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# Principal element lattices

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# PRINCIPAL ELEMENT LATTICES

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## 0. INTRODUCTION

In this paper we introduce (see Section 1) the concept of a weak Noether lattice and prove that a weak $r$-lattice $L$ is a principal element lattice if and only if $L$ is a weak Noether lattice in which every maximal element is weak meet principal. It is shown that if $L$ is a weak Noether lattice in which the zero element is prime and every maximal element is join principal, then every element is principal. In Section 2 we study quasilocal $\pi$-lattices, UFD lattices and Dedekind domains. It is shown that if $L$ is a quasilocal weak $r$-lattice and if $L$ is a $\pi$-lattice, then $L$ is either a domain or $L$ has only finitely many minimal prime elements and every prime element is the join of minimal prime elements. We prove that if $L$ is a UFD lattice and every nontrivial prime element is maximal, then every element is principal. Using these results it is shown that if $L$ is a principally generated Dedekind domain, then every element is principal. In Section 3 we investigate invertible elements and Dedekind domains are characterized in terms of invertible maximal elements. Some equivalent conditions are established for a weak $r$-lattice to be a finite direct product of Dedekind domains.

A multiplicative lattice is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins (i.e., $a(\bigvee_\alpha b_\alpha) = \bigvee_\alpha (ab_\alpha))$ and has greatest element 1 (least element 0) as a multiplicative identity (zero) (see [1]).

Let $L$ be a multiplicative lattice with 1 compact. An element $a \in L$ is said to be *nontrivial* if $a \neq 0, 1$ and $a$ is called *proper* if $a < 1$. A proper element $p \in L$ is called *prime* if $ab \leqslant p$ implies either $a \leqslant p$ or $b \leqslant p$. A proper element $p$ of $L$ is *primary* if for every pair of compact elements $a, b \in L$ $ab \leqslant p$ implies either $a \leqslant p$ or $b^n \leqslant p$ for some positive integer $n$. A proper element $m$ is said to be *maximal* if $m \nleqslant a$ for any other proper element $a$ of $L$.

An element $m$ of a multiplicative lattice $L$ is *weak meet (join) principal* if $a \wedge m = m(a : m)$ $(a \vee (0 : m) = (ma : m))$ for all $a \in L$. Further, $m$ is *meet principal (join principal)* if $am \wedge b = (a \wedge (b : m))m$ $((am \vee b) : m = a \vee (b : m))$ for all $a, b \in L$. The element $m$ is *(weak) principal* if it is both (weak) meet principal and (weak) join principal. A multiplicative lattice $L$ is called a *weak $r$-lattice* if it is principally generated, compactly generated and has 1 compact. A weak $r$-lattice which is also modular is called an *$r$-lattice* and an $r$-lattice in which every element is compact is said to be a *Noether lattice.* Further, if $L$ is a weak $r$-lattice, then for every $a \in L$, $\sqrt{a} = \bigwedge\{p \in L \mid a \leqslant p$, and $p$ is a prime element$\} = \bigwedge\{p \in L \mid p \geqslant a$ is a minimal prime over $a\}$, where $\sqrt{a} = \bigvee\{x \in L \mid x$ is compact and $x^n \leqslant a$ for some $n \in \mathbb{Z}^+\}$.

A multiplicative lattice $L$ is a *domain* if the zero element is prime. $L$ is said to be a *principal element lattice* if every element is principal and it is called a *special principal element lattice* if it has a unique maximal element which is principal and every element is a power of the maximal element. A multiplicative lattice $L$ with 1 compact is said to be *quasilocal* if it contains a unique maximal element.

For all undefined terms used in this paper, the reader is referred to [1] and [4].

## 1. Weak Noether lattices

In this section we introduce the concept of a weak Noether lattice and establish some equivalent conditions for a weak Noether lattice to be a principal element lattice. Throughout this section $L$ represents a weak $r$-lattice unless otherwise noted.

While [1] is concerned mainly with $r$-lattices which are by hypothesis modular, many of the results of [1] do not require modularity. In fact, modularity was not assumed in Section 1. (It should be noted the hypothesis that the greatest element of a quasilocal multiplicative lattice be compact was inadvertently omitted and that in Theorem 1.2 the word "join irreducible" should be replaced by "completely join irreducible".) The process of localization as given in Section 2 of [1] for $r$-lattices carries over with no changes to weak $r$-lattices and we will use it freely.

**Definition 1.1.** *$L$ is said to be a* weak Noether lattice *if $L$ is a weak $r$-lattice which satisfies the ascending chain condition.*

Let $N = (N, +, \cdot)$ be the semiring of nonnegative integers. Then the lattice $L(N)$ of all semiring ideals of $N$ is a weak Noether lattice which is not a modular lattice (see [8]). Therefore a weak Noether lattice need not be a Noether lattice. Clearly a weak Noether lattice is a Noether lattice if and only if it is a modular lattice.

**Lemma 1.1.** *An element $a \in L$ is principal if and only if $a$ is compact and $a_p$ is principal in $L_p$ for every prime element $p$ of $L$.*

P r o o f. The proof of the lemma is similar to the proof of Theorem 2.9 of [1]. $\square$

**Lemma 1.2.** *Let $S$ be a join principally generated multiplicative lattice with 1 compact. If every maximal element is weak meet principal, then every nonmaximal prime element which is a finite join of join principal elements is weak meet principal.*

P r o o f. Suppose every maximal element is weak meet principal. Let $p$ be a nonmaximal prime element which is a finite join of join principal elements. Suppose $a \leqslant p$ for some $a \in S$. We show that $p \vee (0 : a) = 1$. If $p \vee (0 : a) < 1$, then $p, (0 : a) \leqslant m$ for some maximal element $m$ of $S$. Since $p < m$ and $m$ is weak meet principal, we have $p = pm$, so by Lemma 1.1 of [2], $m \vee (0 : p) = 1$. As $a \leqslant p$, $(0 : p) \leqslant (0 : a) \leqslant m$, $m = 1$, a contradiction. Therefore $p \vee (0 : a) = 1$ and hence $a = ap$. Thus every nonmaximal prime element which is a finite join of join principal elements is weak meet principal. $\square$

An element $q \in L$ is said to be *$p$-primary* if $q$ is primary and $\sqrt{q} = p$ is a prime element.

**Lemma 1.3.** *Let $L$ be a weak Noether lattice in which $m$ is the only prime element. Then, for any proper elements $a, c \in L$, $\bigwedge\limits_{n=1}^{\infty} (a^n \vee c) = c$. In particular $\bigwedge\limits_{n=1}^{\infty} a^n = 0$.*

P r o o f. Let $b = \bigwedge\limits_{n=1}^{\infty} (a^n \vee c)$. We show that $b \leqslant c \vee ab$. As $m$ is the only prime element, $L$ is quasilocal and every element is $m$-primary. Since $ab \leqslant c \vee ab$ and $c \vee ab$ is primary, it follows that either $b \leqslant c \vee ab$ or $a^n \leqslant c \vee ab$ for some $n \in \mathbb{Z}^+$ and so in any case $b \leqslant c \vee ab$. Again by Theorem 1.4 of [1], $b \leqslant c$ and hence $b = c$. $\square$

**Lemma 1.4.** *Let $p$ be a nonminimal principal prime element of $L$ and $q = \bigwedge\limits_{n=1}^{\infty} p^n$. Then* (i) *$q$ is prime,* (ii) *$pq = q$ and* (iii) *any prime element properly contained in $p$ is contained in $q$.*

P r o o f. The proof of the lemma is similar to the proof of Theorem 2.2 of [3]. $\square$

**Lemma 1.5.** *Let $L$ be a quasilocal weak Noether lattice. If the maximal element $m$ of $L$ is principal, then every nonzero element is a power $m^k$ $(k \geqslant 0)$ of $m$.*

P r o o f. Suppose the maximal element $m$ of $L$ is principal. We show that $\bigwedge\limits_{n=1}^{\infty} m^n = 0$. If $m$ is the only prime element of $L$, then by Lemma 1.3, $\bigwedge\limits_{n=1}^{\infty} m^n = 0$. Suppose there are prime elements in $L$ different from the maximal element. Then

$m$ is a nonminimal principal prime element of $L$. So by Lemma 1.4, $m\Big(\bigwedge_{n=1}^{\infty} m^n\Big) = \bigwedge_{n=1}^{\infty} m^n$. As $L$ is a quasilocal weak Noether lattice, by Theorem 1.4 of [1], $\bigwedge_{n=1}^{\infty} m^n = 0$.

Now let $a$ be a nonzero element of $L$. Then $a \leqslant m^n$ and $a \nleqslant m^{n+1}$ for some $n \in \mathbb{Z}^+$. So $a = m^n c$ for some $c \nleqslant m$. As $L$ is a quasilocal, $c = 1$ and hence $a = m^n$. □

The following Theorem 1.1 gives an equivalent condition for $L$ to be a principal element lattice.

**Theorem 1.1.** *$L$ is a principal element lattice if and only if $L$ is a weak Noether lattice in which every maximal element is weak meet principal.*

Proof. Suppose $L$ is a weak Noether lattice in which every maximal element is weak meet principal. By Lemma 1.2 every prime element is weak meet principal and hence by Theorem 1.5 of [5], $L$ is a principal element lattice. The converse is clear. □

The following theorem is an extension of Theorem 2 of [6].

**Theorem 1.2.** *Suppose $L$ is a domain and for every prime element $p$ of $L$, $L_p$ is a weak Noether lattice. If every maximal element is compact and join principal, then every element is principal.*

Proof. By Theorem 1.5 of [1], every maximal element is locally principal and hence principal. By Lemma 1.5, $\dim L = 1$ and therefore every prime element is principal. Consequently every element is principal. □

**Corollary 1.1.** *If $L$ is a weak Noether lattice in which $0$ is prime and every maximal element is join principal, then every element is principal.*

## 2. Dedekind domains

In this section, we study $\pi$-lattices, UFD lattices and Dedekind domains. Throughout this section $L$ denotes a multiplicative lattice with 1 compact. For any $a \in L$, $L/a = \{b \in L \mid a \leqslant b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \vee a$.

According to [1], a multiplicative lattice $L$ will be called a *$\pi$-lattice* if there exists a set $S$ of elements of $L$ (not necessarily principal) which generate $L$ under joins such that every element of $S$ is a finite product of prime elements.

For example, let $N = (N, +, \cdot)$ be the semiring of nonnegative integers. Let $L$ be the lattice of semiring ideals of $N$ and $S$ be the set of principal ideals of $N$. Then $L$ is a nonmodular quasilocal $\pi$-lattice.

**Lemma 2.1.** *Let $a$ be a weak principal element of $L$ and $e \in L$ with $(e : a) = e$. If $m$ is a factor of $a$, then for any $c \leqslant m$, $c \vee e = md \vee e$ for some $d \in L$.*

Proof. Let $m$ be a factor of $a$. So $mk = a$ for some $k \in L$. We claim that for any $x, y \in L$, $xk \leqslant yk$ implies $x \leqslant y \vee e$. Suppose $xk \leqslant yk$. Then $xmk \leqslant ymk$, so $x \leqslant y \vee (0 : a)$. But $(0 : a) \leqslant (e : a) = e$, so that $x \leqslant y \vee e$. Now assume that $c \leqslant m$. Then $ck \leqslant mk = a$, so $ck = ad = mkd$ for some $d \in L$. Therefore by the above argument, $c \leqslant md \vee e$, $md \leqslant c \vee e$ and hence $c \vee e = md \vee e$. $\square$

**Lemma 2.2.** *Let $L$ be a join principally generated, quasilocal multiplicative lattice. Let $a$ be a weak principal element and $q$ be a factor of $a$. Suppose $(e : a) = e$ where $e$ is join principal and $q = \bigvee\limits_{\alpha} x_\alpha$. If $q \nleqslant e$, then $q \vee e = x_\alpha \vee e$ for some $\alpha$.*

Proof. Let $a_\alpha = x_\alpha \vee e$. By Lemma 2.1, $a_\alpha = x_\alpha \vee e = qd \vee e$. Since $qd \leqslant a_\alpha$, $d \leqslant (a_\alpha : q)$, so $qd \leqslant (a_\alpha : q)q$ and hence $qd \vee e \leqslant q(a_\alpha : q) \vee e$. Since $(a_\alpha : q)q \leqslant a_\alpha$, we have $(a_\alpha : q)q \vee e \leqslant a_\alpha \vee e = a_\alpha$ and therefore $a_\alpha = q(a_\alpha : q) \vee e$. Again $q \vee e = (\bigvee x_\alpha) \vee e = \bigvee\limits_{\alpha}(x_\alpha \vee e) = \bigvee\limits_{\alpha} a_\alpha = \bigvee\limits_{\alpha}(q(a_\alpha : q) \vee e) = q(\bigvee\limits_{\alpha}(a_\alpha : q)) \vee e$. If $\bigvee\limits_{\alpha}(a_\alpha : q) = 1$, then $q \vee e = x_\alpha \vee e$ for some $\alpha$. Suppose $\bigvee\limits_{\alpha}(a_\alpha : q) < 1$. As $L$ is quasilocal, $\bigvee\limits_{\alpha}(a_\alpha : q) \leqslant m$ ($m$ is the maximal element), so $q(\bigvee(a_\alpha : q)) \leqslant qm$ and hence $q \vee e = qm \vee e$. Now let $b$ be any join principal element of $L$ such that $b \leqslant q$. Put $a^* = b \vee e$. Then $a^* = q(a^* : q) \vee e = ((q \vee e)(a^* : q)) \vee e = ((qm \vee e)(a^* : q)) \vee e = m(a^* : q)q \vee e \leqslant ma^* \vee e$. As $a^*$ is a finite join of join principal element, by Theorem 1.4 of [1], $a^* \leqslant e$ and so $b \leqslant e$. Consequently, $q \leqslant e$, a contradiction. Therefore $q \vee e = x_\alpha \vee e$ for some $\alpha$. $\square$

**Theorem 2.1.** *Let $L$ be a quasilocal weak r-lattice with maximal element $m$. If $L$ is a $\pi$-lattice, then either $L$ is a domain or $L$ has only finitely many minimal prime elements and every prime element is the join of the minimal prime elements contained in it.*

Proof. If $\dim L = 0$, then the result is true. So assume that $\dim L > 0$. Observe that every principal element, being completely join irreducible, is a product of primes. Since $\dim L > 0$, $L$ contains a finite number of minimal primes $p_1, p_2, \ldots, p_l$. Note that each $p_i$ is principal and $m \nleqslant p_1, p_2, \ldots, p_l$. Suppose $L$ is not a domain. Let $q$ be a prime element of $L$ and let $p_1, p_2, \ldots, p_n \leqslant q$ $(1 \leqslant n \leqslant l)$. We show that $q = \bigvee\limits_{i=1}^{n} p_i$.

Suppose $\bigvee_{i=1}^{n} p_i < q$. Then there exists a principal element $a \leqslant q$ such that $a \nleqslant p_j$ for $j = 1, 2, \ldots, n$. Let $e = p_1 \wedge \cdots \wedge p_n$. Then $(e : a) = (p_1 \wedge \cdots \wedge p_n : a) = (p_1 : a) \wedge \cdots \wedge (p_n : a) = p_1 \wedge \cdots \wedge p_n = e$ since $(p_i : a) = a$ for $i = 1, 2, \ldots, n$.

Suppose $a = q_1 \cdots q_m$ for some prime elements $q_1, q_2, \ldots, q_m \in L$. Then $q_i \leqslant q$ for some $i$, say $q_1 \leqslant q$. Note that $p_j \leqslant q_1$ for some $j \in \{1, 2, \ldots, n\}$, say $p_1 \leqslant q_1$. By Lemma 2.1, $p_1 = p_1 \vee e = q_1 d \vee e$ for some $d \in L$. Since $q_1 d \leqslant p_1$ and $p_1 < q_1$, it follows that $d \leqslant p_1$ and hence $p_1 = q_1 d \vee e \leqslant q_1 p_1 \vee e$. By Theorem 1.4 of [1], $p_1 \leqslant e$ and therefore $q$ contains only one minimal prime element, say $p_1$. Again by Lemma 2.2, $q_1 = q_1 \vee p_1 = x_\alpha \vee p_1$ for some principal element $x_\alpha \leqslant q_1$. We claim that $q_1 = x_\alpha$. As $L$ is a $\pi$-lattice, $x_\alpha = k_1 k_2 \cdots k_r$ for some prime elements $k_1, k_2, \ldots, k_r \in L$. Since $x_\alpha \leqslant q_1$, we get $k_i \leqslant q_1$ for some $i$, say $k_1 \leqslant q_1$. Since $p_1$ is the only minimal prime element contained in $q_1$, it follows that $p_1 \leqslant k_1$, so $q_1 = x_\alpha \vee p_1 \leqslant k_1$ and hence $q_1 = k_1$. Therefore $x_\alpha = q_1 d$ for some $d \in L$. Again since $q_1 = x_\alpha \vee p_1 = q_1 d \vee p_1$ and $q_1 > p_1$, by Theorem 1.6 of [1], $d = 1$. This shows that $q_1 = x_\alpha$. As $p_1 < q_1$ and $q_1$ is weak meet principal, $p_1 = q_1 p_1$ and hence $p_1 = 0$ which contradicts the fact that $L$ is not a domain. Therefore $q = \bigvee_{i=1}^{n} p_i$. Thus every prime element is the join of minimal prime elements. □

A multiplicative lattice $L$ is said to satisfy the *union condition on primes* if for any set $p_1, p_2, \ldots, p_n$ of primes in $L$ and any $a \in L$, with $a \nleqslant p_1, \ldots, p_n$, there exists a principal element $b \leqslant a$ with $b \nleqslant p_1, \ldots, p_n$.

**Corollary 2.1.** *Let $L$ be a quasilocal weak $r$-lattice. If $L$ is a $\pi$-lattice satisfying the union condition on primes, then $L$ is either a domain or special principal element lattice.*

**Corollary 2.2.** *Let $L$ be a quasilocal weak $r$-lattice. If $L$ is a $\pi$-lattice, then $L$ is either a domain or $L$ has only finitely many prime elements and every prime element is compact.*

According to [1], a principally generated multiplicative lattice domain is said to be a *UFD lattice* if every principal element is a product of principal primes.

**Theorem 2.2.** *Let $L$ be a weak $r$-lattice. Then $L$ is a UFD if and only if every nonzero prime of $L$ contains a nonzero principal prime.*

Proof. The proof of the theorem is similar to the proof of Theorem 4.6 of [1]. □

**Corollary 2.3.** *Let $L$ be a weak $r$-lattice. Then $L$ is a UFD is and only if $L$ is a $\pi$-domain.*

Proof. The proof of the corollary is similar to the proof of Corollary 4.7 of [1]. □

**Lemma 2.3.** *Let $x$ be a principal element of $L$ and $(0 : x) = 0$. Then any factor of $x$ is a principal element of $L$.*

Proof. Let $x = mk$ for some $m, k \in L$. Obviously $m$ is weak join principal. We show that $m$ is join principal. Let $a, b \in L$. Then $(am \vee b : m)m \leqslant am \vee b$, so $(am \vee b : m)x = (am \vee b : m)mk \leqslant (am \vee b)k = amk \vee bk = ax \vee bk$. Thus $(am \vee b : m) = (am \vee b : m)x : x \leqslant (ax \vee bk : x) = a \vee (bk : x) = a \vee (bk : mk) = a \vee (b : m) \leqslant (am \vee b : m)$. Therefore $m$ is join principal.

Now we claim that $m$ is meet principal. Let $a, b \in L$. Then $(a \wedge mb)k \leqslant ak \wedge xb = ((ak : x) \wedge b)x = ((ak : x) \wedge b)mk$, so $a \wedge mb \leqslant ((ak : x) \wedge b)m = ((ak : mk) \wedge b)m = ((a : m) \wedge b)m \leqslant a \wedge mb$. This shows that $m$ is meet principal. Thus any factor of $x$ is principal. □

**Theorem 2.3.** *Suppose $L$ is a principally generated multiplicative lattice domain. Then $L$ is a UFD lattice if and only if every principal element is a product of prime elements of $L$.*

Proof. The proof of the theorem follows from Lemma 2.3. □

The following Theorem 2.4 and Theorem 2.5 establish some equivalent conditions for a principally generated multiplicative domain to be a principal element lattice.

**Theorem 2.4.** *Suppose $L$ is principally generated. If $L$ is a UFD and every nontrivial prime element is maximal, then every element is principal.*

Proof. Let $p$ be a nontrivial prime element. Choose any nonzero principal element $a \leqslant p$. As $L$ is UFD, $a = p_1 \cdots p_n$ where the $p_i$'s are principal primes. Since $L$ is a domain, $p_i \neq 0$ for $i = 1, 2, \ldots, n$, so by hypothesis each $p_i$ is maximal. Again since $a = p_1 \cdots p_n \leqslant p$ and $p$ is prime, $p_i \leqslant p$ for some $i$ and hence $p_i = p$. Consequently $p$ is principal. Thus every nontrivial prime element is principal and hence every element is principal. □

**Theorem 2.5.** *Suppose $L$ is principally generated. If $L$ is a domain in which every nontrivial principal element of $L$ is the product of a finite number of maximal elements, then every element is principal.*

P r o o f. Let $p$ be a nontrivial prime element. Then there exists a nonzero principal element $a \leqslant p$. By hypothesis, $a = q_1 \cdots q_n$ where the $q_i$'s are maximal elements. Since $a = q_1 \cdots q_n \leqslant p$, $q_i \leqslant p$ for some $i$, so $q_i = p$ and hence $a = pA$ (where $A = \prod_{j \neq i} q_j$). Again since $a$ is principal, by Lemma 2.3, it follows that $p$ is principal. Thus every prime element is principal and hence every element is principal. □

**Definition 2.1.** *A domain $L$ is a* Dedekind domain *if every element of $L$ is a finite product of prime elements.*

We show that if $L$ is a Dedekind domain, then every element of $L$ is principal. For this we require some lemmas.

**Lemma 2.4.** *For $i = 1, 2, \ldots, k$, let $p_i$ be a weak join principal nontrivial prime element of a domain $L$. Let $a = p_1 \cdots p_k$. Then this is the only way of writing $a$ as a product of nontrivial prime elements of $L$ except for the order of the factors.*

P r o o f. Let $a = p'_1 \cdots p'_n$ where each $p'_i$ is a nontrivial prime element of $L$ for $i = 1, 2, \ldots, n$. Assume $p_2$ is minimal among $p_2, \ldots, p_k$. Since $p'_1 \cdots p'_n \leqslant p_1$, some $p'_i$ is contained in $p_1$, say $p'_1 \leqslant p_1$. Since $p_1 \cdots p_k \leqslant p'_1$, we have $p_i \leqslant p'_1$ for some $i$, so $i = 1$ and hence $p_1 = p'_1$. Again since $p_1 p_2 \cdots p_k = p_1 p'_2 \cdots p'_n$ and $p_1$ is weak join principal, we get $p_2 \cdots p_k = p'_2 \cdots p'_n$. Since each $p_i$ and each $thisargument, weget$n=k$and j \leqslant k$. □

**Lemma 2.5.** *Suppose $L$ is a domain and $a, p$ are any two principal elements of $L$. If $p \leqslant (p \vee a)^2$, $a \nleqslant p$ and $p$ is a prime element, then $p \vee a = 1$.*

P r o o f. Suppose $p \leqslant (p \vee a)^2$, $a \nleqslant p$ and $p$ is a prime element of $L$. Since $p \leqslant (p \vee a)(p \vee a) = p^2 \vee a(a \vee p)$, we get $1 = ((p^2 \vee a(a \vee p)) : p)$. As $p$ is principal, $1 = (p^2 \vee a(a \vee p) : p) = p \vee (a(a \vee p) : p)$. Also $(a(a \vee p) : p) = (a^2 \vee ap : p) = a \vee (a^2 : p)$. Therefore $1 = p \vee (a(a \vee p) : p) = p \vee a \vee (a^2 : p)$. Now it is enough if we show that $(a^2 : p) \leqslant a$. Let $x$ be any element of $L$ such that $xp \leqslant a^2$. Then $xp \leqslant a$. As $a$ is weak meet principal, $xp = ad$ for some $d \in L$. Since $ad \leqslant p$, $a \nleqslant p$ and $p$ is prime, we get $d \leqslant p$, so $xp = ad \leqslant ab$ and hence $x \leqslant a$ as $p$ is weak join principal and $L$ is a domain. Therefore $(a^2 : p) \leqslant a$. Consequently $p \vee a = 1$. □

**Theorem 2.6.** *Suppose $L$ is principally generated. If $L$ is a Dedekind domain, then every nontrivial prime element of $L$ is a maximal element.*

P r o o f. Suppose $L$ is a Dedekind domain. First we show that every principal nontrivial prime element of $L$ is maximal. Let $p$ be a principal nontrivial prime element of $L$. Suppose $p$ is not maximal. Then $p < m$ for some maximal element $m$ of $L$. If $L$ is principally generated, $a \nleqslant p$ for some principal element $a \leqslant m$. As $L$ is a Dedekind domain, $p \vee a = p_1 \cdots p_k$ and $p \vee a^2 = q_1 \cdots q_n$, where the $p_i$ and $q_i$ are nontrivial prime elements of $L$. Then $\bar{a} = a \vee p \in L/p$, $\bar{a}^2 \in L/p$ and $\bar{a} = \bar{p}_1 \circ \cdots \circ \bar{p}_k$, $\bar{a}^2 = \bar{q}_1 \circ \cdots \circ \bar{q}_n$, where $\bar{p}_i = p_i \vee p = p_i$, $\bar{q}_j = q_j \vee p = q_j$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$. Since $\bar{a} \neq \bar{0}$ and $\bar{a}, \bar{a}^2$ are weak join principal and $L/p$ is a domain, it follows that each $\bar{p}_i$ and each $\bar{q}_i$ are weak join principal elements in $L/p$. Again we have $\bar{p}_1^2 \circ \cdots \circ \bar{p}_k^2 = \bar{q}_1 \circ \cdots \circ \bar{q}_n$. Hence by Lemma 2.4, $n = 2k$ and we may number the $q_i$ so that for $i = 1, \ldots, k$, $q_{2i-1} = q_{2i} = p_i$. Thus $(p \vee a)^2 = p \vee a^2$. Again since $p \leqslant (p \vee a)^2$, and $a \nleqslant p$, by Lemma 2.5, $p \vee a = 1$, so $m = 1$, a contradiction. Therefore every principal nontrivial prime element of $L$ is maximal.

Now we show that every nontrivial prime element is maximal. Let $p$ be a nontrivial prime element. Since $p \neq 0$, there is a nonzero principal element $a \leqslant p$. As $L$ is a Dedekind domain, $a = p_1 \cdots p_n$, where $p_i$'s are nontrivial prime elements. Since $L$ is a domain and $a$ is principal, by Lemma 2.3, each $p_i$ is principal and hence maximal. As $a = p_1 \cdots p_n \leqslant p$, $p_i \leqslant p$ and hence $p = p_i$ is maximal. □

**Theorem 2.7.** *Suppose $L$ is principally generated. If $L$ is a Dedekind domain, then every element is principal.*

Proof. The proof of the theorem follows from Theorem 2.5 and Theorem 2.6. □

## 3. Invertible elements in lattices

Throughout this section $L$ denotes a weak $r$-lattice. In this section, we introduce the concept of invertible elements in multiplicative lattices and study invertible prime elements. Using invertible prime elements, we establish some equivalent conditions for a weak $r$-lattice $L$ to be a finite direct product of Dedekind domains.

**Definition 3.1.** *An element $a$ of a multiplicative lattice $S$ is said to be* regular *if there is a principal element $b \in S$ such that $(0 : b) = 0$ and $b \leqslant a$.*

**Definition 3.2.** *An element $a$ of a multiplicative lattice $S$ is said to be* invertible *if $ac = d$ for some $c \in S$ and for some principal regular element $d$ of $S$.*

**Lemma 3.1.** *Let $S$ be a multiplicative lattice. Then an element $a \in S$ is an invertible element if and only if $a$ is a principal regular element of $S$.*

Proof. The proof of the lemma follows from Lemma 2.3. □

**Lemma 3.2.** *Let $p$ be a proper invertible prime element of $L$. Then*

(a) *If $p = ab$, where $a, b \in L$, then either $a = 1$ or $b = 1$.*

(b) *If $a$ is an invertible element of $L$ and $a > p$, then $a = 1$.*

(c) *If $p' = \bigwedge_{n=1}^{\infty} p^n$, then $p'$ is a prime element and $p'p = p'$ and if $p''$ is a prime element and $p'' < p$, then $p'' \leqslant p'$. If $p'$ is compact and $q$ is a primary element contained in $p$, then $p' \leqslant q$; in fact $p' = q$ or $\sqrt{q} = p$. In particular, if $p'$ is compact, then $p'$ is the only prime element properly contained in $p$.*

(d) *An element $q$ is $p$-primary if and only if $q$ is a power of $p$.*

(e) *The only invertible elements between $p$ and $p^n$, where $n$ is a positive integer, are powers of $p$.*

Proof. By using Lemma 1.1 of [2] and by imitating the proof of Lemma 21 of [7], we can get the result. □

Now we characterize Dedekind domains in terms of invertible elements. Note that, by Theorem 2.7, a domain $L$ is a Dedekind domain if and only if every nonzero element is invertible.

**Theorem 3.1.** *A domain $L$ is a Dedekind domain if and only if every prime element is compact and every nonzero maximal element is invertible.*

Proof. If $L$ is a Dedekind domain, then by Theorem 2.7, every element is principal and hence every prime element is compact and every nonzero element is invertible.

Conversely, assume that every prime element is compact and every nonzero maximal element is invertible. By Lemma 3.2(c), every prime element is principal and hence every element is principal. Consequently $L$ is a Dedekind domain. □

A multiplicative lattice domain is said to be a proper domain if it is not a two element chain. The following Theorem 3.2 establishes an equivalent condition for $L$ to be a finite direct product of proper Dedekind domains.

**Theorem 3.2.** *Suppose $L$ is not a two element chain. Then $L$ is a finite direct product of proper Dedekind domains if and only if every prime element is compact and every maximal element is invertible.*

Proof. Suppose $L = L_1 \times \cdots \times L_n$, where each $L_i$ is a proper Dedekind domain. Then each $L_i$ is a principal element domain and so $L$ is a principal element lattice. If $m$ is a maximal element of $L$, then $m = (1, 1, \ldots, m_i, \ldots, 1)$, where $m_i$ is a maximal element of $L_i$ and so $0 : m = 0$. Hence every maximal element is invertible.

Conversely, assume that every prime element is compact and every maximal element is invertible. Then $L$ is a principal element lattice and so $\dim L \leqslant 1$. As $L$ is a

Noether lattice, the zero element has a normal decomposition. Let $0 = q_1 \wedge \cdots \wedge q_n$ be a normal decomposition and let $p_i = \sqrt{q_i}$. Suppose for $i = 1, 2, \ldots, k$, the $p_i$'s are nonmaximal and for $i = k+1, \ldots, n$, the $p_i$'s are maximal. By Theorem 3.2 of [2], $q_i = p_i$ for $i = 1, 2, \ldots, k$. By Lemma 3.2(c), $p_i' < q_i$ for $i = k+1, \ldots, n$ where $p_i' = \bigwedge_{n=1}^{\infty} p_i^n$ is a prime element. But this contradicts the hypothesis that a normal decomposition is redundant unless $k = n$. Hence $0 = p_1 \wedge \cdots \wedge p_n$. Further these prime elements are comaximal and so $L \cong L/p_1 \times \cdots \times L/p_n$. Note that each factor is a proper principal element domain, and hence $L$ is a finite direct product of proper Dedekind domains. □

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