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EXISTENCE AND UNIQUENESS OF (L, φ)-REPRESENTATIONS OF ALGEBRAS

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1. INTRODUCTION

Let $\operatorname{Con}(\mathbb{A})$ denote the set of all congruence relations on an algebra $\mathbb{A}$. The least and largest congruences of $\mathbb{A}$ are denoted by $0_{\mathbb{A}}$ and $1_{\mathbb{A}}$. (Occasionally, they are denoted simply by 0 and 1.)

Let $\langle \mathbb{A}_i : i \in I \rangle$ be a system of similar algebras, and let $\mathbb{B} = \prod(\mathbb{A}_i : i \in I)$ denote the direct product of the $\mathbb{A}_i$, $i \in I$.

A subalgebra $\mathbb{A}$ of $\mathbb{B}$ is called a weak direct product of $\mathbb{A}_i, i \in I$, if the following two conditions are satisfied:

$(A1)$ if $x, y \in \mathbb{A}$, then $\{i \in I : x(i) \neq y(i)\}$ is finite,

$(A2)$ if $x \in \mathbb{A}$, $y \in \mathbb{B}$ and if $\{i \in I : x(i) \neq y(i)\}$ is finite, then $y \in \mathbb{A}$ (see [2] or [4]).

A full subdirect product of the $\mathbb{A}_i$, $i \in I$ (see e.g. [2]), is a subalgebra $\mathbb{A}$ of $\mathbb{B}$ satisfying the following conditions:

$(B1)$ $\mathbb{A}$ is a subdirect product of $\mathbb{A}_i$, $i \in I$,

$(B2)$ for any $i \in I$ and $x, y \in \mathbb{A}$, the element $z \in \mathbb{B}$ defined by $z(i) = x(i)$ and $z(j) = y(j)$ for $j \neq i$ belongs to $\mathbb{A}$.

Let I be a nonvoid set. $P(I)$ and $F(I)$ denote the set of all subsets of I and the set of all finite subsets of I, respectively. We denote by $\mathbb{P}(I)$ the Boolean algebra $\langle P(I), \cap, \cup, ', \emptyset, I \rangle$. Now we introduce the following concept:

Definition 1. Let $\langle \mathbb{A}_i : i \in I \rangle$ be a system of similar algebras, L an ideal of $\mathbb{P}(I)$, and let φ be a binary relation on $\mathbb{B} = \prod(\mathbb{A}_i : i \in I)$. A subalgebra $\mathbb{A}$ of $\mathbb{B}$ is called an (L, φ)-product of algebras $\mathbb{A}_i$, $i \in I$, if it is a subdirect product of these algebras and if the following conditions hold:

$(C1)$ for every $x, y \in \mathbb{A}$, $\{i \in I : x(i) \neq y(i)\} \in L$,

$(C2)$ for any $i \in I$ and any $x, y \in \mathbb{A}$, if $\langle x, y\rangle \in \varphi$, then the element $z \in \mathbb{B}$ defined by

$$z(j) = \begin{cases} x(i) & \text{if} \quad j = i, \\ y(j) & \text{if} \quad j \neq i \end{cases}$$

belongs to $\mathbb{A}$.

We write $\mathbb{A} = \prod\limits_{(L,\varphi)} (\mathbb{A}_i : i \in I)$ to denote that $\mathbb{A}$ is an (L, φ)-product of $\mathbb{A}_i$, $i \in I$.

Let $\mathbb{A}$ be a subalgebra of the direct product $\mathbb{B} = \prod(\mathbb{A}_i : i \in I)$ and let L be an ideal of $\mathbb{P}(I)$. We say that $\mathbb{A}$ is an L-restricted subdirect product (cf. [5], p. 92) of the $\mathbb{A}_i$, if $\mathbb{A}$ satisfies conditions $(B1)$ and $(C1)$, i.e., if $\mathbb{A} = \prod\limits_{(L,0_{\mathbb{B}})} (\mathbb{A}_i : i \in I)$. In particular, if $L = P(I)$, then an $(L, 0_{\mathbb{B}})$-product is a subdirect product.

A is a full subdirect product iff $\mathbb{A} = \prod\limits_{(P(I),1_{\mathbb{B}})} (\mathbb{A}_i : i \in I)$. Finally, a weak direct product of $\mathbb{A}_i$ $(i \in I)$ is an $(F(I), 1_{\mathbb{B}})$-product of these algebras.

2. PRELIMINARIES ON C-DECOMPOSITIONS IN LATTICES

Let $\mathbb{L}$ be a complete lattice. Lattice join, meet, inclusion and proper inclusion are denoted respectively by the symbols $\vee, \wedge, \leqslant$ and $<$. Let 0 be the least element and 1 the greatest element of $\mathbb{L}$. By $[a, b]$ $(a \leqslant b,\ a, b \in \mathbb{L})$ we denote an interval that is the set of all $c \in \mathbb{L}$ for which $a \leqslant c \leqslant b$.

A subset M in $\mathbb{L}$ is called join irredundant iff for all proper subsets M' of M we have $\bigvee M' < \bigvee M$. Meet irredundance is the dual notion.

We write $a \prec b$ $(a, b \in \mathbb{L})$ if $[a, b]$ is a two-element set. An element $a \in \mathbb{L}$ is an atom (coatom) if $0 \prec a$ $(a \prec 1)$. We call a lattice $\mathbb{L}$ atomic iff for every $a \in \mathbb{L}$, $a \neq 0$, there is an atom $p \leqslant a$.

An element $a \in \mathbb{L}$ is called compact iff for all $X \subseteq \mathbb{L}$, if $a \leqslant \bigvee X$, then $a \leqslant \bigvee Y$ for a finite $Y \subseteq X$. $\mathbb{L}$ is said to be algebraic (or: compactly generated) iff each of its elements is a join of compact elements. Define a complete lattice $\mathbb{L}$ to be upper continuous if for every $a \in \mathbb{L}$ and every chain C in $\mathbb{L}$, $a \wedge \bigvee C = \bigvee(a \wedge x : x \in C)$. The lattice $\mathbb{L}$ is lower continuous if its dual lattice is upper continuous. It can be shown that every algebraic lattice is upper continuous (see [1], Theorem 2.3).

Recall that a lattice $\mathbb{L}$ is modular if, for all $a, b, c \in \mathbb{L}$, $c \leqslant a$ implies $a \wedge (b \vee c) = (a \wedge b) \vee c$.

Let c be a distributive element of $\mathbb{L}$. Then c satisfies the following condition:

$$c \vee (x \wedge y) = (c \vee x) \wedge (c \vee y) \quad \text{for all} \quad x, y \in \mathbb{L}.$$

By Theorem III.2.2 in [3] the binary relation θ_c on $\mathbb{L}$ defined by

$$\langle x, y\rangle \in \theta_c \quad \text{iff} \quad x \vee c = y \vee c$$

is a congruence relation. Obviously, θ_c has the property that the congruence class containing zero is a principal ideal, i.e., θ_c satisfies condition $(*)$ of Lemma 4[9].

A subset T of $\mathbb{L}$ is said to be c-independent (or: θ_c-independent in the terminology of the papers [8] and [9]) if T is join irredundant and for every $t \in T$,

$$t \wedge \bigvee(T - \{t\}) \leqslant c.$$

If $a \in \mathbb{L}$ and $T = \{t_i \colon i \in I\} \subseteq \mathbb{L}$, then we say that a is a c-join (or: θ_c-join in [8]), and we write

$$a = \sum_c T \quad \text{or} \quad a = \sum_c (t_i \colon i \in I)$$

it T is c-independent and $a = \bigvee T$. The c-join of finitely many elements $t_1, \ldots, t_n$ is also written as $t_1 +_c \ldots +_c t_n$. An element $a \in \mathbb{L}$ $(a \neq 0)$ is said to be c-indecomposable if it cannot be represented as a c-join of two elements of $\mathbb{L}$.

In the sequel we will need

Lemma 1. (cf. [9], Theorem 3). *Let $\mathbb{L}$ be an upper continuous modular lattice and let c be a distributive element of $\mathbb{L}$. If*

$$1 = \sum_c (a_i \colon i \in I) = \sum_c (b_j \colon j \in J)$$

are two c-decompositions of 1 such that each $[0, a_i]$ and each $[0, b_j]$ is of finite length and a_i, b_j are c-indecomposable, then there exists a bijection λ of I onto J such that, for each $i \in I$,

$$1 = a_i +_c \sum_c (b_j \colon j \neq \lambda(i)).$$

3. (L, φ)-REPRESENTATIONS OF ALGEBRAS

Definition 2. Let $\mathbb{A}_i (i \in I)$ and $\mathbb{A}$ be similar algebras, φ a binary relation on $\mathbb{A}$, and let L be an ideal of the Boolean algebra $\mathbb{P}(I)$. Let f be an embedding from $\mathbb{A}$ into $\prod(\mathbb{A}_i \colon i \in I)$. The ordered pair $\langle(\mathbb{A}_i \colon i \in I), f\rangle$ is called an (L, φ)-representation of $\mathbb{A}$ iff $f(\mathbb{A}) = \prod\limits_{(L, f(\varphi))} (\mathbb{A}_i \colon i \in I)$.

For each $i \in I$, we denote by p_i the ith projection function from $\prod(\mathbb{A}_i : i \in I)$ onto $\mathbb{A}_i$. The mapping $f_i = p_i \circ f$ which is a homomorphism of $\mathbb{A}$ onto $\mathbb{A}_i$ will be referred to as the i th f-projection.

An (L,φ)-representation $\langle(\mathbb{A}_i : i \in I), f\rangle$ of $\mathbb{A}$ is called

(i) subdirect, if $L = P(I)$ and $\varphi = 0_{\mathbb{A}}$,

(ii) finitely restricted subdirect, if $L = F(I)$ and $\varphi = 0_{\mathbb{A}}$,

(iii) full subdirect, if $L = P(I)$ and $\varphi = 1_{\mathbb{A}}$,

(iv) weak direct, if $L = F(I)$ and $\varphi = 1_{\mathbb{A}}$.

We shall now correlate (L,φ)-representations of an algebra $\mathbb{A}$ with congruence relations on $\mathbb{A}$. Let θ_i $(i \in I)$ be congruences of $\mathbb{A}$. For any set $M \subseteq I$ we define

$$\theta(M) = \bigwedge(\theta_j : j \in I - M).$$

We will use the notion $\bar{\theta}_i$ for $\theta(\{i\})$, $i \in I$.

The next result characterizes (L,φ)-representations internally.

Theorem 1. *Let $\mathbb{A}$ be an algebra, $\varphi \subseteq \mathbb{A}^2$, and let $\langle\theta_i : i \in I\rangle$ be a system of congruences on $\mathbb{A}$. Let L be an ideal of $\mathbb{P}(I)$. We put $\mathbb{A}_i = \mathbb{A}/\theta_i$ for $i \in I$ and define a mapping $f : \mathbb{A} \longrightarrow \prod(\mathbb{A}_i : i \in I)$ by setting $f(x) = \langle x/\theta_i : i \in I\rangle (x/\theta_i$ is the congruence class containing $x)$. Then $\langle(\mathbb{A}_i : i \in I), f\rangle$ is an (L,φ)-representation of $\mathbb{A}$ iff the following conditions hold:*

(a) $0_{\mathbb{A}} = \bigwedge(\theta_i : i \in I)$,

(b) $1_{\mathbb{A}} = \bigvee(\theta(M) : M \in L)$,

(c) for all $i \in I$, $\varphi \subseteq \theta_i \circ \bar{\theta}_i$ ($\theta_i \circ \bar{\theta}_i$ denotes the relational product of congruences θ_i and $\bar{\theta}_i$).

Proof. Necessity. Since the mapping f is one-to-one we conclude that (a) is satisfied. To prove (b), let $x, y \in \mathbb{A}$. Clearly, $M = \{i \in I : f_i(x) \neq f_i(y)\} \in L$ and $\langle x, y\rangle \in \theta(M)$. Then $\langle x, y\rangle \in \bigvee(\theta(M) : M \in L)$ and hence (b) holds.

Moreover, (c) immediately follows from (C2).

Sufficiency. It is obvious that f is an embedding and that $\bar{\mathbb{A}} = f(\mathbb{A})$ is a subdirect product of algebras $\mathbb{A}_i$, $i \in I$. Let $x, y \in \mathbb{A}$. Now we prove that

$$\{i \in I : f_i(x) \neq f_i(y)\} \in L. \tag{1}$$

By condition (b), $\langle x, y\rangle \in \bigvee(\theta(M) : M \in L)$. So there are finitely many sets $M_1, \ldots, M_n \in L$ such that $\langle x, y\rangle \in \theta(M_1) \vee \ldots \vee \theta(M_n)$. It is easy to see that

$$\{i \in I : f_i(x) \neq f_i(y)\} \subseteq M_1 \cup \ldots \cup M_n.$$

From this by the definition of an ideal we deduce that (1) is satisfied. Now let i be an element of I and let $\bar{x}, \bar{y} \in \bar{\mathbb{A}}$ be such that $\langle \bar{x}, \bar{y} \rangle \in \psi = f(\varphi)$. By (c), the element $\bar{z}$ defined by $\bar{z}(i) = \bar{x}(i)$ and $\bar{z}(j) = \bar{y}(j)$ for $j \neq i$ belongs to $\bar{\mathbb{A}}$. Therefore, $\bar{\mathbb{A}} = \prod_{(L,\psi)} (\mathbb{A}_i : i \in I)$, which was to be proved. □

Let $\langle \theta_i : i \in I \rangle \in (\mathrm{Con}(\mathbb{A}))^I$. Denote by f_θ the function from $\mathbb{A}$ to $\prod(\mathbb{A}/\theta_i : i \in I)$ defined by the rule $f_\theta(x) = \langle x/\theta_i : i \in I \rangle$ $(x \in \mathbb{A})$. We know (see [9], Lemma 4) that $1_{\mathbb{A}} = \bigvee(\theta(M) : M \in P(I))$. Now, it is easy to see that Theorem 1 implies

Corollary 1. (see [2], Lemma 1.1, and [6], Lemma 11). *Let $\langle \theta_i : i \in I \rangle$ be a system of congruences on an algebra $\mathbb{A}$ such that $0_{\mathbb{A}} = \bigwedge(\theta_i : i \in I)$. Then*

(i) *$\langle (\mathbb{A}/\theta_i : i \in I), f_\theta \rangle$ is a finitely restricted subdirect representation of $\mathbb{A}$ iff $1_{\mathbb{A}} = \bigvee(\theta(M) : M \in F(I))$,*

(ii) *$\langle (\mathbb{A}/\theta_i : i \in I, f_\theta \rangle$ is a full subdirect representation of $\mathbb{A}$ iff $1_{\mathbb{A}} = \theta_i \circ \bar{\theta}_i$ for all $i \in I$,*

(iii) *$\langle (\mathbb{A}/\theta_i : i \in I, f_\theta \rangle$ is a weak direct representation of $\mathbb{A}$ iff $1_{\mathbb{A}} = \bigvee(\theta(M) : M \in F(I))$ and $1_{\mathbb{A}} = \theta_i \circ \bar{\theta}_i$ for all $i \in I$.*

4. φ-IRREDUCIBLE CONGRUENCE RELATIONS: SOME LEMMAS

Let $\langle \theta_i : i \in I \rangle$ be system of congruences on an algebra $\mathbb{A}$, $\varphi \subseteq \mathbb{A}^2$, and let L be an ideal of $\mathbb{P}(I)$. For $\alpha \in \mathrm{Con}(\mathbb{A})$, we write

$$\alpha = \prod_{(L,\varphi)} (\theta_i : i \in I)$$

iff $\alpha = \bigwedge(\theta_i : i \in I)$ and the conditions (b) and (c) of Theorem 1 are satisfied. If $L = P(I)$, we will write $\prod_{\varphi}(\theta_i : i \in I)$ for $\prod_{(L,\varphi)} (\theta_i : i \in I)$. In this case, if $I = \{1, \ldots, n\}$, we will write $\alpha = \theta_1 \times_\varphi \ldots \times_\varphi \theta_n$.

Definition 3. Let φ be a binary relation on an algebra $\mathbb{A}$. An element $\alpha \in \mathrm{Con}(\mathbb{A})$ is called φ-irreducible iff $\alpha \neq 1$ and if $\alpha = \theta_1 \times_\varphi \theta_2$, then $\alpha = \theta_1$ or $\alpha = \theta_2$.

Lemma 2. *Let $\alpha \in \mathrm{Con}(\mathbb{A})$.*

(i) *α is 0-irreducible iff α is a meet irreducible element of* Con($\mathbb{A}$) *(i.e., α satisfies the conditions $\alpha \neq 1$ and $\alpha = \theta_1 \wedge \theta_2$ implies $\alpha = \theta_1$ or $\alpha = \theta_2$).*

(ii) *α is 1-irreducible iff $\alpha \neq 1$ and for any $\theta_1, \theta_2 \in$* Con($\mathbb{A}$)*, if $\alpha = \theta_1 \times_1 \theta_2$, then $\theta_1 = 1$ or $\theta_2 = 1$ (i.e., α is indecomposable, see* [7]*, p. 269).*

Lemma 3. *Let φ be a dually distributive element of* $\mathrm{Con}(\mathbb{A})$, *and let $\{\theta_i : i \in I\}$ be a meet irredundant subset of $Con(\mathbb{A})$. If $0_{\mathbb{A}} = \prod_{\varphi}(\theta_i : i \in I)$, then $1_{\mathbb{A}} = \sum_{\varphi}(\theta_i : i \in I)$ (in the dual* $\mathrm{Con}(\mathbb{A})$*).*

Proof. Let $\mathbb{L}$ be the dual of $\mathrm{Con}(\mathbb{A})$. The congruence φ is distributive in $\mathbb{L}$ and $\{\theta_i : i \in I\}$ is a join irredundant subset of $\mathbb{L}$. Since $0_{\mathbb{A}} = \prod_{\varphi}(\theta_i : i \in I)$, we conclude that $0_{\mathbb{A}} = \bigwedge(\theta_i : i \in I)$ and $\varphi \leqslant \theta_i \vee \bigwedge(\theta_j : j \neq i)$ for each $i \in I$. In other words, $1_{\mathbb{A}} = \bigvee(\theta_i : i \in I)$ and $\theta_i \wedge \bigvee(\theta_j : j \neq i) \leqslant \varphi$ in $\mathbb{L}$ for all $i \in I$. Therefore, $1_{\mathbb{A}} = \sum_{\varphi}(\theta_i : i \in I)$. □

Let $\varphi \in \mathrm{Con}(\mathbb{A})$. We say that the congruences of an algebra $\mathbb{A}$ φ-permute iff $\alpha \wedge \varphi$ and $\beta \wedge \varphi$ permute for every $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$.

It is obvious that for every algebra $\mathbb{A}$ the congruences of $\mathbb{A}$ $0_{\mathbb{A}}$-permute and that $1_{\mathbb{A}}$-permuting is the same thing as permuting.

Lemma 4. *Let φ be a dually distributive element of* $\mathrm{Con}(\mathbb{A})$. *Suppose that congruences of $\mathbb{A}$ φ-permute and denote by $\mathbb{L}$ the dual lattice of* $\mathrm{Con}(\mathbb{A})$. *Then*

(i) *for a congruence relation α, if $\alpha = \theta_1 +_{\varphi} \theta_2$ (in $\mathbb{L}$), then $\alpha = \theta_1 \times_{\varphi} \theta_2$ in* $\mathrm{Con}(\mathbb{A})$;

(ii) *if $\alpha \in$* $\mathrm{Con}(\mathbb{A})$ *is φ-irreducible, then it is φ-indecomposable in $\mathbb{L}$.*

Proof. Let $\alpha = \theta_1 +_{\varphi} \theta_2$. Therefore, $\alpha = \theta_1 \vee \theta_2$ and $\theta_1 \wedge \theta_2 \leqslant \varphi$ in $\mathbb{L}$. In other words, $\alpha = \theta_1 \wedge \theta_2$ and $\varphi \leqslant \theta_1 \vee \theta_2$ in $\mathrm{Con}(\mathbb{A})$. Then $\varphi = \varphi \wedge (\theta_1 \vee \theta_2)$ and since φ is dually distributive in $\mathrm{Con}(\mathbb{A})$,

$$\varphi = (\varphi \wedge \theta_1) \vee (\varphi \wedge \theta_2).$$

Hence we have $\varphi = (\varphi \wedge \theta_1) \circ (\varphi \wedge \theta_2)$, because congruences $\varphi \wedge \theta_1$ and $\varphi \wedge \theta_2$ permute. Consequently, $\varphi \subseteq \theta_1 \circ \theta_2$ and therefore, $\alpha = \theta_1 \times_{\varphi} \theta_2$.

The second statement follows immediately from (i). □

Let $\varphi \in \mathrm{Con}(\mathbb{A})$. Congruences α and β on $\mathbb{A}$ are said to be φ-isotopic, written $\alpha \sim \varphi\beta$, iff $0 = \alpha \times_{\varphi} \gamma = \beta \times_{\varphi} \gamma$ for some $\gamma \in \mathrm{Con}(\mathbb{A})$ with $\gamma \neq 0$.

As a preparation, we need two lemmas:

Lemma 5. (cf. [11], Lemma 6). *Let an algebra $\mathbb{A}$ have a one-element subalgebra and let α, β be congruences on $\mathbb{A}$ such that $\alpha \sim_1 \beta$. Then $\mathbb{A}/\alpha \cong \mathbb{A}/\beta$.*

Lemma 6. (cf. [11], Lemma 7). *Let the congruence lattice of an algebra $\mathbb{A}$ be distributive. Let α and β be meet irreducible elements of* $\mathrm{Con}(\mathbb{A})$. *If $\alpha \sim_0 \beta$, the $\alpha = \beta$.*

A congruence $\alpha \in \mathrm{Con}(\mathbb{A})$ is called a decomposition congruence iff there is $\beta \in \mathrm{Con}(\mathbb{A})$ such that $\alpha \wedge \beta = 0_{\mathbb{A}}$ and $\alpha \circ \beta = 1_{\mathbb{A}}$. $\mathrm{DCon}(\mathbb{A})$ denotes the set of all decomposition congruences of $\mathbb{A}$.

Lemma 7. *Let $\mathbb{A}$ be an algebra such that $\mathrm{DCon}(\mathbb{A})$ is a sublattice of $\mathrm{Con}(\mathbb{A})$. If θ is a coatom of DCon($\mathbb{A}$), then $\mathbb{A}/\theta$ is directly indecomposable.*

Proof. Suppose on the contrary that there exist two congruences α, β such that $\theta < \alpha$, $\beta < 1_{\mathbb{A}}$, $\alpha \circ \beta = 1_{\mathbb{A}}$ and $\alpha \wedge \beta = \theta$. Let θ' be a congruence ssatisfying $0_{\mathbb{A}} = \theta \wedge \theta'$ and $1_{\mathbb{A}} = \theta \circ \theta'$. Obviously,

$$\alpha \wedge (\beta \wedge \theta') = 0_{\mathbb{A}} \quad \text{and} \quad \alpha \circ (\beta \wedge \theta') = 1_{\mathbb{A}}.$$

Therefore, $\alpha \in \mathrm{DCon}(\mathbb{A})$, contradicting the fact that θ is a coatom of $\mathrm{DCon}(\mathbb{A})$. Then $\mathbb{A}/\theta$ is directly indecomposable. □

Definition 4. Let $\mathbb{A}$ be an algebra and φ a binary relation on $\mathbb{A}$. Let I be a nonvoid set and L an ideal of $\mathbb{P}(I)$. An (L,φ)-representation $\langle(\mathbb{A}_i : i \in I), f\rangle$ of $\mathbb{A}$ is called irredundant iff the set $\{\ker(f_i) : i \in I\}$ is meet irredundant (in $\mathrm{Con}(\mathbb{A})$), where $\ker(f_i)$ is the kernel of the ith f-projection f_i.

It is easy to see that the following lemma holds.

Lemma 8. *If $\langle(\mathbb{A}_i : i \in I), f\rangle$ is an $(L, 1_{\mathbb{A}})$-representation of $\mathbb{A}$ with $|\mathbb{A}_i| > 1$ for each $i \in I$, then this representation of $\mathbb{A}$ is irredundant.*

We call a sublattice of a complete lattice $\bigvee$-closed whenever it is closed under arbitrary joins.

The existence result is given in the following theorem.

Theorem 2. *Let φ be a dually distributive element of $\mathrm{Con}(\mathbb{A})$. Suppose that the congruences of $\mathbb{A}$ φ-permute and $\mathrm{DCon}(\mathbb{A})$ is a modular $\bigvee$-closed sublattice of $\mathrm{Con}(\mathbb{A})$. Then there is a system $\langle \mathbb{A}_i : i \in I\rangle$ of directly indecomposable algebras and an embedding $f : \mathbb{A} \longrightarrow \prod(\mathbb{A}_i : i \in I)$ such that $\langle(\mathbb{A}_i : i \in I), f\rangle$ is an irredundant (L,φ)-representation of $\mathbb{A}$, where L is an ideal of $\mathbb{P}(I)$ containing all finite subsets of I.*

Proof. It follows from the proof of Lemma 4.3 [1] that $\mathrm{DCon}(\mathbb{A})$ is atomic. Let Γ be the set of all atoms of $\mathrm{DCon}(\mathbb{A})$ and let $\{\alpha_i : i \in I\}$ be a maximal subset of Γ such that $\alpha_i \wedge \bigvee(\alpha_j : j \in I - \{i\}) = 0_{\mathbb{A}}$ for all $i \in I$. (The existence of such a maximal subset of Γ follows easily by Zorn's Lemma). For $i \in I$, we set $\theta_i = \bigvee(\alpha_j \colon j \neq i)$ and $\bar{\theta}_i = \bigwedge(\theta_j \colon j \neq i)$. Applying Theorem 4.3 of [1] we conclude that every element of $\mathrm{DCon}(\mathbb{A})$ is a join of atoms. Furthermore, we know that every atom of an upper continuous lattice is compact (see [1], p. 15). Then $\mathrm{DCon}(\mathbb{A})$ is an algebraic lattice. Now, by Theorem 6.6 of [1] we deduce that

$$0_{\mathbb{A}} = \bigwedge(\theta_i \colon i \in I).$$

From Theorem 6.5 of [1] it follows that

$$1_{\mathbb{A}} = \bigvee(\alpha_i \colon i \in I).$$

Let L be an ideal of $\mathbb{P}(I)$ containing all finite subsets of I. Since $\alpha_i \leqslant \bar{\theta}_i$ for all $i \in I$, we obtain

$$1_{\mathbb{A}} \leqslant \bigvee(\bar{\theta}_i \colon i \in I) = \bigvee(\theta(\{i\}) \colon i \in I) \leqslant \bigvee(\theta(M) \colon M \in L).$$

Hence $1_{\mathbb{A}} = \bigvee(\theta(M) \colon M \in L)$, and therefore the condition (b) of Theorem 1 is satisfied. Let i be an element of I. Obviously we have $1_{\mathbb{A}} = \theta_i \vee \alpha_i \leqslant \theta_i \vee \bar{\theta}_i$. Since φ is dually distributive and the congruences of $\mathbb{A}$ φ-permute, we get

$$\varphi = \varphi \wedge (\theta_i \vee \bar{\theta}_i) = (\varphi \wedge \theta_i) \vee (\varphi \wedge \bar{\theta}_i) = (\varphi \wedge \theta_i) \circ (\varphi \wedge \bar{\theta}_i).$$

From this we conclude that $\varphi \subseteq \theta_i \circ \bar{\theta}_i$, i.e., (c) holds. Thus the system $\langle \theta_i \colon i \in I \rangle$ of congruences on $\mathbb{A}$ satisfies conditions (a), (b), and (c). By Theorem 1, $\langle (\mathbb{A}/\theta_i \colon i \in I), f_\theta \rangle$ is an (L, φ)-representation of $\mathbb{A}$. This representation of $\mathbb{A}$ is irredundant, because the set $\{\theta_i \colon i \in I\}$ is meet irredundant. Since θ_i is a coatom of $\mathrm{DCon}(\mathbb{A})$, Lemma 7 implies that every $\mathbb{A}/\theta_i$ is directly indecomposable. This completes the proof of Theorem 2. $\square$

It is well known that every algebra whose congruences permute has a modular congruence lattice. Therefore, as a consequence of Theorem 2 we get the following

Corollary 2. (see [5], Theorem 4.5). *Let $\mathbb{A}$ be any algebra whose congruences permute and whose decompositon congruences form a $\bigvee$-closed sublattice of* $\mathrm{Con}(\mathbb{A})$. *Then $\mathbb{A}$ is isomorphic to a weak direct product of directly indecomposable algebras.*

We also have

Corollary 3. (see [5], Theorem 4.2). *Let $\mathbb{A}$ be an algebra such that* $\mathrm{DCon}(\mathbb{A})$ *is a modular $\bigvee$-closed sublattice of* $\mathrm{Con}(\mathbb{A})$. *Then there exists a system $\langle \mathbb{A}_i \colon i \in I \rangle$ of directly indecomposable algebras and an embedding*

$f\colon \mathbb{A} \longrightarrow \prod(\mathbb{A}_i\colon i \in I)$ *such that* $\langle(\mathbb{A}_i\colon i \in I), f\rangle$ *is an irredundant finitely restricted subdirect representation of* $\mathbb{A}$.

6. A uniqueness theorem

Let $\mathbb{A}$ be an algebra and φ a congruence relation on $\mathbb{A}$. For two algebras $\mathbb{B}$ and $\mathbb{C}$ we write $\mathbb{B} \sim_\varphi \mathbb{C}$ iff there exist φ-isotopic congruences β and γ on $\mathbb{A}$ such that $\mathbb{B} \cong \mathbb{A}/\beta$ and $\mathbb{C} \cong \mathbb{A}/\gamma$.

Remark 1. By Lemma 5 we conclude that if an algebra $\mathbb{A}$ has a one-element subalgebra and if $\mathbb{B} \sim 1_{\mathbb{A}}\mathbb{C}$, then $\mathbb{B} \cong \mathbb{C}$.

Remark 2. Lemma 6 implies that if $\mathrm{Con}(\mathbb{A})$ is a distributive lattice and if $\mathbb{B} \sim 0_{\mathbb{A}}\mathbb{C}$, then $\mathbb{B} \cong \mathbb{C}$.

Now we present our uniqueness theorem.

Theorem 3. *Let* $\mathbb{A}$ *be any algebra,* φ *a dual distributive element of* $\mathrm{Con}(\mathbb{A})$. *Suppose the congruences on* $\mathbb{A}$ φ*-permute and the lattice* $\mathrm{Con}(\mathbb{A})$ *is modular and lower continuous. Let* $\{\alpha_i\colon i \in I\}$ *and* $\{\beta_j\colon j \in J\}$ *be two sets of* φ*-irreducible congruences on* $\mathbb{A}$, *and let* L_1, L_2 *be ideals of the Boolean algebras* $\mathbb{P}(I)$, $\mathbb{P}(J)$, *respectively. Assume that* $\langle(\mathbb{A}_i\colon i \in I), f\rangle$ *is an irredundant* (L_1, φ)*-representation of* $\mathbb{A}$ *with* $\ker(f_i) = \alpha_i$, *and* $\langle(\mathbb{B}_j\colon j \in J), g\rangle$ *is an irredundant* (L_2, φ)*-representation of* $\mathbb{A}$ *with* $\ker(g_j) = \beta_j$. *If the intervals* $[\alpha_i, 1]$ *and* $[\beta_j, 1]$ $(i \in I,\ j \in J)$ *in* $\mathrm{Con}(\mathbb{A})$ *are of finite length, then there is a bijection* $\lambda\colon I \longrightarrow J$ *such that, for all* $i \in I$, $\mathbb{A}_i \sim_\varphi \mathbb{B}_{\lambda(i)}$.

Proof. Let $\mathbb{L}$ be the dual of $\mathrm{Con}(\mathbb{A})$. By assumption, $\mathbb{L}$ is modular and upper continuous. From Theorem 1 it follows that

$$0 = \prod_{(L_1,\varphi)} (\alpha_i\colon i \in I) = \prod_{(L_2,\varphi)} (\beta_j\colon j \in J).$$

Hence

$$0 = \prod_{\varphi}(\alpha_i\colon i \in I) = \prod_{\varphi}(\beta_j\colon j \in J). \tag{2}$$

Moreover, $\{\alpha_i\colon i \in I\}$ and $\{\beta_j\colon j \in J\}$ are meet irredundant subsets of $\mathrm{Con}(\mathbb{A})$. By Lemma 3,

$$1 = \sum_{\varphi}(\alpha_i\colon i \in I) = \sum_{\varphi}(\beta_j\colon j \in J) \tag{3}$$

in $\mathbb{L}$, and by Lemma 4 (ii) we know that each α_i and β_j are φ-indecomposable. Obviously, the intervals $[0, \alpha_i]$ and $[0, \beta_j]$ contained in $\mathbb{L}$ are of finite lengths. Applying Lemma 1 for two φ-decompositions (3) we conclude that there is a bijection $\lambda\colon I \longrightarrow J$ such that, for each $i \in I$,

$$1 = \alpha_i +_\varphi \sum_\varphi (\beta_j : j \neq \lambda(i)).$$

Hence $1 = \alpha_i +_\varphi \bigvee(\beta_j : j \neq \lambda(i))$ and using Lemma 4(i) we get

$$0 = \alpha_i \times_\varphi \bigwedge(\beta_j : j \neq \lambda(i)) \tag{4}$$

in $\mathrm{Con}(\mathbb{A})$. From (2) we infer, in particular, that

$$0 = \beta_{\lambda(i)} \times_\varphi \bigwedge(\beta_j : j \neq \lambda(i)). \tag{5}$$

By (4) and (5) we obtain that

$$\alpha_i \sim_\varphi \beta_{\lambda(i)} \tag{6}$$

for all $i \in I$. Since $\mathbb{A}_i \cong \mathbb{A}/\alpha_i$ and $\mathbb{B}_j \cong \mathbb{A}/\beta_j$, it follows from (6) that $\mathbb{A}_i \sim_\varphi \mathbb{B}_{\lambda(i)}$ $\square$

Proposition 1. *Let $\mathbb{A}$ have permuting congruences. Suppose that $\mathbb{A}$ has a one-element subalgebra and $\mathrm{Con}(\mathbb{A})$ is lower continuous. Let L_1, L_2 be ideals of the Boolean algebras $\mathbb{P}(I)$, $\mathbb{P}(J)$, respectively. Let $\langle(\mathbb{A}_i : i \in I), f\rangle$ be an $(L_1, 1)$-representation of $\mathbb{A}$ and let $\langle(\mathbb{B}_j : j \in J), g\rangle$ be an $(L_2, 1)$-representation of $\mathbb{A}$. If factors $\mathbb{A}_i, \mathbb{B}_j$ are directly indecomposable and intervals $[\ker(f_i), 1]$ and $[\ker(g_j), 1]$ in $\mathrm{Con}(\mathbb{A})$ are of finite lengths, then there is a bijection $\lambda\colon I \longrightarrow J$ such that $\mathbb{A}_i \cong \mathbb{B}_{\lambda(i)}$ for each $i \in I$.*

P r o o f. Since $\mathbb{A}_i \cong \mathbb{A}/\alpha_i$ and $\mathbb{B}_j \cong \mathbb{A}/\beta_j$ are directly indecomposable, α_i and β_j are indecomposable (see [7], Lemma 2). Hence Lemma 2 implies that each α_i and β_j are 1-irreducible. By Lemma 8, the representations $\langle(\mathbb{A}_i : i \in I), f\rangle$ and $\langle(\mathbb{B}_j : j \in J), g\rangle$ of $\mathbb{A}$ are irredundant. Thus the assumptions of Theorem 3. are satisfied, and therefore, there is a bijection $\lambda\colon I \longrightarrow J$ such that $\mathbb{A}_i \sim_1 \mathbb{B}_{\lambda(i)}$ for each $i \in I$. From this together with Remark 1 we deduce that $\mathbb{A}_i \cong \mathbb{B}_{\lambda(i)}$. $\square$

By Proposition 1 we obtain

Corollary 4. *Let $\mathbb{A}$ be any algebra whose congruences permute and whose congruence lattice is lower continuous. Suppose that $\mathbb{A}$ has a one-element subalgebra.*

If $\langle(\mathbb{A}_i\colon i \in I), f\rangle$ and $\langle(\mathbb{B}_j\colon j \in J), g\rangle$ are two weak direct representations (in particular: full subdirect representations) of $\mathbb{A}$ with all factors directly indecomposable and such that the intervals $[\ker(f_i), 1]$ and $[\ker(g_j), 1]$ in Con($\mathbb{A}$) are of finite lengths, then there is a bijection $\lambda\colon I \longrightarrow J$ such that $\mathbb{A}_i \cong \mathbb{B}_{\lambda(i)}$ for each $i \in I$.

In particular, we have

Corollary 5. (see [7], Theorem 5.3). *If $\mathbb{A}$ has permuting congruences,* Con($\mathbb{A}$) *is of finite length, and $\mathbb{A}$ has a one-element subalgebra, then for every two weak direct representations (direct representations) $\langle(\mathbb{A}_1, \ldots, \mathbb{A}_m), f\rangle$ and $\langle(\mathbb{B}_1, \ldots, \mathbb{B}_n), g\rangle$ of $\mathbb{A}$ with directly indecomposable factors we have $m = n$ and, after renumbering, $\mathbb{A}_i \cong \mathbb{B}_i$ for $1 \leqslant i \leqslant n$.*

From Theorem 3 we also obtain

Proposition 2. *Assume that $\mathbb{A}$ is an algebra whose congruence lattice is distributive and lower continuous. Let $\{\alpha_i\colon i \in I\}$ and $\{\beta_j\colon j \in J\}$ be two sets of congruences on $\mathbb{A}$ such that the intervals $[\alpha_i, 1]$ and $[\beta_j, 1]$ in* Con($\mathbb{A}$) *are of finite lengths. Let L_1, L_2 be ideals of $\mathbb{P}(I)$, $\mathbb{P}(J)$, respectively. If $\mathbb{A}$ has an irredundant $(L_1, 0)$-representation $\langle(\mathbb{A}_i\colon i \in I), f\rangle$ with* $\ker(f_i) = \alpha_i$*, and also has an irredundant $(L_2, 0)$-representation $\langle(\mathbb{B}_j\colon j \in J), g\rangle$ with* $\ker(g_j) = \beta_j$*, and if the factors $\mathbb{A}_i$, $\mathbb{B}_j$ are subdirectly irreducible, then there is a bijection $\lambda\colon I \longrightarrow J$ such that $\mathbb{A}_i \cong \mathbb{B}_{\lambda(i)}$ for all $i \in I$.*

Proof. Since $\mathbb{A}_i \cong \mathbb{A}/\alpha_i$ and $\mathbb{B}_j \cong \mathbb{A}/\beta_j$ are subdirectly irreducible, we conclude that congruences α_i and β_j are meet irreducible, i.e., that α_i and β_j are 0-irreducible (see Lemma 2). By Theorem 3, there is a bijection $\lambda\colon I \longrightarrow J$ such that $\mathbb{A}_i \sim_0 \mathbb{B}_{\lambda(i)}$ for all $i \in I$. From this together with Remark 2 we deduce that $\mathbb{A}_i \cong \mathbb{B}_{\lambda(i)}$. □

As an immediate consequence of Proposition 2 we get

Corollary 6. *Let $\mathbb{A}$ be any algebra and suppose that* Con($\mathbb{A}$) *is distributive and lower continuous. Let $\langle(\mathbb{A}_i\colon i \in I), f\rangle$ and $\langle(\mathbb{B}_j\colon j \in J), g\rangle$ be two irredundant finitely restricted subdirect representations of $\mathbb{A}$ with subdirectly irreducible factors. If the intervals $[\ker(f_i), 1]$ and $[\ker(g_j), 1]$ are of finite lengths, then there is a bijection $\lambda\colon I \longrightarrow J$ such that $\mathbb{A}_i \cong \mathbb{B}_{\lambda(i)}$ for $i \in I$.*

We also have

Corollary 7. *Let $\mathbb{A}$ be an algebra whose congruence lattice is distributive and lower continuous. If $\langle(\mathbb{A}_i\colon i \in I), f\rangle$ and $\langle(\mathbb{B}_j\colon j \in J), g\rangle$ are two irredundant subdirect representations of A with simple factors, then there is a bijection $\lambda\colon I \longrightarrow J$ such that $\mathbb{A}_i \cong \mathbb{B}_{\lambda(i)}$ for all $i \in I$.*

References

[1] *P. Crawley and R. P. Dilworth*: Algebraic Theory of Lattices, Prentice Hall, Englewood Cliffs. New Jersey, (1973).

[2] *H. Draškovičová*: Weak direct product decomposition of algebras, in: Contributions to General Algebra 5, Proc. of Salzburg Conf. 1986. Verlag Holder-Pichler-Tempsky, Wien (1987), 105–121.

[3] *G. Grätzer*: General Lattice Theory. Akademie-Verlag, Berlin, 1978.

[4] *G. Grätzer*: Universal Algebra. Springer-Verlag, New York, 1979.

[5] *J Hashimoto*: Direct, subdirect decompositions and congruence relations. Osaka Math. J. *9* (1957), 87–112.

[6] *T. K. Hu*: Weak products of simple universal algebras. Math. Nachr. *42* (1969), 157–171.

[7] *R. McKenzie, G. McNulty and W. Taylor*: Algebras, Lattices, Varieties, Volume I, Wadsworth Brooks/Cole. Menterey-California, 1987.

[8] *A. Walendziak*: Infinite θ-decomposition in modular lattices, in: Universal and Applied Algebra, Proc. of Turawa Symposium 1988. Vorld Sci. Publishing, Teaneck, NJ, (1989), 321–333.

[9] *A. Walendziak*: Infinite θ-decompositions in upper continuous lattices. Comment. Math *29* (1990), 313–324.

[10] *A. Walendziak*: L-restricted φ–representations of algebras. Period. Math. Hung *23* (1991), 219–226.

[11] *A. Walendziak*: Irredundant φ-representations of algebras—existence and some uniqueness. Algebra Universalis *30* (1993), 319–330.

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