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ON THE NUMBER OF FORESTS(1)

FRANK HARARY, ED PALMER, Ann Arbor (U.S.A.)

In [4], Ore posed the following unsolved problem: „For n given vertices determine the total number of circuit free graphs with m edges." One of us [2] had already found a formula for the generating function which enumerates such graphs. In this note we present a more explicit form of the result. For definitions we refer to [3, 4].

A circuit free graph is simply a forest, i. e. a graph in which each (connected) component is a tree. Let the counting polynomial for forests with p points be

$$f_p(x) = \sum_{q=0}^{p-1} f_{pq}x^q \,, \tag{1}$$

where f_{pq} is the number of forests with p points and q lines. Then the generating function for forests is

$$f(x, y) = \sum_{p=1}^{\infty} y^p f_p(x) \,. \tag{2}$$

To derive formulas for $f_p(x)$ and $f(x, y)$, use is made of the counting series for trees:

$$t(y) = \sum_{p=1}^{\infty} t_p y^p \,, \tag{3}$$

where t_p is the number of trees with p points. Various expressions for t_p and $t(y)$ have been found by Cayley [1], Pólya [6] and Otter [5]. Here are the first ten terms:

$$t(y) = y + y^2 + y^3 + 2y^4 + 3y^5 + 6y^6 + 11y^7 + 23y^8 + 47y^9 + 106y^{10} + \dots \,. \tag{4}$$

The formula in [2] for $f(x, y)$ is obtained by the appropriate application of Pólya's theorem [6] and the following well known combinatorial identity

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for any function $g(x, y)$:

(5) $$1 + \sum_{n=1}^{\infty} Z(S_n, g(x, y)) = \exp \sum_{n=1}^{\infty} \frac{1}{n} g(x^n, y^n).$$

Thus we have the number of forests in terms of the number of trees in the form:

(6) $$1 + f(x, y) = \exp \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{t_k}{n} (x^{k-1}y^k)^n .$$

Using logarithms it is easily seen that this can also be expressed as

(7) $$1 + f(x, y) = \prod_{k=1}^{\infty} (1 - x^{k-1}y^k)^{-t_k},$$

which resembles the form of Cayley's solution [1] for the number of rooted trees.

Now we give a formula for $f_p(x)$ expressed in terms of the numbers t_k.

Theorem. *The counting polynomial for forests with p points is*

(8) $$f_p(x) = \sum_{(j)} \prod_{k=1}^{p} \binom{t_k + j_k - 1}{j_k} x^{(k-1)j_k},$$

and the sum is over all partitions (j) of p.

Proof. Using the familiar identity for combinations with repetition (see [7]), we find that the number of forests consisting of exactly j_k trees, each of which has exactly k points, is the binomial coefficient:

$$\binom{t_k + j_k - 1}{j_k}.$$

Since each of these trees has $k - 1$ lines, we have

(9) $$f_{pq} = \sum_{(j)} \prod_{k=1}^{p} \binom{t_k + j_k - 1}{j_k},$$

where the sum is over those partitions $(j) = (j_1, j_2, \cdots, j_p)$ of p such that

(10) $$q = \sum_{k=1}^{p} (k - 1)j_k$$

The formula (8) for $f_p(x)$ may now be obtained by summing over all partition of p.

For example, using (4) and (8) one easily finds:

$$f_6(x) = 1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5.$$

On multiplying equation (8) by y^p and summing over all positive integers p, one can obtain (6) or (7) by straightforward manipulation.

We conclude by pointing out that the corresponding problem for directed graphs is unsolved. That is, a formula for the number of acyclic digraphs (containing no directed cycles) with a given number of points and lines has not been found. This problem appears to be more difficult.

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The University of Michigan
Department of Mathematics,
Ann Arbor, Michigan, U.S.A.