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RATIONALS AS A NON-TRIVIAL COMPLETE CONVERGENCE GROUP

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It is a well-known fact that convergence spaces, despite of their seemingly close relation to first countable topological spaces, do not admit a reasonable notion of completeness. This obstacle may be overcome by imposing more structure on the underlying set: As proved by J. Novák, there is a sound notion of a Cauchy filter on a convergence group, and every convergence group has a completion [N]. However, numerous papers pointed out that even in the most elementary setting, namely $(\mathbb{Q}, +)$, things may go weird (see e.g. [F1, F2]). Since—up to the author's knowledge—nobody has paid attention to those group convergences on rationals which are strictly finer than the usual metric one, we want to show that it may even happen that $\mathbb{Q}$ is complete in such a case. (Another example of this kind may be found in [DFZ], with the convergence coarser than the metric convergence and the induced closure antiHausdorff.) We do not consider the result just another bizzare example, because we feel that it provides some information on the complexity of those sequences of rationals which converge to an irrational number.

However, it is also true that for every irrational number x there is a group convergence $\mathscr{C}$ on the rationals such that for its categorical completion $(\widetilde{\mathbb{Q}}, \widetilde{\mathscr{C}})$ one has $x \in \widetilde{\mathbb{Q}}$ and still $\mathbb{R} \setminus \widetilde{\mathbb{Q}} \neq \emptyset$. This is a special case of our Theorem 2, where we characterize compact subsets X of $\mathbb{R}$ such that for some group convergence $\mathscr{C}$ on $\mathbb{Q}$, finer than the usual metric one, $\mathbb{Q} \cup X \subseteq \widetilde{\mathbb{Q}} \subsetneqq \mathbb{R}$.

For the reader's convenience, let us recall the basic notions from the theory of convergence groups. Let X be a set. A subset $\mathscr{C} \subseteq {}^{\omega}X \times X$ is called a *convergence* on X provided the following holds:

(S) for each $x \in X$ we have $\langle\langle x \colon n \in \omega\rangle, x\rangle \in \mathscr{C}$, where $\langle x \colon n \in \omega\rangle$ denotes the constant sequence with value x;

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(F) whenever $\langle\langle x(n)\colon n \in \omega\rangle, x\rangle \in \mathscr{C}$ and $f \in {}^\omega\omega$ is strictly increasing, then $\langle\langle x(f(n))\colon n \in \omega\rangle, x\rangle \in \mathscr{C}$;

(H) if $\langle\langle x(n)\colon n \in \omega\rangle, x\rangle \in \mathscr{C}$ and $\langle\langle x(n)\colon n \in \omega\rangle, y\rangle \in \mathscr{C}$, then $x = y$;

(U) if $\langle\langle x(n)\colon n \in \omega\rangle, x\rangle \in {}^\omega X \times X$ is such that for every strictly increasing $f \in {}^\omega\omega$ there is a strictly increasing $g \in {}^\omega\omega$ such that $\langle\langle x(f(g(n)))\colon n \in \omega\rangle, x\rangle \in \mathscr{C}$, then $\langle\langle x(n)\colon n \in \omega\rangle, x\rangle \in \mathscr{C}$.

To avoid unnecessary repetitions, let us denote by Mon the set of all strictly increasing functions in ${}^\omega\omega$. If $\langle\langle x(n)\colon n \in \omega\rangle, x\rangle \in \mathscr{C}$, we will say that the sequence $\langle x(n)\colon n \in \omega\rangle$ *converges* to x (or, more precisely, $\mathscr{C}$*-converges* to x) and abbreviate it to $x(n) \longrightarrow x$. So the axioms of convergence mean that constant sequences converge, subsequences of a convergent sequence converge to the same limit point, the limits are unique and, if for some sequence we find a point such that every subsequence contains a subsubsequence converging to that point, then the sequence itself converges.

Every convergence on a set X induces a closure operation on X; in general, it need not be a topology, i.e., $\operatorname{cl}\operatorname{cl} M = \operatorname{cl} M$ may fail.

Suppose now that $\cdot$ is a group operation on X with a neutral element e. We shall say that a convergence $\mathscr{C}$ is a *group convergence* (and the triple $(X, \cdot, \mathscr{C})$ is a *convergence group*), if $\mathscr{C}$ moreover satisfies

(L) if $\langle\langle x(n)\colon n \in \omega\rangle, x\rangle \in \mathscr{C}$ and $\langle\langle y(n)\colon n \in \omega\rangle, y\rangle \in \mathscr{C}$, then $\langle\langle x(n) \cdot y(n)^{-1}\colon n \in \omega\rangle, x \cdot y^{-1}\rangle \in \mathscr{C}$ and $\langle\langle x(n)^{-1} \cdot y(n)\colon n \in \omega\rangle, x^{-1} \cdot y\rangle \in \mathscr{C}$ as well.

If the group operation $\cdot$ is clear from the context, we write simply $(X, \mathscr{C})$.

If $\mathscr{C}$ is a group convergence, then a sequence $\langle x(n)\colon n \in \omega\rangle$ is called a *Cauchy sequence* ($\mathscr{C}$*-Cauchy sequence*, if it is necessary to express $\mathscr{C}$), if for every $f, g \in \mathsf{Mon}$, $\langle\langle x(f(n)) \cdot x(g(n))^{-1}\colon n \in \omega\rangle, e\rangle \in \mathscr{C}$ and $\langle\langle x(f(n))^{-1} \cdot x(g(n))\colon n \in \omega\rangle, e\rangle \in \mathscr{C}$. A convergence group is called *complete*, if every Cauchy sequence converges. For an abelian group $(X, \cdot)$ and for every group convergence $\mathscr{C}$ on X, there is a group $\widetilde{X}$ containing X as a cl-dense subgroup, and a group convergence $\widetilde{\mathscr{C}}$ on $\widetilde{X}$ such that $(\widetilde{X}, \widetilde{\mathscr{C}})$ is a complete convergence group, $\mathscr{C} = \widetilde{\mathscr{C}} \cap ({}^\omega X \times X)$ and $\widetilde{\mathscr{C}}$ is the smallest group convergence with these properties. The convergence group $(\widetilde{X}, \widetilde{\mathscr{C}})$ is unique (up to an isomorphism) and is called a *categorical completion* of $(X, \mathscr{C})$; its existence and unicity was proved by J. Novák in [N].

Let us begin with a technical lemma. Since we shall mention here a few topological terms, let us agree that all references to the topology of rationals or reals always concern the usual metric one. Next, as usually adopted, for $G, H \subseteq \mathbb{R}$ denote by $G + H$ the set $\{p + q\colon p \in G, q \in H\}$. Similarly for $G - H$, and for $q \in \mathbb{R}$, $q + H$ means $\{q\} + H$. To avoid confusion, the set theoretical difference will be denoted by $G \setminus H$.

Lemma. *Let $\mathscr{C}$ be a group convergence on $\mathbb{Q}$ finer than the metric convergence, i.e., $\langle\langle x(n)\colon n\in\omega\rangle, 0\rangle \in \mathscr{C}$ implies $\lim_{n\to\infty} x(n) = 0$, let $\{S_k\colon k\in\omega\}$ be a family of closed subsets of the reals. Suppose that for each $\langle\langle x(n)\colon n\in\omega\rangle, 0\rangle\in\mathscr{C}$ there are $m, k\in\omega$ such that for all $n\geqslant m$, the point $x(n)$ belongs to the set S_k.*

Then for every $\mathscr{C}$-Cauchy sequence $\langle x(n)\colon n\in\omega\rangle$ there is some $p\in\mathbb{Q}$ and $k, r\in\omega$ such that for every $n > r$, $x(n)\in p - S_k$. In particular, the metric limit of each $\mathscr{C}$-Cauchy sequence belongs to the set $\mathbb{Q} - \bigcup_{k\in\omega} S_k$.

Proof. Let $\langle x(n)\colon n\in\omega\rangle$ be an arbitrary $\mathscr{C}$-Cauchy sequence. Since the group $(\mathbb{Q}, +)$ is abelian, it is enough to assume that for every $f\in \mathsf{Mon}$, the sequence $\langle x(n) - x(f(n))\colon n\in\omega\rangle$ $\mathscr{C}$-converges to 0. Indeed, if $f, g\in\mathsf{Mon}$, then $x(f(n)) - x(g(n)) = x(f(n)) - x(n) + x(n) - x(g(n)) = 0 - (x(n) - x(f(n))) + (x(n) - x(g(n)))$. Both sequences $\langle x(n) - x(f(n))\colon n\in\omega\rangle$ and $\langle x(n) - x(g(n))\colon n\in\omega\rangle$ converge to 0 by the assumption, the constant sequence $\langle 0\colon n\in\omega\rangle$ converges to 0 by (S), so $\langle x(f(n)) - x(g(n))\colon n\in\omega\rangle$ converges to 0 since $\mathscr{C}$ satisfies (L).

For every $f\in\mathsf{Mon}$, we have $\langle\langle x(n) - x(f(n))\colon n\in\omega\rangle, 0\rangle\in\mathscr{C}$. By the assumption of the Lemma, there are natural numbers $m = m_f$ and $k = k_f$ such that for all $n\geqslant m$, $x(n) - x(f(n))\in S_k$. For $(m,k)\in\omega\times\omega$ let us put $Z(m,k) = \{f\in\mathsf{Mon} : (m,k) = (m_f, k_f)\}$.

Consider ${}^{\omega}\omega$ as a Tychonoff product of countably many countable discrete spaces. Then ${}^{\omega}\omega$ is a complete metric space and Mon, as a closed subspace of it, is a complete metric space, too. In particular, Mon is not of the first category in itself. Hence there is some $(m,k)\in\omega\times\omega$ such that the set $Z(m,k)$ is not nowhere dense. Let this (m,k) be fixed for the rest of the proof.

Before we proceed further, let us introduce some rather standard notation. For $n\in\omega$ and $\varphi\in{}^{n}\omega$, denote $[\varphi] = \{f\in{}^{\omega}\omega\colon f\supseteq\varphi\}$. Recall that $\{[\varphi]\colon n\in\omega, \varphi\in{}^{n}\omega\}$ is an open basis for the topology of the space ${}^{\omega}\omega$. With this notation, we may state a claim.

Claim. *There is a strictly increasing $\varphi\in\bigcup_{n\in\omega}{}^{n}\omega$ such that for every strictly increasing $\psi\in\bigcup_{n\in\omega}{}^{n}\omega$ with $\psi\supseteq\varphi$, $[\psi]\cap Z(m,k)\neq\emptyset$.*

Suppose this is not the case and consider an arbitrary non-void open U in the space Mon. Choose a strictly increasing φ with $[\varphi]\subseteq U$. Then one will be able to find a strictly increasing $\psi\supseteq\varphi$ with $[\psi]\cap Z(m,k) = \emptyset$. As U was arbitrary, we see that the set $Z(m,k)$ is nowhere dense, which is a contradiction.

Choose φ as in the claim and denote by m_0 the maximum of m, $\mathrm{dom}(\varphi)$. Let ψ be a strictly increasing function, $\mathrm{dom}(\psi) = m_0$, $\mathrm{rng}(\psi)\subseteq\omega$ and $\psi\supseteq\varphi$. Let $r = \psi(m_0 - 1)$. Next, for every $n > r$, let $\psi_n\in{}^{m_0+1}\omega$, $\psi_n\supseteq\psi$, $\psi_n(m_0) = n$. According to the claim, $[\psi_n]\cap Z(m,k)\neq\emptyset$, so there is $f_n\in Z(m,k), f_n\supseteq\psi_n$. As

$m_0 \geqslant m$ and $f_n \in Z(m,k)$, we have $x(m_0) - x(f_n(m_0)) \in S_k$ for all $n > r$. However, $f_n(m_0) = \psi_n(m_0) = n$.

Therefore, whenever $n > r$, then $x(m_0) - x(n) \in S_k$. So all values $x(n)$ for $n > r$ belong to the closed set $x(m_0) - S_k$. It remains to set $p = x(m_0)$.

Notice now that every $\mathscr{C}$-Cauchy sequence is Cauchy also in the usual metric of the reals. The sequence $\langle x(n) \colon n \in \omega\rangle$ is thus an ordinary Cauchy sequence of rationals and eventually ranges in a closed set $p - S_k$. Hence it has a limit x and this x must belong to $p - S_k \subseteq \mathbb{Q} - \bigcup\limits_{k\in\omega} S_k$, which completes the proof. □

Now, let us state the title result.

Theorem 1. *Let $\mathscr{C}$ be the smallest group convergence on the additive group of rational numbers such that $\langle\langle \frac{1}{n+1} \colon n \in \omega\rangle, 0\rangle \in \mathscr{C}$. Then the convergence group $(\mathbb{Q}, +, \mathscr{C})$ is complete.*

Proof. Let us recall Prof. M. Dolcher's paper [D], where the minimal convergence containing a given subset $A \subseteq {}^{\omega}X \times X$, is described. Consider three mappings α, β, γ from the powerset of ${}^{\omega}X \times X$ into itself. For $A \subseteq {}^{\omega}X \times X$, let $\alpha(A) = A \cup \{\langle\langle x \colon n \in \omega\rangle, x\rangle \colon x \in X\}$. Next, let $\beta(A) = A \cup \{\langle\langle x(f(n)) \colon n \in \omega\rangle, x\rangle \colon f \in \mathsf{Mon}, \langle\langle x(n) \colon n \in \omega\rangle, x\rangle \in A\}$. Finally, let $\gamma(A) = A \cup \{\langle\langle x(n) \colon n \in \omega\rangle, x\rangle \colon \forall f \in \mathsf{Mon}\, \exists g \in \mathsf{Mon}$ such that $\langle\langle x(f(g(n))) \colon n \in \omega\rangle, x\rangle \in A\}$.

Then for every $A \subseteq {}^{\omega}X \times X$, the set $\gamma(\beta(\alpha(A)))$ is stable under α, β and γ and satisfies FSU. If (H) is true for sequences in A, then $\gamma(\beta(\alpha(A)))$ satisfies (H), too. Moreover, $\gamma(\beta(\alpha(A)))$ is the smallest convergence containing A. We refer the reader to [D] for the proof.

In our case, A is the two-element set $\{\langle\langle \frac{1}{n+1} \colon n \in \omega\rangle, 0\rangle, \langle\langle -\frac{1}{n+1} \colon n \in \omega\rangle, 0\rangle\}$.

Denote by $\mathscr{D}$ the set of all pairs $\langle\langle x(n) \colon n \in \omega\rangle, 0\rangle \in {}^{\omega}X \times X$ with $\langle\langle x(n) \colon n \in \omega\rangle, 0\rangle \in \gamma(\beta(\alpha(A)))$ and let $\mathscr{C}_0$ be the set of all $\langle\langle x(n) \colon n \in \omega\rangle, 0\rangle$ such that there is a natural number k and for every $i < k$ an element $\langle\langle x_i(n) \colon n \in \omega\rangle, 0\rangle \in \mathscr{D}$ such that $x(n) = \sum\limits_{i<k} x_i(n)$ for every $n \in \omega$. Finally, let $\mathscr{C} = \{\langle\langle x(n) + y(n) \colon n \in \omega\rangle, y\rangle \colon \langle\langle x(n) \colon n \in \omega\rangle, 0\rangle \in \mathscr{C}_0, \langle\langle y(n) \colon n \in \omega\rangle, y\rangle \in \gamma(\beta(\alpha(A)))\}$. It is an easy exercise to verify that $\mathscr{C}$ is a group convergence and that it is the minimal one with $\frac{1}{n+1} \longrightarrow 0$; we leave it to the reader.

We have to prove that $\mathscr{C}$ is complete. We wish to apply the lemma, so our first observation concerns the possible values of sequences which $\mathscr{C}$-converge to 0. Denote by S_0 the set $\{0\} \cup \{\frac{1}{n+1} \colon n \in \omega\} \cup \{-\frac{1}{n+1} \colon n \in \omega\}$. Then by induction, if S_k is known, let $S_{k+1} = S_k \cup (S_k + S_k)$.

Claim. *If $\langle\langle y(n) \colon n \in \omega\rangle, 0\rangle \in \mathscr{C}$, then there is some $m \in \omega$ and some $k \in \omega$ such that for all $n \geqslant m$ we have $y(n) \in S_k$.*

The proof follows the pattern how $\mathscr{C}$ is built. First consider $\alpha(A)$. The sequences $\langle \frac{1}{n+1} : n \in \omega \rangle$, $\langle -\frac{1}{n+1} : n \in \omega \rangle$ and the constant sequence $\langle 0 : n \in \omega \rangle$ satisfy the claim with $m = k = 0$. But these three are the only sequences in $\alpha(A)$, converging to 0. Whenever $\langle y(n) : n \in \omega \rangle$ is a subsequence of any of them, then it satisfies the claim as well and again with $k = m = 0$. Hence the claim holds for all sequences from $\beta(\alpha(A))$ converging to 0. Next, let $\langle \langle y(n) : n \in \omega \rangle, 0 \rangle \in \gamma(\beta(\alpha(A)))$ and, aiming for the contrary, suppose that for each $n \in \omega$ there is $f(n) \in \omega$ with $f(n) \geqslant n$ and such that $y(f(n)) \notin S_0$. We may assume that $f \in \mathsf{Mon}$. Obviously there is no $g \in \mathsf{Mon}$ with $\langle \langle y(f(g(n))) : n \in \omega \rangle, 0 \rangle \in \beta(\alpha(A))$. So $\langle \langle y(n) : n \in \omega \rangle, 0 \rangle \notin \gamma(\beta(\alpha(A)))$, a contradiction.

We have verified that for all $\langle \langle y(n) : n \in \omega \rangle, 0 \rangle \in \mathscr{D}$, all but finitely many values $y(n)$ belong to S_0. Thus if $\langle \langle y_i(n) : n \in \omega \rangle, 0 \rangle \in \mathscr{D}$ for all $i < k$, and if m_i is such that $n \geqslant m_i$ implies $y_i(n) \in S_0$, then for $m = \max\limits_{i<k} m_i$ and $n \geqslant m$ we have $\sum\limits_{i<k} y_i(n) \in \sum\limits_{i<k} S_0 \subseteq S_k$. So all sequences in $\mathscr{C}_0$ satisfy the claim. It remains to observe that $\langle \langle y(n) : n \in \omega \rangle, 0 \rangle \in \mathscr{C}_0$ if and only if $\langle \langle y(n) : n \in \omega \rangle, 0 \rangle \in \mathscr{C}$, and the claim is proved.

Our next observation is trivial.

Observation. *Every set S_k is compact.*

Indeed, S_0 is compact. If S_k is compact, then S_{k+1}, being the union of a compact set S_k and of a continuous image of a compact set $S_k \times S_k$, must be compact as well.

Now, let $\langle x(n) : n \in \omega \rangle$ be an arbitrary $\mathscr{C}$-Cauchy sequence. As already noticed, it is also Cauchy in the metric of the reals, let q be its metric limit. Since we have verified all the assumptions of the lemma, we know that there is a rational number p and $k, r \in \omega$ such that for all $n > r$, $x(n) \in p - S_k$ and $q \in p - S_k$, too. Since $S_k \subseteq \mathbb{Q}$, the number q is rational.

It remains to show that $\langle \langle x(n) : n \in \omega \rangle, q \rangle \in \mathscr{C}$, i.e., that the sequence $\langle x(n) : n \in \omega \rangle$ also $\mathscr{C}$-converges to q. But this is fairly easy now. Let $t(n) = q - x(n)$ for $n \in \omega$. We have $t(n) \longrightarrow 0$ in the usual metric topology. However, we may notice that $t(n) = q - x(n) = q - p + p - x(n)$, $p - q \in S_k$ and $p - x(n) \in S_k$ for all $n > r$, therefore $t(n) \in S_k - S_k \subseteq S_{k+1}$ for all $n > r$.

According to the definition of the set S_{k+1}, for every $n > r$ there are numbers $a(n, i) \in S_0$ such that $t(n) = \sum \{a(n, i) : i < 2^{k+1}\}$. For $n \leqslant r$, let $a(n, 0) = t(n)$ and $a(n, i) = 0$ for $1 \leqslant i < 2^{k+1}$.

For a fixed i with $i < 2^{k+1}$, consider the sequence $\langle a(n, i) : n \in \omega \rangle$. With finitely many exceptions, it ranges in S_0 and converges to 0. Therefore for every $f \in \mathsf{Mon}$ we can find $g \in \mathsf{Mon}$ such that either all $a(f(g(n)), i)$ equal zero, or all $a(f(g(n)), i)$ are positive and pairwise distinct or all $a(f(g(n)), i)$ are negative and pairwise distinct. In any of these three cases, $\langle \langle a(f(g(n), i) : n \in \omega \rangle, 0 \rangle \in \beta(\alpha(A))$. So $\langle \langle a(n, i) : n \in$

$\omega\rangle, 0\rangle \in \gamma(\beta(\alpha(A)))$ and then, $\langle\langle t(n) \colon n \in \omega\rangle, 0\rangle \in \mathscr{C}_0 \subseteq \mathscr{C}$. Since $\langle\langle q \colon n \in \omega\rangle, q\rangle \in \mathscr{C}$, too, we have that $\langle\langle x(n) \colon n \in \omega\rangle, q\rangle = \langle\langle q - t(n) \colon n \in \omega\rangle, q - 0\rangle \in \mathscr{C}$. □

Remark. In the above proof, we did not use any particular property of the sequence $\langle \frac{1}{n+1} \colon n \in \omega\rangle$ except that it converges to 0. In fact, whenever $\langle q_n \colon n \in \omega\rangle$ is a sequence of rationals converging to 0, then the smallest group convergence on $\mathbb{Q}$ containing $\langle\langle q_n \colon n \in \omega\rangle, 0\rangle$ is complete. After an appropriate modification of the proof, the same is true if we consider not just one, but a finite number of sequences at the start.

Now, we want to find another group convergence $\mathscr{C}$ on $\mathbb{Q}$ such that some irrationals, but not all, will be the limits of $\mathscr{C}$-Cauchy sequences. Before doing so, we shall collect several easy facts on subsets of the reals.

Definition. Let $X \subseteq \mathbb{R}$. Denote by X^{+0} the set $X \cup -X$ and for $n > 0$, let $X^{+n} = X^{+(n-1)} + X$. Let us call a set $X \subseteq \mathbb{R}$ *additively nowhere dense*, if for each $n \in \omega$, the set X^{+n} is nowhere dense.

Fact 1. *Let $x, y \in \mathbb{R}$. If $X \subseteq \mathbb{R}$ is compact and additively nowhere dense, then so are the sets $X \cup \{0\}$, $X \cup -X$ and $(X - x) \cup (X - y)$.*

Proof. Indeed, for $Y = X \cup \{0\}$ we have $Y^{+1} = X^{+1}$, for $Y = X \cup -X$ we have $Y = X^{+0}$, which immediately implies the statement.

For $Y = (X - x) \cup (X - y)$ we have $Y^{+n} \subseteq X^{+n} - M_n$, where M_n is the set of all sums $ix + jy$ with $i, j \in \mathbb{Z}$, $|i| + |j| \leqslant n + 1$. Since X^{+n} is nowhere dense and M_n is finite, the set $X^{+n} - M_n$ is nowhere dense, too, hence also Y^{+n} is. □

Fact 2. *If $X \subseteq [0, 1]$ is compact and for every real $0 < r \leqslant 1$, $X \cap [r, 1]$ is additively nowhere dense, then X is additively nowhere dense.*

Proof. It should be clear that X is nowhere dense. So we already have X^{+0} nowhere dense.

Induction step: Suppose X^{+n} to be nowhere dense and let us consider X^{+n+1}. Choose an arbitrary nondegenerate interval (a, b); we may suppose that $a > 0$ (the proof is symmetrical for $b < 0$). By the induction hypothesis, there are real numbers c, d with $a < c < d < b$ such that $(c, d) \cap X^{+n} = \emptyset$. Choose $r > 0$ so small that $c + r = e < f = d - r$ and consider $t \in X^{+n+1} \cap (e, f)$. Then $t = x_0 + x_1 + \ldots + x_{n+1}$ with all x_i belonging to X^{+0}. For every $i \leqslant n + 1$ we obviously have $t - x_i \in X^{+n}$. If there is some $i \leqslant n + 1$ with $|x_i| < r$, then either $t - x_i \leqslant c$ or $t - x_i \geqslant d$, since X^{+n} does not meet (c, d). In both cases $t \notin (e, f)$.

We have just shown that if $t \in X^{+n+1} \cap (e, f)$, then $t \in (X \cap [r, 1])^{+n+1}$, or stating it differently, $X^{+n+1} \cap (e, f) = (X \cap [r, 1])^{+n+1} \cap (e, f)$. Since the set $X \cap [r, 1]$

is additively nowhere dense, we get $X^{+n+1} \cap (e,f)$ nowhere dense, hence there are some e_1, f_1 with $e \leqslant e_1 < f_1 \leqslant f$ such that $(e_1, f_1) \cap X^{+n+1}$ is empty. As (a,b) was arbitrary, this shows that X^{+n+1} is nowhere dense. $\square$

Fact 3. *Let $M, X \subseteq [0,1]$. If the set X is compact and additively nowhere dense and if every neighborhood of a set X contains all but finitely many points of M, then $X \cup M$ is compact and additively nowhere dense.*

Proof. The compactness of $X \cup M$ is trivial: If $\mathscr{U}$ is an open cover of $X \cup M$, then some finite $\mathscr{V} \subseteq \mathscr{U}$ covers X. What remains still uncovered is, by the assumption, a finite subset of M.

The set $X \cup M$ is nowhere dense: Indeed, if $a < b$ are arbitrary real numbers, then there are some real c, d with $a < c < d < b$ such that the closed interval $[c,d]$ is disjoint from X. Since $\mathbb{R} \setminus [c,d]$ is an open neighborhood of the set X, the intersection $[c,d] \cap (X \cup M)$ is finite. So there must be a nondegenerate open interval (c_1, d_1) contained in (c,d) and disjoint from $X \cup M$.

Being a union of two nowhere dense sets, the set $(X \cup M)^{+0}$ is nowhere dense. In order to show that $(X \cup M)$ is additively nowhere dense, proceed by induction: Let $n \in \omega$ and suppose that the sets $(X \cup M)^{+k}$ are nowhere dense for all $k < n+1$.

Since X is compact, for every open neighborhood U of a set X^{+n+1} there is an open neighborhood V of a set X such that $V^{+n+1} \subseteq U$. We leave the verification of this simple fact to the reader.

Let real numbers $a < b$ be arbitrary. By the assumption, X^{+n+1} is nowhere dense. Hence there are some $c, d \in \mathbb{R}$ with $a < c < d < b$ and such that the closed interval $[c,d]$ does not meet the set X^{+n+1}. So there is an open neighborhood $V \supseteq X$ with $V^{+n+1} \subseteq \mathbb{R} \setminus [c,d]$. Denote by F the finite set $(X \cup M) \setminus V$. If $x \in (X \cup M)^{+n+1} \setminus (\mathbb{R} \setminus [c,d]) = (X \cup M)^{+n+1} \cap [c,d]$, then $x = \sum\limits_{j<n+1} x_j$ with all $x_j \in X \cup M$ and at least one x_j must belong to the set F — otherwise $x \in V^{+n+1} \subseteq \mathbb{R} \setminus [c,d]$. From this observation we immediately get $[c,d] \cap (X \cup M)^{+n+1} \subseteq \bigcup\limits_{j<n+1} (X \cup M)^{+j} + F^{+n+1-j}$. Since F is finite and since $(X \cup M)^{+j}$ is nowhere dense for all $j < n+1$ by the inductive assumption, the set $\bigcup\limits_{j<n+1} (X \cup M)^{+j} + F^{+n+1-j}$ is a finite union of nowhere dense sets. Therefore there are reals c_1, d_1 with $c \leqslant c_1 < d_1 \leqslant d$ such that $(c_1, d_1) \cap (X \cup M)^{+n+1} = \emptyset$.

Since the open interval (a,b) was arbitrary, this shows that the set $(X \cup M)^{+n+1}$ is nowhere dense. Having completed the induction step, we conclude that the set $X \cup M$ is additively nowhere dense. $\square$

It is the right time now to make profit from these ridiculous facts.

Theorem 2. *The following are equivalent for a closed subset X of the unit interval* $[0,1]$*:*

(i) *The set X is additively nowhere dense;*

(ii) *there is a group convergence $\mathscr{C}$ on $\mathbb{Q}$ such that its categorical completion $(\widetilde{\mathbb{Q}}, \widetilde{\mathscr{C}})$ satisfies $\mathbb{Q} \cup X \subseteq \widetilde{\mathbb{Q}} \subsetneq \mathbb{R}$.*

P r o o f. The implication (ii)$\Longrightarrow$(i) is easier, so let us start with it. Suppose that X is not additively nowhere dense, let $\mathscr{C}$ be a group convergence on $\mathbb{Q}$ such that $\widetilde{\mathbb{Q}} \supseteq \mathbb{Q} \cup X$. Thus for every $x \in X$ there is a $\mathscr{C}$-Cauchy sequence $\langle x(n) : n \in \omega \rangle$ such that $\langle \langle x(n) : n \in \omega \rangle, x \rangle \in \widetilde{\mathscr{C}}$.

Since we assume that X is not additively nowhere dense, there is $k \in \omega$ and a nonempty open interval (a,b) such that $X^{+k+1} \cap (a,b)$ is dense in (a,b). Since we also assume that X is compact, we get $(a,b) \subseteq X^{+k+1}$. Therefore whenever $y \in (a,b)$, we may find $x_0, x_1, \ldots, x_k \in X$ such that $y = x_0 + x_1 + \ldots x_k$. As all sequences $\langle x_i(n) : n \in \omega \rangle$ are $\mathscr{C}$-Cauchy $(i < k+1)$, the sequence $\left\langle \sum\limits_{i<k+1} x_i(n) : n \in \omega \right\rangle$ is $\mathscr{C}$-Cauchy, too. So $y \in \widetilde{\mathbb{Q}}$, hence $\widetilde{\mathbb{Q}} \supseteq (a,b)$. But for every rational q, the constant sequence $\langle q : n \in \omega \rangle$ $\mathscr{C}$-converges to q and so $\widetilde{\mathbb{Q}} \supseteq (a,b) + \mathbb{Q}$, because $\mathscr{C}$ satisfies (L). However, (a,b) is nonempty open, so $(a,b) + \mathbb{Q} = \mathbb{R}$, which contradicts the sharp inclusion in (ii).

For the opposite implication suppose that $X \subseteq [0,1]$ is compact and additively nowhere dense.

Before proceeding further, let us introduce some notation. For $x \in [0,1]$, let $\boldsymbol{x} \in {}^{\omega}2$ be the function such that $x = \sum\limits_{j=0}^{\infty} \boldsymbol{x}(j) \cdot 2^{-j-1}$. In the ambiguous case, choose the $\boldsymbol{x}$ with a tail of 1's.

Let us define a set $A \subseteq {}^{\omega}\mathbb{Q} \times \{0\}$ as follows:

$$A = \left\{ \left\langle \left\langle \sum_{j=n}^{f(n)} \boldsymbol{x}(j) \cdot 2^{-j-1} : n \in \omega \right\rangle, 0 \right\rangle : x \in X, f \in \mathsf{Mon} \right\}.$$

Let $\mathscr{C}$ be the smallest group convergence on $\mathbb{Q}$ with $A \subseteq \mathscr{C}$. The reader has undoubtedly noticed that our definition of A was tailored as to ensure that for every $x = \sum\limits_{j=0}^{\infty} \boldsymbol{x}(j) \cdot 2^{-j-1} \in X$, the sequence of all partial sums

$$\left\langle \sum_{j=0}^{n} \boldsymbol{x}(j) \cdot 2^{-j-1} : n \in \omega \right\rangle$$

is $\mathscr{C}$-Cauchy. Hence, if $(\widetilde{\mathbb{Q}}, \widetilde{\mathscr{C}})$ is the categorical completion of $(\mathbb{Q}, \mathscr{C})$, we have $\mathbb{Q} \cup X \subseteq \widetilde{\mathbb{Q}}$.

In order to show that there are real numbers which *are not* limits of $\mathscr{C}$-Cauchy sequences, we shall use the lemma. The convergence $\mathscr{C}$ is obviously finer than the metric convergence on $\mathbb{Q}$. For the second assumption of Lemma, we need to find suitable compact sets S_k.

To this end, let Φ be a mapping from the powerset of $[0,1]$ into itself defined by

$$\Phi(Z) = \{0\} \cup \left\{ \sum_{j=n}^{\infty} \boldsymbol{x}(j) \cdot 2^{-j-1} : x \in Z, n \in \omega \right\}.$$

Observe that $\Phi(Z)$ may be also expressed as follows: Let Z_0=Z; then, if Z_n is known, let $Z_{n+1} = Z_n \cup ((Z_n \cap [2^{-n-1}, 2^{-n}]) - 2^{-n-1})$. Then $\Phi(Z) = \{0\} \cup \bigcup_{n \in \omega} Z_n$. From this we immediately get that if Z is compact, then $\Phi(Z)$ is compact, because for every $n \in \omega$, $\Phi(Z) \cap [2^{-n-1}, 1]$ is a finite union of compact sets. It is however also true that if Z is compact and additively nowhere dense, then $\Phi(Z)$ is additively nowhere dense. Indeed, every $\Phi(Z) \cap [2^{-n-1}, 1]$ is additively nowhere dense by Fact 1, hence Fact 2 applies.

Similarly as before, for a set $Y \subseteq [0,1]$, let

$$\Psi(Y) = Y \cup \left\{ \sum_{j=0}^{n} \boldsymbol{x}(j) \cdot 2^{-j-1} : x \in Y, n \in \omega \right\}.$$

Notice that whenever Y is compact and U is an open neighborhood of Y, then $\Psi(Y) \setminus U$ is finite. To see this, let U be an open set containing X. Since Y is compact, there is $\varepsilon > 0$ such that for all $r \notin U$ and $x \in Y$ we have $|r - x| > \varepsilon$. Choose $n \in \omega$ such that $2^{-n} < \varepsilon$. The set $M = \left\{ \sum_{i=0}^{n} \varphi(i) \cdot 2^{-i-1} : \varphi \in {}^{n+1}2 \right\}$ is finite and $M \cup U \supseteq \Psi(Y)$ — if not, then there is some $m > n$ and $\varphi \in^{m+1} 2$ with $\varphi(m) = 1$ and such that $r = \sum_{i=0}^{m} \varphi(i) \cdot 2^{-i-1} \in \Psi(Y) \setminus U$. For $x \in Y$ with $\boldsymbol{x} \restriction m+1 = \varphi$ we get $|r - x| < \varepsilon$, which contradicts our choice of ε.

Now we are able to define $S = \Psi(\Phi(X))$. We already know that if X is compact and additively nowhere dense, then so is $\Phi(X)$, and we have just verified that $\Psi(\Phi(X))$ satisfies the assumptions of Fact 3, so S is compact and additively nowhere dense. It remains to put $S_0 = S \cup \{0\} \cup -S$; by Fact 1, S_0 is compact and additively nowhere dense, too.

Knowing S_0, let $S_k = (S_0)^{+k}$ for $0 < k < \omega$. Observe that for an arbitrary $x \in X$, all sums $\sum_{j=n}^{n+k} \boldsymbol{x}(j) \cdot 2^{-j-1}$ with $n, k \in \omega$ belong to the set S_0. This immediately follows from the definition of $\Phi(X)$ and of $\Psi(\Phi(X))$.

Exactly in the same way as in the proof of Theorem 1, i.e., following the procedure $A \to \alpha(A) \to \beta(\alpha(A)) \to \ldots$, we can show that the assumptions of the lemma are satisfied with the choice of S_k as indicated. Now, by the lemma, if $\langle x(n) \colon n \in \omega \rangle$ is a $\mathscr{C}$-Cauchy sequence, then its metric limit belongs to the set $\mathbb{Q} - \bigcup_{k\in\omega} S_k$. Since all sets S_k are nowhere dense, the set $\mathbb{Q} - \bigcup_{k\in\omega} S_k$ is of the first category and hence every real $r \in \mathbb{R} \setminus (\mathbb{Q} - \bigcup_{k\in\omega} S_k)$ is not a limit of any $\mathscr{C}$-Cauchy sequence. Therefore $\widetilde{\mathbb{Q}} \subseteq \mathbb{Q} - \bigcup_{k\in\omega} S_k \subsetneqq \mathbb{R}$, which was to be proved. □

Remarks. It is clear that any compact countable subspace of reals is additively nowhere dense. For a perfect example of an additively nowhere dense set, consider e.g. the set $\left\{ \sum_{n\in\omega} f(n)\cdot 2^{-(n+1)!} \colon f \in {}^{\omega}2 \right\}$. We do not know any example of a compact subset of $\mathbb{R}$ with all finite sums $X+X+\ldots+X$ nowhere dense, but $X-X$ containing a non-degenerate interval. (There exists a compact $X \subset \mathbb{R}$ with $X+X$ nowhere dense, $X-X \supseteq [-1,1]$, see [CGM].)

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