in \mathscr{R}(n)$. Recall that $m > n$. If we make the same observation as in (7), we get a contradiction.

Thus

$$d(r,s) = n-1. \tag{10}$$

Recall that $d(r,y) \geqslant n$. As follows from (10), $d(r,y) = n$. We see that

$$\text{there exists } \psi \in \Sigma \text{ such that } r\psi sy \in \mathscr{S}.$$

By virtue of (8), $r\psi sy \in \mathscr{R}$. Since $\zeta_k \in \mathscr{R}(n)$, $\mathbf{D}_n$ implies that

$$xr\psi sy \in \mathscr{R}(n).$$

As follows from $\mathbf{B}_n$, $ys\bar{\psi}rx \in \mathscr{R}(n)$. According to $\mathbf{C}_n$, $s\bar{\psi}rx \in \mathscr{R}$. Since $d(s,x) = d(x,s) = n-1$, (1) implies that

$$s\bar{\psi}rx \in \mathscr{S}(n-1).$$

Hence $s\bar{\psi}r \in \mathscr{S}(n-2)$. We get $d(r,s) = d(s,r) = n-2$, which contradicts (10).

Thus $\zeta_0 \in \mathscr{S}$. We have proved that $\mathscr{R}(n) \subseteq \mathscr{S}$.

It follows from (8) that $\mathscr{R}(n) = \mathscr{S}(n)$, which completes the proof. □

Remark 1. Recall that G is a graph in the sense of [1]. This means that V is finite. However, the finiteness of V was not exploited in the proof of Theorem 1.

We will utilize Theorem 1 in the following proof of Theorem 0.

Proof of Theorem 0. I. Let first $\mathscr{R} = \mathscr{S}$. Consider arbitrary $u, v, w, z \in V$ and $\alpha, \beta \in \Sigma$. It is easy to see that $\mathbf{A} - \mathbf{D}$, $\mathbf{F}$ and $\mathbf{G}$ are fulfilled.

Assume that $uv\alpha w, vu\beta z \in \mathscr{R}$ and $\{w, z\} \in E$. Then $v\alpha w \in \mathscr{S}$, $d(u,w) = d(v,w)+1$, $d(v,z) = d(u,z)+1$, $d(u,w) \leqslant d(u,z)+1$ and $d(v,z) \leqslant d(v,w)+1$. This implies that $d(v,z) = d(v,w)+1$. Since $v\alpha w \in \mathscr{S}$, we get $v\alpha wz \in \mathscr{S}$ and therefore, $v\alpha wz \in \mathscr{R}$. Thus $\mathbf{E}$ is fulfilled, too.

II. Conversely, let $\mathbf{A} - \mathbf{G}$ be fulfilled (for arbitrary $u, v, w, z \in V$ and $\alpha, \beta \in \Sigma$). We are to prove that $\mathscr{R}(n) = \mathscr{S}(n)$ for every integer $n \geqslant 0$. We proceed by induction on n. Since $\mathscr{R} \subseteq \mathscr{P}$, it follows from $\mathbf{G}$ that $\mathscr{R}(0) = \mathscr{P}(0) = \mathscr{S}(0)$. Combining $\mathbf{G}$ and $\mathbf{A}$, we get $\mathscr{R}(1) = \mathscr{S}(1)$.

Let $n \geqslant 2$, and let $\mathscr{R}(n-1) = \mathscr{S}(n-1)$. Clearly, $\mathbf{B}_n - \mathbf{G}_n$ are fulfilled. Consider arbitrary $r, s, t \in V$ and $\kappa, \mu, \nu \in \Sigma$. Assume that $r\kappa t\mu s, t\mu s\nu r \in \mathscr{R}(n)$. According to $\mathbf{B}$, $s\bar{\mu}t\bar{\kappa}r \in \mathscr{R}$. First, let $\mu = *$. Then $st\bar{\kappa}r, ts\nu r \in \mathscr{R}$. According to $\mathbf{D}$, $sts\nu r \in \mathscr{R}$, which contradicts the assumption that $\mathscr{R} \subseteq \mathscr{P}$. Now, let $\mu \neq *$. There exist $x \in V$ and $\pi \in \Sigma$ such that $\mu = x\pi$. We have

$$s\bar{\pi}xt\bar{\kappa}r,\ \ tx\pi s\nu r \in \mathscr{R}.$$

As follows from $\mathbf{C}$, $xt\bar{\kappa}r \in \mathscr{R}$. According to $\mathbf{D}$, $xtx\pi s\nu r \in \mathscr{R}$, which is a contradiction. This implies that $\mathbf{H}_n$ is fulfilled, too. It follows from Theorem 1 that $\mathscr{R}(n) = \mathscr{S}(n)$, which completes the proof of Theorem 0. □

Remark 2. Theorem 0 (more exactly, a theorem similar to it) was generalized in [3]. Note that the idea of that generalization is very different from the idea of Theorem 1.

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