Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 46 (1996), No. 1, 1–20

Persistent URL: http://dml.cz/dmlcz/127265

Czechoslovak Mathematical Journal, 46 (121) 1996, Praha

PERRON TYPE INTEGRATION ON n-DIMENSIONAL INTERVALS AS AN EXTENSION OF INTEGRATION OF STEPFUNCTIONS BY STRONG EQUICONVERGENCE

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(Received September 29, 1992, enlarged version June 1, 1994)

0. INTRODUCTION

The integral of a stepfunction (on an n-dimensional interval I) is given by an elementary formula so that a linear functional Λ on the linear space S of stepfunctions is defined in the natural way. The idea of obtaining an integration theory directly as an extension of Λ from S to a wider space has been used several times. Monotone convergence was used by E. J. McShane in [8] and by F. Riesz and B. Sz.-Nagy in [9] to develop Lebesgue integration. If S is made a normed space by putting $\|x\| = \Lambda(|x|)$ for $x \in S$, then the Lebesgue integral is the continuous extension of Λ to the completion of S (cf. the approach to Bochner integration in [10]). In the case $n = 1$ Lee and Chew in [7] proved that for every Denjoy integrable (in the restricted sense) $f\colon I \to \mathbb{R}$ there exists a sequence of stepfunctions $f_k\colon I \to \mathbb{R}$ such that f_k is control convergent to f for $k \to \infty$ (and, consequently, $(D_*)\int_I f\,\mathrm{d}t$ is the limit of $\Lambda(f_k)$). On the other hand, any $g\colon I \to \mathbb{R}$ which is the limit in the control convergence of a sequence of Denjoy integrable g_k is itself Denjoy integrable and $(D_*)\int_I g_k\,\mathrm{d}t \to (D_*)\int_I g\,\mathrm{d}t$ for $k \to \infty$. In this paper an analogue to the result of Lee and Chew is proved in the multidimensional case. The concept of integral involved is the strong ϱ-integral which was introduced by the authors in [3]; in the onedimensional case the strong ϱ-integral reduces to the Henstock-Kurzweil integral which is equivalent to the Denjoy and Perron integrals. The paper is organized in three sections. In Section 1 the relevant notions and results from [3] are recalled and a suitable convergence concept (strong ϱ-equiconvergence) is introduced. In Section 2

[1] This paper was supported by grant No. 201/94/1068 of the GA of the Czech Republic

we formulate the main result and establish some auxiliary facts. Section 3 is devoted to the proof of the main result.

1. The strong ϱ-equiconvergence

The same notation and concepts as in [3] will be used throughout the paper, in particular

$$I = [a, b] = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$$

is a nondegenerate compact interval and

$$\varrho \colon I \times (0, \infty) \to [0, 1)$$

fulfils

$$\limsup_{\sigma \to 0+} \varrho(t, \sigma) < 1 \quad \text{for } t \in I, \tag{1.1}$$

$$\inf\{\varrho(t, \sigma)\,;\; t \in I, \sigma > 0\} > 0, \tag{1.2}$$

(cf. [3], (2.1), (3.1)).

As usual, ∂J, $\operatorname{Int} J$, $m(J)$ and $d(J)$ denote the boundary, the interior, the measure and the diameter of $J \subset \mathbb{R}^n$.

1.1 Definition. A function $f \colon I \to \mathbb{R}^n$ is called *strongly ϱ-integrable* if there exists an additive interval function F such that for every $\varepsilon > 0$ there is a gauge δ such that

$$\sum_{\Delta, \mathbb{M}} |f(t) m(M) - F(M)| \leqslant \varepsilon \tag{1.3}$$

holds for every δ-fine ϱ-regular system $\Delta = \{(t, J)\}$ and every set $\mathbb{M} = \{M\}$ of intervals such that the inclusion $M \subset J$ defines a one-to-one correspondence between Δ and $\mathbb{M}$. For brevity, such a set of intervals will be called *an associated (with Δ) family.*

Of course, a strongly ϱ-integrable function f is ϱ-integrable and F is its primitive. For $n = 1$ the two integrals coincide and, moreover, reduce to the Perron integral.

A convergence theorem concerning a pointwise convergent sequence of strongly ϱ-integrable functions, which was proved in [3], Theorem 4.6, is the starting point for further convergence results.

1.2. Theorem. *Let $f_j\colon I \to \mathbb{R}$ be strongly ϱ-integrable for $j \in \mathbb{N}$, F_j being the primitives, let $f\colon I \to \mathbb{R}$. Assume that*

(1.4) *for every $\xi > 0$ there is a gauge ω such that*

$$\left|\sum_{\Delta,\mathbb{M}} f_j(t)m(M) - F_j(M)\right| \leqslant \xi$$

for any ω-fine ϱ-regular system $\Delta = \{(t, J)\}$, any j and any associated family of intervals $\mathbb{M}$,

and that

$$f_j(t) \to f(t) \quad \text{for } t \in I, j \to \infty. \tag{1.5}$$

Then f is strongly ϱ-integrable and $F_j(K) \to F(K)$ for $j \to \infty$, $K \subset I$ being an interval, F being the primitive of f.

Our aim is to prove that any strongly ϱ-integrable function g is the limit of a sequence of stepfunctions in a suitable convergence. Of course, $f\colon I \to \mathbb{R}$ is called a stepfunction if there exist intervals $J_1, J_2, \dots, J_k \subset I$ such that $\cup_i J_i = I$, $\mathrm{Int}J_i \cap \mathrm{Int}J_l = \emptyset$ for $i \neq l$ and if the restriction of f to any $\mathrm{Int}J_i$ is a constant function. The convergence from Theorem 1.2 cannot be directly applied to our purpose since the condition (1.5) is too restrictive. The assumption of pointwise convergence can be weakened to the assumption of convergence almost everywhere as a consequence of the following theorem, the proof of which is straightforward.

1.3 Theorem. *Assume that there is $N \subset I$, $m(N) = 0$ such that*

$$g_j\colon I \to \mathbb{R} \quad \text{for } j \in \mathbb{N},\ g\colon I \to \mathbb{R}, \tag{1.6}$$

$$\begin{aligned} &f_j(t) = g_j(t), f(t) = g(t) \quad \text{for } t \in I \setminus N,\ j \in \mathbb{N},\\ &f_j(t) = 0, f(t) = 0 \quad \text{for } t \in N,\ j \in \mathbb{N}. \end{aligned} \tag{1.7}$$

Then the following two properties are equivalent:

(1.8) (i) $g_j(t) \to g(t)$ *for* $j \to \infty, t \in I \setminus N$,
(ii) *there exists an additive interval function* G_j *on* I *for* $j \in \mathbb{N}$ *and for every* $\eta > 0$ *there is a gauge* ϑ *such that*

$$\sum_{\Delta,\mathbb{M}} |g_j(t)m(M) - G_j(M)| \leqslant \eta$$

for $j \in \mathbb{N}$ *and for any* ϑ*-fine* ϱ*-regular* $(I \setminus N)$*-tagged system* $\Delta = \{(t, J)\}$ *and any associated family* $\mathbb{M}$ *of intervals* M, *and*

$$\sum_{\Delta\mathbb{M}} |G_j(M)| \leqslant \eta$$

for $j \in \mathbb{N}$ *and for any* ϑ*-fine* ϱ*-regular* N*-tagged system* $\Delta = \{(t, J)\}$ *and any associated family* $\mathbb{M}$ *of intervals* M;

(1.9) f_j *is strongly* ϱ*-integrable for* $j \in \mathbb{N}$ *and both* (1.4) *and* (1.5) *hold.*

1.4. Remark. Let (1.8) hold. Then G_j is the primitive of g_j so that $G'_j = g_j$ a.e. (cf. [3], Definition 2.6 and Theorem 2.8). Moreover, [3], Lemma 1.8 implies that (1.8) holds if N is replaced by N_1 provided $N \subset N_1 \subset I$, $m(N_1) = 0$, since $\{g_j(t)\colon j \in \mathbb{N}\}$ is bounded for $t \in I \setminus N$.

1.5 Definition. Let $g_j\colon I \to \mathbb{R}$ for $j \in \mathbb{N}$, $g\colon I \to \mathbb{R}$. The sequence g_j is said to be *strongly* ϱ*-equiconvergent to* g for $j \to \infty$ if there exists $N \subset I$ such that $m(N) = 0$ and (1.8) holds.

The next theorem is a direct consequence of Theorems 1.2 and 1.3.

1.6 Theorem ([3], Theorem 4.9). *Let* $g_j\colon I \to \mathbb{R}$ *for* $j \in \mathbb{N}$, $g\colon I \to \mathbb{R}$ *and let* g_j *be strongly* ϱ*-equiconvergent to* g *for* $j \to \infty$. *Then* g *is strongly* ϱ*-integrable and* $G_j(K) \to G(K)$ *for* $j \to \infty$ *and any interval* $K \subset I$ *(*G_j *and* G *being the primitives of* g_j *and* g*, respectively).*

The concept of strong ϱ-equiconvergence plays the crucial role in Theorem 1.3; observe that $\{g_j(t);\, j \in \mathbb{N}\}$ need not be bounded if $t \in N$.

2. Density of the set of stepfunctions

Let $\varrho_k\colon I \times (0,\infty) \to (0,1)$ fulfil (1.1), (1.2) for $k \in \mathbb{N}$ and let

$$\varrho_k(t,\sigma) \geqslant \varrho_{k+1}(t,\sigma) \quad \text{for } k \in \mathbb{N}, t \in I, \sigma > 0. \tag{2.1}$$

2.1 Theorem (Main Result). *Let $g\colon I \to \mathbb{R}$ be strongly ϱ_k-integrable for $k \in \mathbb{N}$. Then there exists a sequence of stepfunctions $g_j\colon I \to \mathbb{R}$, $j \in \mathbb{N}$ such that g_j is strongly ϱ_k-equiconvergent to g for $j \to \infty$ and every $k \in \mathbb{N}$.*

The proof will be given in Section 3. Now we will only establish some auxiliary results.

2.2 Remark. If $\varrho_k(t,\sigma) = \varrho(t,\sigma)$ for $k \in \mathbb{N}$, then by Theorem 2.1 any strongly ϱ-integrable g can be obtained as the limit of a strongly ϱ-convergent sequence of stepfunctions g_j, $j \in \mathbb{N}$ and the primitive G of g is the limit of the sequence G_j of the primitives of g_j. If we put $\varrho_k(t,\sigma) = \frac{1}{k+1}$ for $t \in I$, $\sigma > 0, k \in \mathbb{N}$, then in an analogous way any g may be obtained which is strongly ϱ-integrable for every constant function ϱ, $\varrho \in (0,1)$. Such a g need not be Perron integrable, see [5].

2.3 Remark. Let $n = 1$. The strong ϱ-integral reduces to the Henstock-Kurweil integral (cf. [3], Note 4.3) independently of ϱ and the Henstock-Kurzweil integral is known to be equivalent both to the Perron integral and to the Denjoy integral. Lee and Chew in [6] introduced the control convergence for sequences of Denjoy integrable functions and proved the corresponding convergence theorem. In [7] they proved that every Denjoy integrable function is the limit of a control convergent sequence of stepfunctions. Another concept of convergence was introduced and studied by R. A. Gordon, [1], [2]; it follows from his results that any control convergent sequence is equiconvergent.

2.4. Lemma. *Let $K \subset \mathbb{R}^n$ be a nondegenerate compact interval, let $0 < A < 1$ and $\operatorname{reg} K \geqslant A$. Denote by $\Omega(S,r)$ the neighbourhood of a set S with radius r.*

Then there exists a constant $\kappa = \kappa(n) > 0$ such that

$$m\big(\Omega\big(\partial K, \zeta d(K)\big)\big) \leqslant \kappa A^{1-n} \zeta m(K) \tag{2.2}$$

provided $0 < \zeta < \frac{1}{2}A$.

Proof. Without loss of generality, let us assume

$$\begin{aligned} K &= [0,a_1] \times [0,a_2] \times \ldots \times [0,a_n], \\ d(K) &= a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant a_1 A. \end{aligned} \tag{2.3}$$

Then

$$\begin{aligned}
\Omega(\partial K, \zeta d(K)) &= [-\zeta a_1, a_1+\zeta a_1]\times[-\zeta a_1, a_2+\zeta a_1]\times\ldots\times[-\zeta a_1, a_n+\zeta a_1]\\
&\qquad \setminus [\zeta a_1, a_1-\zeta a_1]\times[\zeta a_1, a_2-\zeta a_1]\times\ldots\times[\zeta a_1, a_n-\zeta a_1],\\
m(\Omega(\partial K, \zeta d(K))) &= (a_1+2\zeta a_1)(a_2+2\zeta a_1)\ldots(a_n+2\zeta a_1)\\
&\qquad -(a_1-2\zeta a_1)(a_2-2\zeta a_1)\ldots(a_n-2\zeta a_1)\\
&= 2\sum(2\zeta a_1)a_{i_1}\ldots a_{i_{n-1}} + 2\sum(2\zeta a_1)^3 a_{i_1}\ldots a_{i_{n-3}}+\ldots\\
&\leqslant \kappa a_1^n\zeta \leqslant \kappa A^{1-n}\zeta m(K).
\end{aligned}$$

□

2.5. Lemma. *Let K, H_1, H_2, …, H_p be nondegenerate compact intervals in $\mathbb{R}^n$, let $0 < A < 1$, $\zeta > 0$, $\operatorname{reg} H_i \geqslant A$ and $d(H_i) \geqslant \zeta d(K)$ for $i = 1, 2, \ldots, p$. Let H_1, H_2, …, H_p be nonoverlapping.*

Then

$$\#\{H_i;\ H_i\cap K\neq\emptyset\}\leqslant 3^n A^{1-n}\max\{1,\zeta^{-n}\}.$$

Proof. Assume that K has the form (2.3) from the proof of Lemma 2.4. If $H_i\cap K\neq\emptyset$ then

$$m(H_i\cap[-\zeta a_1, a_1+\zeta a_1]^n)\geqslant(\zeta a_1)^n A^{n-1},$$

hence

$$\#\{H_i, H_i\cap K\neq\emptyset\}\leqslant (a_1+2\zeta a_1)^n/\zeta^n a_1^n A^{n-1}\leqslant 3^n A^{1-n}\max\{1,\zeta^{-n}\}.$$

□

Put

$$V(t,\nu)=[t_1-\nu, t_1+\nu]\times\ldots\times[t_n-\nu, t_n+\nu]\qquad\text{for}\quad t\in\mathbb{R}^n,\ \nu>0.$$

The authors proved in [4], Corollary 2 and Theorem 1, the following result:

Let G be an additive function of interval on I, $g\in\mathbb{R}$, $t\in\operatorname{Int} I$. Let G be regularly differentiable to g at t. Then for every $\varepsilon>0$ there is $r>0$ such that

$$|G(J)-gm(J)|\leqslant\varepsilon(2\nu)^n \tag{2.4}$$

for every interval $J\subset V(t,\nu)$, where $\nu\leqslant r$.

(2.4) can be rewritten in the following way. For $L = [c_1, d_1] \times \ldots \times [c_n, d_n]$, $s \in \mathbb{R}^n$ let $\psi(s, L)$ be the smallest ν such that $L \subset V(s, \nu)$ (i.e. $\psi(s, L) = \max\{|d_i - c_i|, |c_i - s_i|, |d_i - s_i|\,;\ i = 1, 2, \ldots, n\}$. Obviously $\psi(t, J)$ can be substituted for ν in (2.4) so that (2.4) can be replaced by

$$|G(J) - gm(J)|\bigl(2\psi(t, J)\bigr)^{-n} \leqslant \varepsilon, \tag{2.5}$$

provided $\psi(t, J) \leqslant r$.

From this result we prove in a standard manner:

2.6. Lemma. *Let F be an additive function of interval on I. Denote by D_F the set of such $t \in I$ that F is regularly differentiable to (some) $F'(t) \in \mathbb{R}$ at t and put $N_F = I \setminus D_F$. Asume that $m(N_F) = 0$ and that F is continuous at any interval $L \subset \operatorname{Int} I$ (i. e. for every $\varepsilon > 0$ there is $\eta > 0$ such that $|F(K) - F(L)| \leqslant \varepsilon$ for every interval $K \subset I$ satisfying $m(K \setminus L) + m(L \setminus K) \leqslant \eta$). Put*

$$f(t) = F'(t) \qquad \text{for} \qquad t \in D_F.$$

Then f is measurable and there exist

$$\begin{gathered} N \subset I, \quad N \supset N_F \cup \partial I, \quad m(N) = 0, \quad \xi \in (0, 1/4), \\ \eta\colon [0, \xi] \to [0, 1) \quad \text{increasing}, \ \eta(0) = 0, \ \eta(\sigma) > \sigma \ \text{ for } \sigma \in (0, \xi], \ \lim_{\sigma \to 0+} \eta(\sigma) = 0, \\ \omega\colon I \setminus N \to (0, \xi] \quad \text{measurable}, \ V\bigl(t, \omega(t)\bigr) \subset I \ \text{ for } t \in I \setminus N, \end{gathered}$$

such that

$$|F(K) - f(t)m(K)| \leqslant \eta(\nu)\nu^n \tag{2.6}$$

for every $t \in I \setminus N$, $\nu \in (0, \omega(t)]$, $K \subset \operatorname{Int} V(t, \nu)$ (K being an interval).

Proof. Let $I = [a_1, b_1] \times \ldots \times [a_n, b_n]$. f is measurable since $f(t) = \lim_{\sigma \to 0+} F(V(t, \sigma))(2\sigma)^{-n}$ for $t \in D_F \cap \operatorname{Int} I$ and $F(V(t, \sigma))$ is continuous with respect to t on $(a_1 + \sigma, b_1 - \sigma) \times \ldots \times (a_n + \sigma, b_n - \sigma)$. For $M = [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n]$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ put

$$M(t) = [\alpha_1 + t_1, \beta_1 + t_1] \times \ldots \times [\alpha_n + t_n, \beta_n + t_n].$$

For $\sigma \in \Bigl(0, \frac{1}{2} \min\{b_i - a_i\,;\ i = 1, 2, \ldots, n\}\Bigr)$, $t \in [a_1 + \sigma, b_1 - \sigma] \times \ldots \times [a_n + \sigma, b_n - \sigma]$, put

$$\begin{aligned} \varphi_\sigma(t) = \sup\bigl\{&\bigl|F\bigl(M(t)\bigr) - f(t)m\bigl(M(t)\bigr)\bigr| \cdot \bigl(\psi(0, M)\bigr)^{-n}; \\ &M \subset V(0, \sigma), \alpha_i, \beta_i \text{ rational}\bigr\}. \end{aligned} \tag{2.7}$$

$\varphi_\sigma\colon [a_1+\sigma, b_1-\sigma] \times \ldots \times [a_n+\sigma, b_n-\sigma] \to \mathbb{R}$ is measurable and it follows from (2.5) that

$$\varphi_\sigma(t) \searrow 0 \quad \text{for } \sigma \searrow 0, t \in D_F \cap \operatorname{Int} I.$$

For $\lambda > 0$ put

$$E(\sigma, \lambda) = \{t \in \operatorname{Int} I;\ V(t,\sigma) \subset I, \varphi_\sigma(t) \leqslant \lambda\}.$$

$E(\sigma, \lambda)$ has the following properties:

$$\begin{aligned}
&E(\sigma,\lambda) \quad \text{is measurable,}\\
&E(\sigma_1,\lambda) \supset E(\sigma_2,\lambda) \quad \text{for } 0 < \sigma_1 \leqslant \sigma_2,\\
&\bigcup_{\sigma>0} E(\sigma,\lambda) = D_F \cap \operatorname{Int} I,\\
&\lim_{\sigma\to 0} m\big(E(\sigma,\lambda)\big) = m(I).
\end{aligned}$$

Therefore , for $i \in \mathbb{N}$ there exists $\sigma_i > 0$ such that

$$\sigma_1 < 1/4, \quad 0 < \sigma_{i+1} < \frac{1}{2}\sigma_i, \quad m\big(I \setminus E(\sigma_i, 2^{-i})\big) \leqslant 2^{-i}.$$

Put

$$N = I \setminus \liminf_{i\to\infty} E(\sigma_i, 2^{-i}) = I \setminus \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} E(\sigma_i, 2^{-i}).$$

It can be seen that $m(N) = 0$, $N \supset N_F \cup \partial I$.

Let $\xi = \sigma_1$. For $\nu \in (0,\xi]$ there exists a unique $i \in \mathbb{N}$ such that $\nu \in (\sigma_{i+1}, \sigma_i]$; put $\eta(\nu) = 2^{-i}$. Since $\sigma_i < 2^{-i-1}$ for $i \in \mathbb{N}$, we have $\eta(\sigma_i) > \sigma_i$ for $i \in \mathbb{N}$ and obviously $\eta(\nu) > \nu$ for $\nu \in (0,\xi]$ and $\lim\limits_{\sigma\to 0+} \eta(\sigma) = 0$.

For $t \in I \setminus N$ let $h(t)$ be the smallest j such that $t \in \bigcap\limits_{i=j}^{\infty} E(\sigma_i, 2^{-i})$. Put $\omega(t) = \sigma_{h(t)}$; ω is measurable.

If $t \in I \setminus N$, $\nu \in (0, \omega(t)]$, then $\nu \in (\sigma_{k+1}, \sigma_k]$ for some $k \geqslant h(t)$ so that $t \in E(\sigma_k, 2^{-k})$. Assume that $M = [\alpha_1,\beta_1] \times \ldots \times [\alpha_n,\beta_n] \subset V(0,\nu)$, α_i, β_i being rationals (i.e. $\psi(0,M) < \nu$). By (2.7) we have

$$\begin{aligned}
\eta(\nu) = 2^{-k} &\geqslant \big|F\big(M(t)\big) - f(t)m\big(M(t)\big)\big|\big(\psi(0,M)\big)^{-n}\\
&\geqslant \big|F\big(M(t)\big) - f(t)m\big(M(t)\big)\big|\nu^{-n}.
\end{aligned}$$

Since F is continuous at any interval $L \subset \operatorname{Int} I$, we obtain

$$\big|F\big(M(t)\big) - f(t)m\big(M(t)\big)\big| \leqslant \eta(\nu)\nu^n$$

for every $M = [\alpha_1,\beta_1] \times \ldots \times [\alpha_n,\beta_n] \subset \operatorname{Int} V(0,\nu)$, α_i, β_i being reals, $i = 1, 2, \ldots, n$. (2.6) holds since any $\operatorname{Int} K \subset V(t,\nu)$ is equal to some $M(t)$ with $M \subset \operatorname{Int} V(0,\nu)$. $\square$

2.7. Corollary. *If $t \in I \setminus N$, $t \in H \subset V(t,\omega(t))$, $K \subset H$, H, K being intervals, then*

$$|F(K) - f(t)m(K)| \leqslant \eta(d(H))(d(H))^n; \tag{2.8}$$

if, moreover, $\operatorname{reg} H \geqslant A$ *with* $0 < A < 1$, *then*

$$|F(K) - f(t)m(K)| \leqslant A^{1-n}\eta(d(H))m(H). \tag{2.9}$$

The last inequality follows from the fact that the longest edge of H has the length $d(H)$ while all the others have lengths not less than $Ad(H)$.

2.8. Lemma. *If $g\colon I \to \mathbb{R}$ is Lebesgue integrable, then g is strongly ϱ-integrable.*

Proof. g is ϱ-integrable by [3], Note 1.5. Let G be the primitive of g. By [3], Theorem 3.2 we conclude that $m(I \setminus D_G) = 0$ and $G' = g$ a.e. Since $(L)\int_K f = (\varrho)\int_K f$ for every interval $K \subset I$, the absolute continuity of the Lebesgue integral implies that (4.2) from [3] holds. Take into account that the correct version of condition (B) in [3], Theorem 4.12, is

(B) F is additive, $m(I \setminus D_F) = 0$, (4.2) holds and $F' = f$ a.e.

(by a misprint the incorrect (4.4) appears instead of the correct (4.2) in condition B of [3], Theorem 4.12). Thus (B) from [3], Theorem 4.12, is fulfilled and it follows that g is strongly ϱ-integrable and G is its primitive. □

3. Proof of main result

Let $g\colon I \to \mathbb{R}$ be strongly ϱ_k-integrable for $k \in \mathbb{N}$. For any interval $L \subset I$ we put $F(L) = (\varrho_k)\int_L g$; the right hand side is independent of k (cf. (2.1)) and F is called the primitive of g. F is an additive function of interval on I and it is continuous at any interval $L \subset \operatorname{Int} I$ by [3], Theorem 2.1 (in [3], Theorem 2.1 the correct form of the assumption on L is $L \subset I$ and the corresponding form of continuity of F at L is described even if $L \not\subset \operatorname{Int} I$). By [3], Theorem 2.8 and Definition 2.6 F is regularly differentiable to $F'(t)$ at every $t \in D_F$, $m(N_F) = 0$ where $N_F = I \setminus D_F$ and $F' = g$ a.e. The assumptions of Lemma 2.6 being fulfilled, let f, N, ξ, η, ν, ω have the same meaning as in Lemma 2.6 so that, in particular, (2.6) holds. If necessary the set N can be enlarged so that

$$f(t) = g(t) \qquad \text{for } t \in I \setminus N,\ m(N) = 0. \tag{3.1}$$

Moreover, by [3], Theorem 4.5 we conclude that simultaneously

(3.2) for every $\lambda > 0$ and $i \in \mathbb{N}$ there is a gauge γ such that

$$\sum_{\Xi,\mathbb{M}} |F(M)| \leqslant \lambda$$

for every γ-fine ϱ_i-regular N-tagged system $\Xi = \{(s,K)\}$ and any associated family of intervals $\mathbb{M}$.

Let us choose a sequence $\{\xi_k\}$ such that

$$\xi \geqslant \xi_1 \geqslant \xi_2 \geqslant \ldots > 0, \quad \lim_{k\to\infty} \xi_k = 0, \tag{3.3}$$

($[0,\xi]$ being the domain of η). There is a measurable $\omega_1 : I \setminus N \to (0,1]$ such that

$$|f(t)| \leqslant \left[\eta(2\omega_1(t))\right]^{-\frac{1}{4n}} \tag{3.4}$$

for $t \in I \setminus N$. Let us set

$$\delta_k(t) = \min\left\{\frac{1}{2}\xi_k, \omega_1(t), \omega(t)\right\} \tag{3.5}$$

for $t \in I \setminus N$, $k = 1,2,3,\ldots$, where ω is from Lemma 2.6.

Referring to (3.2) let us choose $\delta_k(t)$ for $t \in N$ such that

$$\delta_k(t) \leqslant \frac{1}{2}\xi_k \tag{3.6}$$

and

$$\sum_{\Xi,\mathbb{M}} |F(M)| \leqslant \xi_k \tag{3.7}$$

provided $\Xi = \{(s,K)\}$ is a δ_k-fine ϱ_k-regular N-tagged system and $\mathbb{M} = \{M\}$ is an associated family of intervals (i.e., the inclusion $M \subset K$ defines a one-to-one correspondence between $\mathbb{M}$ and Ξ).

For the basic interval I let us write

$$I = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n].$$

If $K = [c_1, d_1] \times [c_2, d_2] \times \ldots \times [c_n, d_n] \subset I$, then we write

$$K^0 = [c, d_1]^0 \times [c_2, d_2]^0 \times \ldots \times [c_n, d_n]^0$$

where

$$[c_i, d_i]^0 = \begin{cases} [c_i, d_i) & \text{if } d_i < b_i \\ [c_i, d_i] & \text{if } d_i = b_i. \end{cases}$$

Now we can define the desired sequence of stepfunctions g_k.

For $k \in \mathbb{N}$ let us choose a δ_k-fine ϱ_1-regular partition $\Delta_k = \{(t, J)\}$ of the interval I, and for $s \in I$ let us set

$$g_k(s) = \frac{F(J)}{m(J)} \tag{3.8}$$

where J is such that $(t, J) \in \Delta_k$ for some t and $s \in J^0$. (Evidently, there is a unique J with the property.)

The function g_k is integrable (cf. Lemma 2.8); let G_k be its primitive function, $k \in \mathbb{N}$. For any interval $M \subset I$ we have

$$G_k(M) = \sum_{(t,J)\in\Delta_k} \frac{F(J)}{m(J)} m(J \cap M). \tag{3.9}$$

The result to be established can be formulated as follows.

3.1. Theorem. *For every $i \in \mathbb{N}$ the sequence $\{g_k\}$ is strongly ϱ_i-equiconvergent to g.*

It is a consequence of the following two propositions.

3.2. Proposition. *For every $\varepsilon > 0$ and $i \in \mathbb{N}$ there are $l_1 \in \mathbb{N}$ and $\vartheta_1 \colon N \to (0, 1]$ such that*

$$\Sigma_1 = \sum_{\Theta,\mathbb{M}} |G_k(M)| \leqslant \varepsilon \tag{3.10}$$

for every ϑ_1-fine ϱ_i-regular N-tagged system $\Theta = \{(u, L)\}$, every associated family $\mathbb{M} = \{M\}$ and every $k \geqslant l_1$.

3.3. Proposition. *For every $\varepsilon > 0$ and $i \in \mathbb{N}$ there are $l_2 \in \mathbb{N}$ and $\vartheta_2 \colon I \setminus N \to (0, 1]$ such that*

$$\Sigma_2 = \sum_{\Theta,\mathbb{M}} |G_k(M) - g_k(u)m(M)| \leqslant \varepsilon \tag{3.11}$$

for every ϑ_2-fine ϱ_i-regular $I \setminus N$-tagged system $\Theta = \{(u, L)\}$, every associated family $\mathbb{M}$ and every $k \geqslant l_2$. Moreover,

$$g_k(s) \to g(s) \quad \text{for } s \in I \setminus N,\ k \to \infty. \tag{3.12}$$

3.4. Convention. Since ϱ_k fulfil (1.1), (1.2) and (2.1), for every $k \in \mathbb{N}$ there is A_k, $0 < A_k \leqslant \varrho_k(t,\sigma) \leqslant \varrho_1(t,\sigma)$, and we may assume $A_{k+1} \leqslant A_k$ for $k \in \mathbb{N}$. Hence $\operatorname{reg} J \geqslant A_1 \geqslant A_l$ for $(t,J) \in \Delta_k$ and any $k, l \in \mathbb{N}$, and $\operatorname{reg} L \geqslant A_i$ for $(u,L) \in \Theta$ since Θ is ϱ_i-regular. The index $i \in \mathbb{N}$ is fixed throughout the proofs of Propositions 3.2, 3.3 and $A_1 \geqslant A_i$. Therefore we may and will write A instead of A_1 and A_i, which implies that Δ_k, $k \in \mathbb{N}$, as well as Θ are A-regular. To simplify the formulas we will also assume (without loss of generality) that $m(I) \leqslant 1$.

Proof of Proposition 3.2. Given $\varepsilon > 0$ and $i \in \mathbb{N}$, let us choose $j \in \mathbb{N}$ such that

$$j \geqslant i,\ \xi_j(3 + 2\cdot 3^n A^{1-n}) < \frac{1}{2}\varepsilon \tag{3.13}$$

and denote

$$r(u) = \min\{k \in \mathbb{N};\ \xi_k \leqslant \delta_j(u)\} \qquad \text{for } u \in N. \tag{3.14}$$

For every $k \in \mathbb{N}$ there is an open set $U_k \subset \mathbb{R}^n$ such that $N \subset U_k$ and

$$m(U_k) \leqslant \xi_j\beta_k, \quad \beta_k = \frac{\min\{m(J);\ (t,J) \in \Delta_k\}}{\max\{1 + |F(J)|;\ (t,J) \in \Delta_k\}}. \tag{3.15}$$

For every $k \in \mathbb{N}$ there is a gauge $\mu_k\colon N \to (0,1]$ such that

$$V\big(u, \mu_k(u)\big) \subset U_k \tag{3.16}$$

for $u \in N$. We choose a gauge $\vartheta_1\colon N \to (0,1]$ satisfying the condition

$$\begin{aligned} &\vartheta_1(u) \leqslant \mu_k(u) \quad \text{for } k < r(u),\ u \in N,\\ &\vartheta_1(u) \leqslant \delta_j(u) \quad \text{for } u \in N. \end{aligned} \tag{3.17}$$

Now we start estimates leading to (3.10). Let $\Theta = \{(u,L)\}$ be a ϑ_1-fine ϱ_i-regular N-tagged system and let $\mathbb{M}$ be an associated family of intervals. For $k \in \mathbb{N}$ we have

$$\Sigma_1 \leqslant \Gamma_1 + \Gamma_2 = \sum_{\substack{\Theta,\mathbb{M}\\ \exists (t,J)\in\Delta_k, L\subset J}} |G_k(M)| + \sum_{\substack{\Theta,\mathbb{M}\\ L\setminus J\neq\emptyset, \forall (t,J)\in\Delta_k}} |G_k(M)|.$$

By virtue of (3.9) we obtain

$$\Gamma_1 \leqslant \Gamma_3 + \Gamma_4 = \sum_{\Delta_k} \sum_{\substack{\Theta,\mathbb{M}\\ \exists (t,J)\in\Delta_k, L\subset J\\ k<r(u)}} |F(J)|\frac{m(M\cap J)}{m(J)}$$
$$+ \sum_{\Delta_k} \sum_{\substack{\Theta,\mathbb{M}\\ \exists (t,J)\in\Delta_k, L\subset J\\ k\geqslant r(u)}} |F(J)|\frac{m(M\cap J)}{m(J)}.$$

If $(t,J) \in \Delta_k$, $(u,L) \in \Theta$, $k < r(u)$, $L \subset J$ then $M \cap J \subset L \subset U_k \cap J$ since $u \in N$ (cf. (3.16), (3.17)), and consequently (cf. (3.15))

$$\beta_k^{-1} \sum_{\Delta_k} \sum_{\substack{\theta \\ \exists (t,J) \in \Delta_k, L \subset J \\ k < r(u)}} m(L) \leqslant \beta_k^{-1} \sum_{\Delta_k} m(J \cap U_k). \tag{3.18}$$

We proceed to Γ_4. For $(t,J) \in \Delta_k$ let $\Omega(t,J)$ be the set of $(u,L) \in \Theta$ such that $L \subset J$, $k \geqslant r(u)$. We have

$$\Gamma_4 \leqslant \sum_{\Delta_k} |F(J)| \sum_{\Omega(t,J)} \frac{m(J \cap L)}{m(J)} \leqslant \sum_{\substack{\Delta_k \\ \exists (u,L) \in \Theta, L \subset J \\ k \geqslant r(u)}} |F(J)|.$$

Obviously $u \in N$ since Θ is N-tagged.In the last sum we have $u \in L \subset J$, $d(J) \leqslant 2\delta_k(t) \leqslant \xi_k \leqslant \delta_j(u)$ by (3.6) and (3.14), hence $J \subset V\big(u, \delta_j(u)\big)$. From (3.7) we conclude

$$\Gamma_4 \leqslant \xi_j. \tag{3.19}$$

(We apply (3.7) for a system of pairs (u,J) which is δ_j-fine, ϱ_1-regular and N-tagged, putting $M = J$.)

Now we shall estimate Γ_2. Using (3.9) we obtain

$$\Gamma_2 \leqslant \Gamma_5 + \Gamma_6 = \sum_{\Theta, \mathbb{M}} |F(M)|$$
$$+ \sum_{\Theta, \mathbb{M}} \left| \sum_{\substack{\Delta_k \\ L \setminus J \neq \emptyset, \forall (t,J) \in \Delta_k}} \Big(\frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \Big) \right|.$$

Θ is ϱ_i-regular and ϑ_1-fine; by the first inequality in (3.13) it is ϱ_j-regular and by (3.17) it is δ_j-fine. (3.7) can be applied with k replaced by j, so that

$$\Gamma_5 \leqslant \xi_j. \tag{3.20}$$

Further, we can write

$$\Gamma_6 \leqslant \Gamma_7 + \Gamma_8 = \sum_{\substack{\Theta, \mathbb{M} \\ L \setminus J \neq \emptyset, \forall (t,J) \in \Delta_k \\ t \in N}} \left| \sum_{\Delta_k} \Big(\frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \Big) \right|$$
$$+ \sum_{\substack{\Theta, \mathbb{M} \\ L \setminus J \neq \emptyset, \forall (t,J) \in \Delta_k \\ t \in I \setminus N}} \left| \sum_{\Delta_k} \Big(\frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \Big) \right|.$$

The first sum can be divided into three terms:

$$\Gamma_7 \leqslant \Gamma_9 + \Gamma_{10} + \Gamma_{11} = \sum_{\substack{\Delta_k \\ t \in N}} \frac{|F(J)|}{m(J)} \sum_{\Theta, \mathbb{M}} m(M \cap J)$$
$$+ \sum_{\Theta, \mathbb{M}} \sum_{\substack{\Delta_k \\ d(J) \geqslant d(L)}} |F(M \cap J)| + \sum_{\substack{\Delta_k \\ t \in N}} \sum_{\substack{\Theta, \mathbb{M} \\ d(L) > d(J)}} |F(M \cap J)|.$$

By (3.7) we obtain

$$\Gamma_9 \leqslant \xi_k \tag{3.21}$$

since the inner sum (for fixed $(t, J) \in \Delta_k$) does not exceed $m(J)$. Further,

$$\Gamma_{10} \leqslant \sum_{\Theta} \sup\{|F(K)|;\, K \subset L\} \cdot \#\{(t, J) \in \Delta_k;\, J \cap L \neq \emptyset, d(J) \geqslant d(L)\}.$$

By Lemma 2.5 the number of elements of Δ_k on the righthand side of the inequality has the upper bound $3^n A^{1-n}$ (since $\zeta = 1$), which together with (3.17) and (3.7) yields

$$\Gamma_{10} \leqslant 3^n A^{1-n} \sum_{\Theta} \sup\{|F(K)|;\, K \subset L\} \leqslant 3^n A^{1-n} \xi_j. \tag{3.22}$$

Similarly, with the role of Δ_k and Θ interchanged, we obtain

$$\begin{aligned} \Gamma_{11} \leqslant & \sum_{\Delta_k;\ t \in N} \sup\{|F(H);\, H \subset J\} \cdot \#\{(u, L) \in \Theta;\, L \cap J \neq \emptyset, \\ & d(L) > d(J)\} \leqslant 3^n A^{1-n} \xi_k. \end{aligned} \tag{3.23}$$

Returning to Γ_8, we note that (2.6) and (3.5) yield for $(t, J) \in \Delta_k$ and $t \in I \setminus N$

$$\begin{aligned} |F(J) - f(t)m(J)| &\leqslant A^{1-n} \eta(d(J)) m(J), \\ |F(M \cap J) - f(t)m(M \cap J)| &\leqslant A^{1-n} \eta(d(J)) m(J), \end{aligned} \tag{3.24}$$

hence

$$\left| \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right| \leqslant 2A^{1-n} \eta(d(J)) m(J). \tag{3.25}$$

Consequently,

$$\Gamma_8 \leqslant \Gamma_{12} + \Gamma_{13} = \sum_{\substack{\Delta_k \\ t \in I \setminus N}} \sum_{\substack{\Theta, \mathbb{M} \\ L \cap J \neq \emptyset \\ d(L) \geqslant [\eta(d(J))]^{\frac{3}{4n}} d(J)}} \left| \frac{F(J)}{m(J)} m(M \cap J) - F(J \cap M) \right|$$

$$+ \sum_{\Theta, \mathbb{M}} \sum_{\substack{\Delta_k ;\ t \in I \setminus N \\ L \setminus J \neq \emptyset \\ d(L) < [\eta(d(J))]^{\frac{3}{4n}} d(J)}} \left| \frac{F(J)}{m(J)} m(M \cap J) - F(M \cap J) \right|.$$

Estimating Γ_{12} with help of (3.25) and Lemma 2.5 we arrive at

$$\begin{aligned} \Gamma_{12} \leqslant & \sum_{\Delta_k ;\ t \in I \setminus N} 2A^{1-n} \eta\big(d(J)\big) m(J) \\ & \times \#\big\{(u, L) \in \Theta ;\ L \cap J \neq \emptyset, d(L) \geqslant \big[\eta\big(d(J)\big)\big]^{\frac{3}{4n}} d(J)\big\} \\ \leqslant & \ 2A^{1-n} \sum_{\Delta_k ;\ t \in I \setminus N} \eta\big(d(J)\big) m(J) 3^n A^{1-n} \big[\eta\big(d(J)\big)\big]^{-\frac{3}{4}} \end{aligned}$$

and by (3.5) we obtain

$$\Gamma_{12} \leqslant 2 \cdot 3^n A^{2-2n} [\eta(\xi_k)]^{\frac{1}{4}}. \tag{3.26}$$

In order to estimate Γ_{13} we use the first inequality (3.24):

$$\Gamma_{13} \leqslant \Gamma_{14} + \Gamma_{15} + \Gamma_{16} = \sum_{\substack{\Delta_k \\ t \in I \setminus N}} |f(t)| \sum_{\substack{\Theta, \mathbb{M} \\ L \cap J \neq \emptyset \neq L \setminus J \\ d(L) \leqslant [\eta(d(J))]^{\frac{3}{4n}} d(J)}} m(M \cap J)$$

$$+ A^{1-n} \sum_{\Delta_k} \sum_{\Theta, \mathbb{M}} \eta\big(d(J)\big) m(M \cap J) + \sum_{\Theta, \mathbb{M}} \sum_{\substack{\Delta_k \\ d(J) > d(L)}} |F(M \cap J)|.$$

Now (3.4), (3.5) imply

$$\Gamma_{14} \leqslant \sum_{\Delta_k} \big[\eta\big(d(J)\big)\big]^{-\frac{1}{4n}} \sum_{\substack{\Theta \\ L \cap J \neq \emptyset \neq L \setminus J \\ d(L) \leqslant [\eta(d(J))]^{\frac{3}{4n}} d(J)}} m(L \cap J)$$

and, assuming

$$[\eta(\xi_k)]^{\frac{3}{4n}} < \frac{1}{2} A \tag{3.27}$$

we conclude by (3.4), (3.5) and Lemma 2.4

$$\begin{aligned}(3.28)\qquad \Gamma_{14} &\leqslant \sum_{\Delta_k} \big[\eta\big(d(J)\big)\big]^{-\frac{1}{4n}} \kappa A^{1-n} m(J) \big[\eta\big(d(J)\big)\big]^{\frac{3}{4n}} \\ &\leqslant \kappa A^{1-n} [\eta(\xi_k)]^{\frac{1}{2n}}.\end{aligned}$$

Evidently,

$$(3.29)\qquad \Gamma_{15} \leqslant A^{1-n} \sum_{\Delta_k} \eta\big(d(J)\big) m(J) \leqslant A^{1-n} \eta(\xi_k)$$

and finally, by Lemma 2.5 and (3.7),

$$\begin{aligned}(3.30)\quad \Gamma_{16} &\leqslant \sum_{\Theta} \sup\{|F(K)|;\, K \subset L\} \cdot \#\{(t,J) \in \Delta_k;\, J \cap L \neq \emptyset, d(J) > d(L)\} \\ &\leqslant 3^n A^{1-n} \xi_j.\end{aligned}$$

Putting together the estimates (3.18)–(3.23), (3.26) and (3.28)–(3.30) we obtain

$$\begin{aligned}\Sigma_1 \leqslant\ & 3\xi_j + \xi_k + 3^n A^{1-n} \xi_j + 3^n A^{1-n} \xi_k \\ & + 2 \cdot 3^n A^{2-2n} [\eta(\xi_k)]^{\frac{1}{4}} + \kappa A^{1-n} [\eta(\xi_k)]^{\frac{1}{2n}} \\ & + A^{1-n} \eta(\xi_k) + 3^n A^{1-n} \xi_j.\end{aligned}$$

This together with (3.13) implies that Proposition 3.2 holds for $k \geqslant l_1$ where l_1 is such that (3.27) and

$$\xi_k(1 + 3^n A^{1-n}) + 2 \cdot 3^n A^{2-2n} [\eta(\xi_k)]^{\frac{1}{4}} + \kappa A^{1-n} [\eta(\xi_k)]^{\frac{1}{2n}} + A^{1-n} \eta(\xi_k) < \frac{1}{2}\varepsilon$$

hold for every $k \geqslant l_1$. □

P r o o f o f P r o p o s i t i o n 3.3. Given $\varepsilon > 0$ and $i \in \mathbb{N}$, let us choose $h \in \mathbb{N}$ such that

$$(3.31)\qquad \xi_h + A^{1-n} \eta(\xi_h) + 3^n A^{2-2n} \eta(2\xi_h) < \frac{1}{2}\varepsilon$$

and denote

$$(3.32)\qquad R(s) = \min\{k \in \mathbb{N};\, 2\xi_k \leqslant \delta_h(s)\} \quad \text{for } s \in I \setminus N.$$

For $k \in \mathbb{N}$ let a gauge $\gamma_k \colon I \setminus N \to (0,1]$ be such that

$$(3.33)\qquad \sum_{\Xi, \mathbb{M}} |G_k(M) - g_k(s) m(M)| \leqslant \xi_h$$

is satisfied provided $\Xi = \{(s, K)\}$ is a γ_k-fine ϱ_i-regular $(I \setminus N)$-tagged system and $\mathbb{M}$ an associated family of intervals (cf. Lemma 2.8). We choose a gauge $\vartheta_2 \colon I \setminus N \to (0, 1]$ satisfying the condition

$$\begin{aligned} \vartheta_2(s) &\leqslant \gamma_k(s) \quad \text{for } k < R(s),\ s \in I \setminus N, \\ \vartheta_2(s) &\leqslant \delta_h(s) \quad \text{for } s \in I \setminus N. \end{aligned} \tag{3.34}$$

According to the definition of the functions g_k we have $g_k(s) = F(K)/m(K)$ where $(z, K) \in \Delta_k$, $s \in K^0$. If, moreover, $s \in I \setminus N$, $k \geqslant R(s)$, then $K \subset V(z, \delta_k(z))$, $d(K) \leqslant 2\delta_k(z) \leqslant \xi_k \leqslant \frac{1}{2}\delta_h(s) \leqslant \omega(s)$ (see (3.5) and (3.32)), hence $K \subset V(s, \delta_h(s)) \subset V(s, \omega(s))$, and putting $H = K$ in (2.6) we obtain

$$|F(K) - f(s)m(K)| \leqslant A^{1-n}\eta(d(K))m(K)$$

and consequently,

$$|g_k(s) - f(s)| \leqslant A^{1-n}\eta(\xi_k). \tag{3.35}$$

Now we start estimates leading to (3.11). Let $\Theta = \{(u, L)\}$ be a ϑ_2-fine ϱ_i-regular $(I \setminus N)$-tagged system and let $\mathbb{M}$ be an associated family of intervals. For $k \in \mathbb{N}$ we have (cf. (3.11))

$$\Sigma_2 \leqslant \Gamma_{17} + \Gamma_{18} = \sum_{\substack{\Theta,\mathbb{M} \\ k<R(u)}} |G_k(M) - g_k(u)m(M)| + \sum_{\substack{\Theta,\mathbb{M} \\ k\geqslant R(u)}} |G_k(M) - g_k(u)m(M)|.$$

By (3.34) and (3.33) we have

$$\Gamma_{17} \leqslant \xi_h. \tag{3.36}$$

Further, we can write

$$\Gamma_{18} \leqslant \Gamma_{19} + \Gamma_{20} = \sum_{\substack{\Theta,\mathbb{M} \\ k\geqslant R(u)}} |f(u) - g_k(u)|m(M) + \sum_{\substack{\Theta,\mathbb{M} \\ k\geqslant R(u)}} |G_k(M) - f(u)|m(M)$$

and, by virtue of (3.35) we have

$$\Gamma_{19} \leqslant A^{1-n}\eta(\xi_k) \tag{3.37}$$

since $u \in I \setminus N$. Proceeding to Γ_{20} we estimate it as

$$\begin{aligned} \Gamma_{20} \leqslant \Gamma_{21} + \Gamma_{22} &= \sum_{\Theta,\mathbb{M}} |F(M) - f(u)m(M)| \\ &+ \sum_{\Theta,\mathbb{M}} \sum_{\substack{\Delta_k \\ k\geqslant R(u)}} \left|\frac{F(J)}{m(J)}m(M \cap J) - F(M \cap J)\right|. \end{aligned}$$

Applying (2.9) with M, u, L respectively instead of K, t, H and then (3.34) we conclude ($m(I) \leqslant 1$ by Convention 3.4)

$$\Gamma_{21} \leqslant A^{1-n}\eta(\xi_h). \tag{3.38}$$

The term Γ_{22} is divided into three sums:

$$\begin{aligned}\Gamma_{22} \leqslant \Gamma_{23}+\Gamma_{24}+\Gamma_{25} = \sum_{\Theta,\mathbb{M}} \sum_{\substack{\Delta_k;\ k\geqslant R(u)\\ d(J)\geqslant d(L)}} &\left|\frac{F(J)}{m(J)}m(M\cap J) - F(M\cap J)\right| \\ +\sum_{\Theta,\mathbb{M}} \sum_{\substack{\Delta_k;\ k\geqslant R(u)\\ t\in I\setminus N, d(L)>d(J)}} &\left|\frac{F(J)}{m(J)}m(M\cap J) - F(M\cap J)\right| \\ +\sum_{\Theta,\mathbb{M}} \sum_{\substack{\Delta_k;\ k\geqslant R(u)\\ t\in N, d(L)>d(J)}} &\left|\frac{F(J)}{m(J)}m(M\cap J) - F(M\cap J)\right|,\end{aligned}$$

where

$$\begin{aligned}\Gamma_{23} &\leqslant \Gamma_{26}+\Gamma_{27} \\ &= \sum_{\Theta,\mathbb{M}} \sum_{\substack{\Delta_k;\ k\geqslant R(u)\\ d(J)\geqslant d(L)}} \left|\frac{F(J)}{m(J)} - f(u)\right| m(M\cap J) \\ &\quad +\sum_{\Theta,\mathbb{M}} \sum_{\substack{\Delta_k\\ d(J)\geqslant d(L)}} |f(u)m(M\cap J) - F(M\cap J)|.\end{aligned}$$

Let us estimate Γ_{26}. The partition Δ_k is δ_k-fine so that $d(J) \leqslant \xi_k$ by (3.5) and also $d(L) \leqslant \xi_k$. Moreover, $k \geqslant R(u)$ implies $\xi_k \leqslant \frac{1}{2}\delta_h(u)$ (cf. (3.32)). If a summand in Γ_{26} is nonzero then necessarily $L \cap J \neq \emptyset$, which implies $J \subset V\big(u, d(L)+d(J)\big) \subset V(u, 2\xi_k) \subset V\big(u, \delta_h(u)\big) \subset V\big(u, \omega(u)\big)$ (see (3.5)). Replacing K, t, ν in (2.6) by J, u, $2d(J)$, respectively, we obtain

$$|F(J) - f(u)m(J)| \leqslant \eta\big(2d(J)\big)[2d(J)]^n$$

and taking into account that $m(J) \geqslant A^{n-1}\big(d(J)\big)^n$, we conclude that

$$|F(J) - f(u)m(J)| \leqslant 2^n A^{1-n}\eta\big(2d(J)\big)m(J)$$

and, eventually,

$$\Gamma_{26} \leqslant 2^n A^{1-n}\eta(2\xi_k). \tag{3.39}$$

For the nonvanishing summands of Γ_{27} we again use the fact that $J \cap L \neq \emptyset$, $u \in I \setminus N$, $M \cap J \subset L$, hence (2.9) yields with $t = u$, $K = M \cap J$, $H = L$

$$|f(u)m(M \cap J) - F(M \cap J)| \leqslant A^{1-n}\eta\big(d(L)\big)m(L)$$

since L is A-regular (cf. Convention 3.4). By Lemma 2.5 we find

$$\begin{aligned}\Gamma_{27} \leqslant A^{1-n} \sum_{\Theta,\mathbb{M}} \eta\big(d(L)\big)m(L)\\ \#\{(t,J) \in \Delta_k\,;\ J \cap L \neq \emptyset, d(J) \geqslant d(L)\} \leqslant 3^n A^{2-2n}\eta(2\xi_h)\end{aligned} \tag{3.40}$$

(see (3.34)).

In the sum Γ_{24} we consider only terms with $t \in I \setminus N$, therefore we can use (2.9) replacing H with J and K with J or $J \cap M$. We arrive at the inequalities

$$\begin{aligned}|F(J) - f(t)m(J)| &\leqslant A^{1-n}\eta\big(d(J)\big)m(J),\\ |F(M \cap J) - f(t)m(M \cap J)| &\leqslant A^{1-n}\eta\big(d(J)\big)m(J),\end{aligned}$$

which yield

$$\left|\frac{F(J)}{m(J)}m(M \cap J) - F(M \cap J)\right| \leqslant 2A^{1-n}\eta\big(d(J)\big)m(J)$$

and, eventually, Lemma 2.5 implies

$$\begin{aligned}\Gamma_{24} &\leqslant 2A^{1-n}\sum_{\Delta_k}\eta\big(d(J)\big)m(J)\\ &\qquad \times \#\{(u,L) \in \Theta\,;\ L \cap J \neq \emptyset, d(L) > d(J)\}\\ &\leqslant 2A^{1-n}\sum_{\Delta_k}\eta\big(d(J)\big)m(J) \cdot 3^n A^{1-n}\\ &\leqslant 2 \cdot 3^n A^{2-2n}\eta(\xi_k)\end{aligned} \tag{3.41}$$

since $d(J) \leqslant 2\delta_k(t) \leqslant \xi_k$ by (3.5).

Finally, we write

$$\Gamma_{25} \leqslant \Gamma_{28} + \Gamma_{29} = \sum_{\Theta,\mathbb{M}}\ \sum_{\Delta_k\,;\ t\in N} \frac{|F(J)|}{m(J)}m(M \cap J) + \sum_{\Theta,\mathbb{M}}\ \sum_{\substack{\Delta_k\,;\ t\in N\\ d(L)>d(J)}} |F(M \cap J)|.$$

By (3.7) we have

$$\Gamma_{28} \leqslant \sum_{\Delta_k, t\in N} |F(J)| \sum_{\Theta} \frac{m(L \cap J)}{m(J)} \leqslant \sum_{\Delta_k\,;\ t\in N} |F(J)| \leqslant \xi_k, \tag{3.42}$$

and again by (3.7) and Lemma 2.5 we conclude

$$\begin{aligned}
\Gamma_{29} &\leqslant \sum_{\Delta_k;\ t\in N} \sup\{|F(K)|;\ K\subset J\} \\
&\qquad \times \#\{(u,L)\in\Theta;\ L\cap J\neq\emptyset, d(L)>d(J)\} \\
&\leqslant \xi_k\cdot 3^n A^{1-n}.
\end{aligned} \tag{3.43}$$

Combining (3.36)–(3.43) we obtain

$$\begin{aligned}
\Sigma_2 \leqslant \xi_h &+ A^{1-n}\eta(\xi_k) + A^{1-n}\eta(\xi_h) \\
&+ 2^n A^{1-n}\eta(2\xi_k) + 3^n A^{2-2n}\eta(2\xi_h) \\
&+ 2\cdot 3^n A^{2-2n}\eta(\xi_k) + \xi_k + 3^n A^{1-n}\xi_k.
\end{aligned}$$

Since h satisfies (3.31), it is sufficient to choose l_2 such that

$$(A^{1-n} + 2\cdot 3^n A^{2-2n})\eta(\xi_k) + 2^n A^{1-n}\eta(2\xi_k) + (1+3^n A^{1-n})\xi_k < \frac{1}{2}\varepsilon$$

is satisfied for all $k\geqslant l_2$. (3.11) holds and the proof of Proposition 3.3 is complete, since (3.12) holds by (3.25) and (3.1). □

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