Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 1, 93–98

Persistent URL: http://dml.cz/dmlcz/127273

Czechoslovak Mathematical Journal, 46 (121) 1996, Praha

SOME PROPERTIES OF BOOLEAN ORDERED SETS

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(Received October 19, 1993)

The aim of this paper is to generalize some results known for boolean algebras to the case of boolean ordered sets. First of all, let us recall some basic notions. Let $(S, \leqslant)$ be an ordered set. For $X \subseteq S$ let us denote by

$$U(X) = \{y \in S;\, y \geqslant x \text{ for all } x \in X\}$$

the so called *upper cone* of X in S and, dually

$$L(X) = \{y \in S;\, y \leqslant x \text{ for all } x \in X\}$$

the so called *lower cone* of X in S.

Instead of $L(U(X))$ or $U(L(X))$ we shall write briefly $LU(X)$ or $UL(X)$. An ordered set S is called *distributive* (see [3]) if for all $a, b, c \in S$ the equation $L(U(a,b),c) = LU(L(a,c), L(b,c))$ holds. Complemented ordered sets were introduced in [4]: an ordered set is called *complemented* if for all $x \in S$ there exists $x' \in S$ such that $UL(x,x') = LU(x,x') = S$ holds. Then the element x' is called a *complement* of x in S and vice versa. A distributive and complemented set is called *boolean*.

Let us remark that a complemented set need not necessarily have the greatest and the least elements.

An element $a \in S$ is called an *atom* if

(i) $a \succ 0$, if 0 is the least element of S ($\succ$ is a cover relation),

(ii) a is a minimal element of S if S has no least element.

The set S is called *atomic* if for all $x \in S$, $x \neq 0$, there exists an atom $a \in S$ such that $x \geqslant a$. The set of all atoms of S will be denoted by $\mathrm{At}(S)$ and for $a \in S$ let $\mathrm{At}(a) = \mathrm{At}(S) \cap L(a)$.

There are differences between some properties of complemented lattices and complemented ordered sets which are not lattices, namely, in [2] it is shown that a uniquely complemented atomic set need not be distributive. But conversely, it is true that a distributive complemented set has to be uniquely complemented.

For an ordered set S let $N(S)$ be Dedekind-Mac Neill's completion of S.

Theorem 1. *If S is a boolean ordered set then $N(S)$ is a boolean algebra.*

Proof. Let us denote $A' = \{a';\ a \in A\}$ for $A \subseteq S$. The proof is based on the following five claims:

Claim 1: $\forall\ A \subseteq S\colon UL(A') = (LU(A))'$.

If $x \in UL(A')$, then $x \geqslant y$ for every $y \in L(A')$. Further, $y \leqslant a'$ for every $a \in A$ which is equivalent to $y' \geqslant a$ for every $a \in A$. Hence $x' \leqslant y'$ for every $y' \in U(A)$, $x' \in LU(A)$, so $x \in (LU(A))'$. The converse inclusion can be estabilished similarly.

Claim 2. If $x \leqslant y$ and $y \in LU(A)$, then $x \in LU(A)$.

The proof of this claim is obvious.

Claim 3. If $A \in N(S)$, then $U(A) \cap A' = U(S)$.

If $x \in U(A) \cap A'$, then $x \geqslant a$ for every $a \in A$ and $x' \in A$, henceforth $x \geqslant x'$.

This gives $LU(x, x') = L(x) = S$, i.e. $U(x) = U(S)$ and $x \in U(S)$.

Claim 4. If $A \in N(S)$, then $A \cap L(A') = L(S)$.

If $x \in A$, $x \leqslant a'$ for every $a \in A$, then $x' \geqslant a$ for every $a \in A$ and $x' \geqslant x$. Then $UL(x, x') = U(x) = S$ and $LU(x) = L(x) = L(S), x \in L(S)$. The converse inclusion is obvious.

Claim 5. If $A \in N(S)$, then $L(A')$ is a unique complement of A in $N(S)$.

By Claim 4, $A \cap L(A') = L(S)$. Let us compute the join in $N(S)$: $A \vee L(A') = LU(A, L(A')) = L(U(A) \cap UL(A'))$. Now, by Claim 1, $UL(A') = (LU(A))'$, thus $L(U(A) \cap UL(A')) = L(U(A) \cap (LU(A))')$. Since $A \in N(S)$, $LU(A) = A$ and the righthand side of last equality can be simplified to $L(U(A) \cap A')$. Finally, by Claim 3 this is equal to $LU(S) = S$.

Let us prove the uniqueness of a complement. Let $B \in N(S)$ be a complement of A. Then $A \cap B = S$, $LU(A, B) = S$ and $U(A, B) = U(S)$. Let $b \in B$. Then for every $a \in A$, $L(a, b) \subseteq A \cap B = L(S)$ holds due to Claim 2, so $L(a, b) = L(S)$. But then using distributivity we can derive

$$\begin{aligned} L(b) &= L(b) \cap LU(a, a') = L(b, U(a, a')) = LU(L(a, b), L(b, a')) \\ &= LU(L(S), L(b, a')) = L(b, a'), \end{aligned}$$

henceforth $b \leqslant a'$, $b \in L(A')$ and $B \subseteq L(A')$. Conversely, let $x \in L(A')$, i.e. $x \leqslant a'$ for every $a \in A$. Then $x' \geqslant a$ for every $a \in A$ and so $x' \in U(A)$. Let $y \in U(B)$. We have $U(x', y) \subseteq U(A) \cap U(B) = U(A, B) = U(S)$, hence $LU(x', y) = S$. But then $L(x) = L(x) \cap LU(x', y) = LU(L(x, x'), L(x, y)) = L(x, y)$, so we have $x \leqslant y$ and $x \in LU(B) = B$ (since $B \in N(S)$). Finally, $L(A') \subseteq B$ and the uniqueness of a complement is proved.

Further, let $A, B \in N(S)$ and let us compute the join and intersection of $L(A')$ and $L(B')$ in $N(S)$: $L(A') \vee L(B') = LU(L(A'), L(B')) = L(UL(A') \cap UL(B'))$; using Claim 1, this is equal to $L((LU(A))' \cap (LU(B))') = L(A') \cap L(B') = L((A \cap B)')$ (we used the fact that $A, B \in N(S)$); analogously, the intersection: $L(A') \cap L(B') = LUL(A' \cup B') = LUL((A \cup B)')$; using Claim 1, we have $LUL((A \cup B)') = L((LU(A \cup B))') = L((A \vee B)')$.

Now, if we denote by the symbol A^* the complement of $A \in N(S)$, the last equalities can be rewritten to

$$(A \cap B)^* = A^* \vee B^* \text{ and } (A \vee B)^* = A^* \cap B^*. \tag{6}$$

It is well-known that a uniquely complemented lattice satisfying the condition (6) is a boolean algebra. □

The next question concerning the boolean sets is what is the number of nonisomorphic finite boolean sets having a given number of elements. The key to a partial solution of this problem is

Theorem 2. *If B is an atomic boolean set, then $U(a) = U(\mathrm{At}(a))$ holds for every $a \in B$.*

Proof. Evidently, $U(a) \subseteq U(\mathrm{At}(a))$. Let $z \in U(\mathrm{At}(a))$ and let $\mathrm{At}(a) = \{q_a^\alpha;\ \alpha \in A\}$. Then $z' \leqslant (q_a^\alpha)'$ for every $q_a^\alpha \in \mathrm{At}(a)$. Henceforth $L(z', a) \subseteq L(a, \{(q_a^\alpha)';\ \alpha \in A\})$. Let us suppose that

$$L(a, \{(q_a^\alpha)';\ \alpha \in A\}) \neq L(B).$$

Then there exists an atom $q \in L(a, \{(q_a^\alpha)';\ \alpha \in A\})$. But since $q \in \mathrm{At}(a)$ then $q \leqslant q'$, a contradiction. We proved that $L(a, z') = L(B)$, i.e. $UL(a, z') = B$. Then we can derive $U(z) = U(z, L(a, z')) = UL(U(a, z), U(z, z')) = U(a, z)$, so $z \geqslant a$. Finally, $U(\mathrm{At}(a)) \subseteq U(a)$. □

Corollary 1. *If B is an atomic boolean set, then for every $a, b \in B$, $\mathrm{At}(a) = \mathrm{At}(b)$ implies $a = b$.*

Corollary 2. *If B is a boolean set with k atoms, then $N(B) \cong 2^k$.*

Theorem 3. *For every even natural number n there exists an n-element boolean set. Moreover, there exist non-isomorphic n-element boolean sets for $n \geqslant 8$.*

Proof. Let $2^{k-1} < n \leqslant 2^k$ for some natural number k. Then by Corollary 2 B has k atoms and is isomorphic to $(\{\mathrm{At}(a);\ a \in B\}, \subseteq)$. Now we can give an algorithm for finding n-element boolean sets.

First, we take a 2^k-element boolean algebra. It has just k atoms and k coatoms. There remain just $2^k - 2k$ elements and $2^{k-1} - k$ pairs of complemented elements. By deleting of such a pair we obtain an ordered set $2^k \setminus \{a, a'\}$ which will be still isomorphic to $(\{\mathrm{At}(b);\ b \in 2^k \setminus \{a, a'\}\}, \subseteq)$. This set has to be distributive and complemented. Further, we can repeat this procedure $(2^{k-1} - k)$-times. $\square$

However, if S_1 or S_2 are two n-element boolean sets created by deleting the set of pairs of complemented elements K_1 or $K_2 (K_1 \neq K_2)$, respectively, from the boolean algebra 2^k, the sets S_1 and S_2 need not be isomorphic. Hence, the problem of finding the number of nonisomorphic n-element boolean sets is complicated.

Example 1. An ordered set visualized in Fig. 1 is an 8-element boolean set non-isomorphic to 2^3.

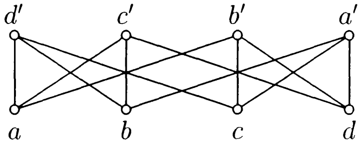

Fig. 1

Example 2. The set depicted in Fig. 2 is a 10-element boolean set. A non-isomorphic one is obtained by adding to the set of Fig. 1 the greatest and least elements.

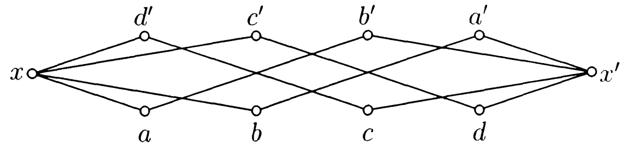

Fig. 2

It is well-known that boolean algebras are $\wedge$ and $\vee$-continuous lattices.

We shall show that boolean sets satisfy modified identities. Let us introduce the following useful notation. If B_i, $i \in I$, are subsets of an ordered set S, let $U(\overrightarrow{B_i})$ be $U(B_1, B_2, \ldots, B_i, \ldots)$, $U(B_1, B_2, \ldots, B_i, \ldots) = U\big(\bigcup\{B_i;\ i \in I\}\big)$ and let $\overrightarrow{x_i}$ be a "vector" of elements of S indexed by I.

Definition. An ordered set S is called *U-continuous* if

$$\forall\ x, \overrightarrow{x_i} \in S\colon L(x, U(\overrightarrow{x_i})) = LU(\overrightarrow{L(x, x_i)})$$

holds for every index set $I \neq \emptyset$. If it satisfies the dual identity it will be called *L-continuous*.

Theorem 4. *Every boolean ordered set is both U- and L-continuous.*

Proof. If we take $I \neq \emptyset$, evidently $L(x, x_i) \subseteq L(x)$ and $L(x, x_i) \subseteq LU(\overrightarrow{x_i})$ holds for every $i \in I$. Then we have $L(x, U(\overrightarrow{x_i})) \supseteq L(x, x_i)$ for every $i \in I$, so

$$L(x, U(\overrightarrow{x_i})) \supseteq \bigcup\{L(x, x_i);\, i \in I\}.$$

Apply an operator LU to the last inclusion we obtain

$$LU(L(x, U(\overrightarrow{x_i}))) = L(x, U(\overrightarrow{x_i})) \supseteq LU(\overrightarrow{L(x, x_i)}).$$

Conversely, let $u \in U(\overrightarrow{L(x, x_i)})$, so $U(u) \subseteq UL(x, x_i)$, $L(u) \supseteq L(x, x_i)$ for every $i \in I$. Let $z \in L(x, U(\overrightarrow{x_i}))$, i.e. $L(z) \subseteq L(x, U(\overrightarrow{x_i}))$. Further $L(x_i) = L(x_i, U(x, x')) = LU(L(x, x_i), L(x_i, x'))$ and since $L(x, x_i) \subseteq L(u), L(x_i, x') \subseteq L(x')$, we obtain the inclusion $L(x_i) \subseteq LU(u, x')$ for every $i \in I$. But then $U(x_i) \supseteq U(u, x')$ and so $U(\overrightarrow{x}_i) \supseteq U(u, x'), LU(\overrightarrow{x}_i) \subseteq LU(u, x')$. Finally, we can derive the following system of inclusions: $L(z) \subseteq L(x, U(\overrightarrow{x}_i)) \subseteq L(x, U(u, x')) = LU(L(x, u), L(x, x')) = L(x, u) \subseteq L(u)$, so $z \leqslant u$ and $z \in LU(\overrightarrow{(x, x_i)})$. The proof of the dual identity can be done similarly. □

Moreover, for atomic boolean algebras it can be shown that they satisfy the general distributive law. The next theorem gives a similiar result for atomic boolean sets.

Definition. A boolean ordered set B is said to be *generally distributive* if the following identity holds:

$$\forall\, B_1, \ldots, B_\alpha \subseteq B\colon L(U(B_1), \ldots, U(B_\alpha)) = LU(\overrightarrow{L(b_1, \ldots, b_\alpha)}),$$

where $b_1 \in B_1, \ldots, b_\alpha \in B_\alpha$ and α is an element of an arbitrary non-empty index set A.

Theorem 5. *Every atomic boolean set B is generally distributive.*

Proof. If $\mathrm{At}(B)$ is the set of all atoms of B, then by Theorem 2, B is isomorphic to a subset of $A = 2^{\mathrm{At}(B)}$. Let $A_1, \ldots, A_\beta$ be some elements of $B \subseteq A$. First, let us show that

$$L_B U_B(A_1, \ldots, A_\beta) = L_A(A_1 \cup \ldots \cup A_\beta) \cap B \tag{1}$$

where the subscript denotes the cone in B or A, respectively. It suffices to show that

$$L_A U_B(A_1, \ldots, A_\beta) = L_A U_A(A_1, \ldots, A_\beta). \tag{2}$$

To this end, let $Z \in A, Z \subseteq W$ for every $W \in B$ such that $W \supseteq A_1, \ldots, A_\beta$. Let $X_\beta = \{x_\beta^\gamma\}$ be the set of all coatoms of A for which $x_\beta^\gamma \supseteq A_\beta$. But by the construction of the boolean atomic sets we have $x_\beta^\gamma \in B$ for every β and γ. So we can prove that $Z \subseteq \cap X_\beta$ and therefore $Z \in L_A U_A(A_1, \ldots, A_\beta)$. The converse inclusion is trivial. Now, the claim (1) is an easy consequence of (2).

So, let $B_1 = \{B_1^{\beta_1} ; \beta_1\}, \ldots, B_\alpha = \{B_\alpha^{\beta_\alpha} ; \beta_\alpha\}$ be the elements of B. Then the left hand side of the general distributive law is equal to (using identity (1))

$$\begin{aligned} L_B(U_B(B_1), \ldots, U_B(B_\alpha)) &= L_B U_B(B_1) \cap \ldots \cap L_B U_B(B_\alpha) \\ &= L_A(\bigcup\{(B_1^{\beta_1} ; \beta_1\}) \cap B \cap \ldots \cap L_A(\bigcup\{B_\alpha^{\beta_\alpha} ; \beta_\alpha\}) \cap B \\ &= L_A(\bigcup\{B_1^{\beta_1} ; \beta_1\}) \cap \ldots \cap L_A(\bigcup\{B_\alpha^{\beta_\alpha} ; \beta_\alpha\}) \cap B. \end{aligned}$$

Analogously, $U_B L_B(B_1^{\beta_1}, \ldots, B_\alpha^{\beta_\alpha}) = U_A(B_1^{\beta_1} \cap \ldots \cap B_\alpha^{\beta_\alpha}) \cap B$ for every choice of $B_1^{\beta_1} \in B_1, \ldots, B_\alpha^{\beta_\alpha} \in B_\alpha$.

Let $Q \in B$, $Q \supseteq B_1^{\beta_1} \cap \ldots \cap B_\alpha^{\beta_\alpha}$ for all possible choices of $B_1^{\beta_1} \in B_1, \ldots, B_\alpha^{\beta_\alpha} \in B_\alpha$. Further, let $Z \in B$, $Z \subseteq \bigcup\{B_1^{\beta_1} ; \beta_1\}, \ldots, \bigcup\{B_\alpha^{\beta_\alpha} ; \beta_\alpha\}$. Then it is evident that

$$Z \subseteq (\bigcup\{B_1^{\beta_1} ; \beta_1\}) \cap \ldots \cap (\bigcup\{B_\alpha^{\beta_\alpha} ; \beta_\alpha\}).$$

Now, using a general distributive law for sets, we have

$$Z \subseteq \bigcup\{B_1^{\beta_1} \cap \ldots \cap B_\alpha^{\beta_\alpha} ; \beta_1, \ldots, \beta_\alpha\}.$$

Further, by the choice of Q we have $Q \supseteq \bigcup\{B_1^{\beta_1} \cap \ldots \cap B_\alpha^{\beta_\alpha} ; \beta_1, \ldots, \beta_\alpha\}$, so $Z \subseteq Q$ and since Z, Q were arbitrary, $L_B U_B(B_1, \ldots, B_\alpha) \subseteq L_B U_B(\overrightarrow{L_B(b_1, \ldots, b_\alpha)})$. The converse inclusion is trivial to prove. □

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