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ON IRREDUCIBLE GRAPHS OF DIAMETER TWO WITHOUT TRIANGLES

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I. INTRODUCTION

In paper [1] there were given some necessary and sufficient conditions for a graph with a diameter r without k-gons, where $3 \leqslant k \leqslant r + 1$ (so called δ_r-graph), to be extended (or reduced, respectively) by one vertex in order to obtain again a δ_r-graph.

In this paper we give some necessary and sufficient conditions for a graph to be a δ_2-graph, or an η-irreducible δ_2-graph, respectively. It is shown that for every graph G of diameter r, $r \geqslant 2$, without triangles there exists a δ_2-graph H such that G is a section graph of the graph H. A list is given of all μ-irreducible δ_2-graphs with the number of vertices n, $n \leqslant 10$. It is shown that for every natural number n, $n = 3p + 4$, $p \geqslant 2$, there exists a μ-irreducible δ_2-graph in which the minimum degree is 3. Moreover, for every n, $n \geqslant 8$, there exists an η-irreducible δ_2-graph with a minimal degree of vertices 3. For the η-irreducible δ_2-graph there is given a bound for the maximum degree which can be obtained. Finally we give some bounds for the number of edges of a δ_2-graph depending on the number of vertices and the maximum degree of vertices.

II. DEFINITIONS AND DENOTATIONS

We use the concepts and the denotations which are not defined here as we used them in [1]. First of all we repeat some necessary notions and then we define some new notions.

Let $G_1 = (U_1, H_1)$ be a graph and $G = (U, H)$ its subgraph. Let $v \in U_1$. Then we denote $\Omega_{G,G_1}(v) = \{x \mid x \in U \wedge \varrho_{G_1}(x, v) = 1\} \cap U$. Instead of $\Omega_{G,G_1}(v)$ we write $\Omega_G(v)$ if it is clear which of the supergraphs of the graph G is considered. By $\mu(G)$ we denote the set of all μ-sets.

Let us have some graphs $G_1 = (U_1, H_1)$, $G_2 = (U_2, H_2)$; $|U_1| = n$. With every $x_i \in U_1$ there is associated a set $X_i \subset U_2$. Let it denote by $\mathscr{X} = \{X_i\}_{i=1}^{n}$.

Then we define the union $G_1 \oplus G_2$ of graphs G_1, G_2 through the system $\mathscr{X} = \{X_i\}_{i=1}^n$ as a graph $G = (U_1 \cup U_2, H_1 \cup H_2 \cup H')$, where $H' = \{(x_i, z) \mid x_i \in U_1,\ z \in X_i\}$.

Let G' be the graph arising from $G = (U, H)$ by omitting a vertex $v \in U$ and all the edges incident with the vertex v. The vertex v is μ-reducible if $d(G) = d(G')$,

η-reducible if $d(G) = d(G') = 2$ and moreover there exists a vertex $u \in U$, $u \neq v$ such that $\Omega(u) = \Omega(v)$. A graph is $\mu(\eta)$-irreducible if every vertex is not $\mu(\eta)$-reducible, respectively. Let $\gamma_2(G)$ be the system of all kernels of the graph G.

Definition 1. *Let $G = (U, H)$ be the graph and n be a natural number. Let for $i = 1, 2, \ldots, n$ be $X_i \in \gamma_2(G)$. Then the system of kernels $\{X_i\}_{i=1}^n$ is called*

A) *an α-covering of the graph G if*

1. $\bigcup_{k=1}^{n} X_k = U$,

2. *for every two vertices $x, y \in U$ with $\varrho_G(x, y) > 2$ there exists k such that $x, y \in X_k$;*

B) *an α_1-covering of the graph G if the conditions* 1., 2. *hold and moreover*

3. $X_i \neq X_j$ *for* $i \neq j$;

C) *an α_2-covering of the graph G if the conditions* 1., 2. *hold and moreover*

4. *for every $k = 1, 2, \ldots, n$ the system $\{X_i\}_{i=1}^n - X_k$ is not an α-covering of the graph G.*

Remark 1. From definition 1 it follows that every α_2-covering is also an α_1-covering.

Remark 2. The existence of coverings from definition 1 is obvious. It is also clear that for a graph more coverings may exist.

Definition 2. *We call the graph $G = (U, H)$, where $U = \{z_0, z_1, \ldots, z_k\}$, $H = \{(z_0, z_i) \mid i = 1, 2, \ldots, k\}$ a star formed by the set of vertices U and denote it by F_k.*

III. RESULTS

First of all we give some necessary and sufficient conditions for a graph in order to be a δ_2-graph or an η-irreducible δ_2-graph, respectively.

Theorem 1. *Let $G = (U, H)$ be a graph; $z_0 \in U$. Then G is a δ_2-graph if and only if G is the union $F_k \oplus R$ of the graphs F_k, $R = (V, E)$ through the system $\{\pi_i\}_{i=1}^k$ where R is a graph without triangles and F_k is the star formed by the vertex set $A = \{z_i\}_{i=0}^k$, $(A \cap V = \emptyset)$ whereby $\pi_0 = \emptyset$; $\{\pi_i\}_{i=1}^k$ is an α-covering of the graph R.*

(The set of vertices $\pi_i \subset V$ is associated to the vertex $z_i \in A$).

Proof. Let $G = (U, H)$ be a δ_2-graph; $z_0 \in U$. Let us put $A = \{z_0\} \cup \Omega(z_0)$, $R = (V, E)$ where $V = U - A$, $E = \{(x, y) \mid (x, y) \in H;\ x, y \in V\}$. It is clear that R does not contain any triangle. Obviously $\Omega_R(z_0) = \emptyset$. It is easy to verify that the set $\Omega_R(z_i)$, $i = 1, 2, \ldots, k$ is a kernel of the graph R. Let us put $\pi_i = \Omega_R(z_i)$, $i = 1, 2, \ldots, k$. The system $\{\Omega_R(z_i)\}_{i=1}^k$ is an α-covering of the graph R. In the opposite case either $V - \bigcup_{i=1}^{k} \Omega_R(z_i) = M \neq \emptyset$ and then $\varrho_G(x, z_0) > 2$ for $x \in M$ (this is a contradiction with the assumption $d(G) = 2$) or there exist vertices $x, y \in V$ such that $\varrho_R(x, y) > 2$ and $x, y \neq \Omega_R(z_i)$ for all i. Then $\varrho_G(x, y) > 2$, which is also a contradiction.

Now we shall prove that the union $F_k \oplus R$ of such graphs F_k, R is a δ_2-graph. It is obvious that $F_k \oplus R$ does not contain a triangle. Hence, we only need to prove that $d(G) = 2$.

It is clear that $\varrho_G(z_0, x) \leq 1$ for $x \notin V$, Let $x \in V$. Then there exists i, $1 \leqslant \leqslant i \leqslant k$ such that $x \in \Omega_R(z_i)$ and hence $\varrho_G(z_0, x) = 2$. The set $\Omega_R(z_j)$ is a kernel of the graph R, hence $\varrho_G(z_0, x) \leqslant 2$. If $x, y \in V$ and $\varrho_R(x, y) > 2$ then from the definition of an α-covering it follows that there exist i, $1 \leqslant i \leqslant k$ such that $x, y \in \Omega_R(z_i)$ and hence $\varrho_G(x, y) = 2$. It is obvious that $\varrho_G(z_i, x) \leqslant 2$ for all $x \in U$.

Theorem 2. *Let $G = (U, H)$ be a δ_2-graph, $|U| \geqslant 4$. Then G is an η-irreducible δ_2-graph if and only if*

1. *$\{\pi_j\}_{j=1}^k$ is an α_1-covering of the graph $R = (V, E)$,*
2. *$\Omega_R(x) \neq \Omega_R(y)$ for all $x, y \in V$, $x \neq y$,*
3. *$\Omega_R(x) \neq \emptyset$ for every vertex $x \in V$.*

Remark. The denotations in this Theorem are used in the same sense as in Theorem 1.

Proof. Let G be an η-irreducible δ_2-graph. By Theorem 1 $G = F_k \oplus R$. If $\Omega_R(x) = \emptyset$ for $x \in V$, then $\Omega_R(x) = \Omega_R(z_0) = \emptyset$ and so we must have $\Omega_G(x) = \Omega_G(z_0)$ which is a contradiction. If $\Omega_R(x) = \Omega_R(y)$ for $x, y \in V$, $x \neq y$ then we would have $\Omega_G(x) = \Omega_G(y)$, but this is impossible. Hence the system $\{\Omega_R(z_i)\}_{i=1}^n$ is an α_1-covering of the graph R.

Let the conditions 1., 2., 3. be fulfilled. For $i \neq j$ we have $\Omega_G(z_i) \neq \Omega_G(z_j)$ because otherwise it would be $\Omega_R(z_i) = \Omega_R(z_j)$ and that would be a contradiction. It is obvious that $\Omega_G(z_0) \neq \Omega_G(z_i)$; $i = 1, 2, \ldots, k$. For $x \in V$ we have $\Omega_G(z_0) \neq \Omega_G(x)$ because otherwise it would be $\Omega_R(x) = \emptyset$ and it is a contradiction. Hence G is an η-irreducible δ_2-graph.

Assertion 1. *Let $G = (U, H)$ be a μ-irreducible δ_2-graph. Let $\mathscr{X} = \{\pi_j\}_{j=1}^k$, $R = (V, E)$ have the same meaning as in Theorem 2. Then $\mathscr{X}$ is an α_2-covering of the graph R.*

Proof. Let us suppose that $\mathscr{X}$ is not an α_2-covering of the graph R. Then there exists a number m such that $\{\pi_j\}_{j=1}^{k} - \pi_m$ is an α-covering, hence the vertex z_m is a μ-reducible vertex and this is not possible.

Remark 3. The reverse assertion does not hold as the graph in Figure 1 shows.

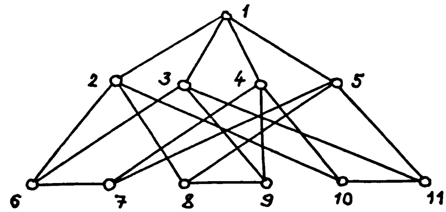

Fig. 1.

The system $\{\{6, 8, 10\}, \{6, 9, 11\}, \{7, 9, 10\}, \{7, 8, 11\}\}$ is an α_2-covering of the graph $R = (V, E)$, where $V = \{6, 7, 8, 9, 10, 11\}$ and $E = \{(6, 7), (8, 9), (10, 11)\}$; but in this graph the vertex 1 is μ-reducible.

Assertion 2. *For every graph G_1 without triangles there exists a δ_2-graph G_2 such that G_1 is a section graph of the graph G_2.*

Proof. The assertion follows immediately from Theorem 1 if we take for the graph R the graph G_1.

Remark 4. Let $G = (U, H)$ be a δ_2-graph and $|U| = n$. Then it is clear that for every vertex $x \in U$ we have $1 \leqslant |\Omega(x)| \leqslant n - 1$, with equalities for the star F_{n-1}.

The following two theorems give some estimations for the degrees of the vertices of η-irreducible δ_2-graphs.

Theorem 3. *Let $G = (U, H)$ be an η-irreducible δ_2-graph. Let us denote $k = \max_{x \in u} |\Omega(x)|$, $|U| = n$. Then $k \leqslant [y]$ where y is the root of the equation $x = 2^{\frac{n-1}{2}} \cdot 2^{\frac{x}{2}}$.*

Proof. According to Theorem 2 we may write $G = F_r \oplus R$, $R = (V, E)$ where z_0 is an arbitrary vertex from U, $r = |\Omega(z_0)|$, $|V| = n - r - 1$. Since the system $\{\Omega_R(z_i)\}_{i=1}^{n}$ is an α_1-covering of the graph R, $r \leqslant |\gamma_2(R)|$ holds. By Theorem 6 from [1], $|\gamma_2(R)| \leqslant 2^{\frac{n-r-1}{2}}$, i. e. $r \leqslant 2^{\frac{n-r-1}{2}}$. Hence $r \leqslant [y]$, where y is the root of the equation $x = 2^{\frac{n-1}{2}} \cdot 2^{-\frac{x}{2}}$.

Remark 5. The values of $[y]$ for some numbers n are in the following table:

n	5	6	7	8	9	10	20	30	40	50
$[y]$	2	2	2	3	4	4	11	20	29	38

Remark 6. There exist η-irreducible δ_2-graphs $S(i)$ with the number of vertices $n = 2^i + 2i + 1$ where i is a natural number, whereby $k = [y]$. We construct the graph $S(i)$ according to Theorem 2 so that $S(i) = F_k \oplus R$, where the graph R consists of i components; every component has one edge and two vertices. It is obvious that $|\gamma_2(R)| = 2^i$ Hence it suffices to put $k = 2^i$. This construction for $i = 3$ is illustrated in Figure 2.

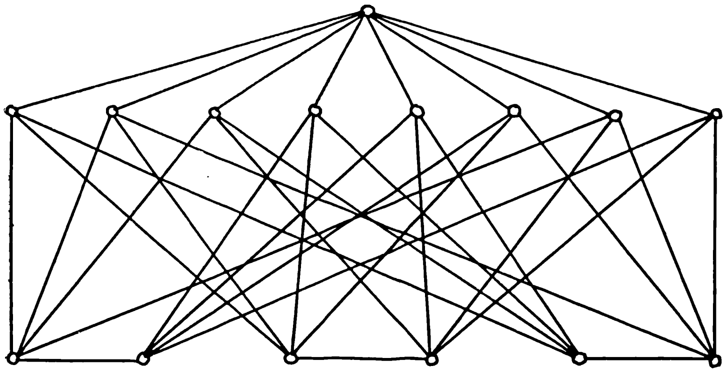

Fig. 2.

Remark 7. For the root y of the equation $x = 2^{\frac{n-1}{2}} \cdot 2^{-\frac{x}{2}}$ the following holds:

a) $\lim\limits_{n\to\infty} \dfrac{y}{n} = \lim\limits_{n\to\infty} \dfrac{y}{y + 2\log y + 1} = 1,$

b) $\lim\limits_{n\to\infty} (n - y) = \lim\limits_{y\to\infty} (y + 2\log y + 1 - y) = \infty.$

Assertion 3. *Let $G = (U, H)$ be an η-irreducible δ_2-graph, $|U| = n$. Let the minimum degree of vertices in G be s. Then:*

a) *If $s = 1$ then G is isomorphic with the star F_2.*

b) *If $s = 2$ then G is isomorphic with the pentagon.*

Proof. a) Let $|\Omega(x)| = 1$; denote $\Omega(x) = \{y\}$. Then for every vertex $z \in U - \{x, y\}$ we have $z \in \Omega(y)$ (because otherwise $\varrho(z, x) > 2$). Hence the graph G is the star F_{n-1} ,which is for every $n > 3$ η-reducible, so the graph G must be isomorphic with the star F_2.

b) Let $|\Omega(x)| = 2$, $\Omega(x) = \{z_1, z_2\}$. Let us denote

$$M_0 = \Omega(z_1) \cap \Omega(z_2),$$
$$M_1 = \{u \mid u \in \Omega(z_1) \wedge u \in \Omega(z_2)\},$$
$$M_2 = \{u \mid u \notin (z_1) \wedge u \in \Omega(z_2)\}$$

It is clear that $x \in M_0$. We have $M_0 = \{x\}$ because if a vertex y would exist such that $y \in M_0$, $y \neq x$ then we would have $\Omega(x) = \Omega(y)$, which is impossible. Let $a \in M_1$. Then $\varrho(a, z) = 1$ holds for every vertex $z \in M_2$ (otherwise it would be $\varrho(a, z) > 2$). Hence $|M_1| = 1$ because G is η-irreducible. Analogously we may prove that $|M_2| = 1$. Thus $M_1 = \{a\}$, $M_2 = \{b\}$, $\varrho(a, b) = 1$ i. e. G is isomorphic with the pentagon.

Theorem 4. *For every natural number N there exists a μ-irreducible δ_2-graph with the minimal degree $s = 3$ and with a number of vertices $n > N$.*

Proof. For every natural number p, $p \geqslant 2$ we construct a μ-irreducible δ_2-graph with $3p + 4$ vertices, (the diagram of this graph is shown in Figure 3).

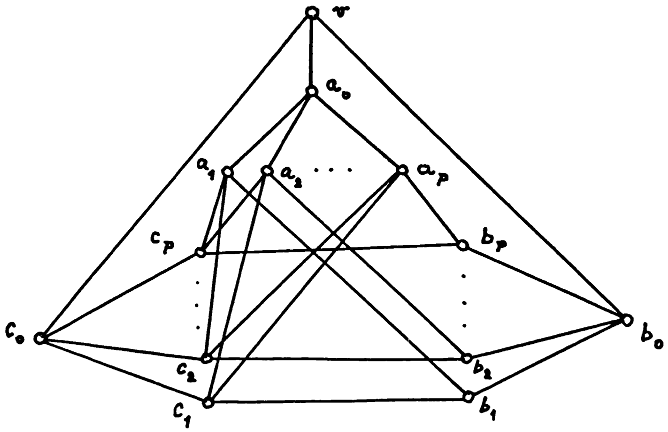

Fig. 3.

We describe the construction of these graphs using the neighbourhoods of the vertices; we denote $A = \{a_i\}_{i=1}^{p}$, $B = \{b_i\}_{i=1}^{p}$, $C = \{c_i\}_{i=1}^{p}$. For every $i = 1, 2, \dots, p$ denote $A_i = A - \{a_i\}$, $C_i = C - \{c_i\}$. $\Omega(a_i) = \{a_0, b_i\} \cup C_i$; $\Omega(b_i) = \{b_0, a_i, c_i\}$, $\Omega(c_i) = \{c_0, b_i\} \cup A_i$. For the remaining vertices we have $\Omega(a_0) = \{v\} \cup A$; $\Omega(b_0) = \{v\} \cup B$; $\Omega(c_0) = \{v\} \cup C$; $\Omega(v) = \{a_0, b_0, c_0\}$. Now we show that every vertex is μ-irreducible. We cannot μ-reduce the vertex

v, because then $\varrho(a_0, b_0) = 3$,

a_0, because then $\varrho(v, a_i) = 3$ for $i = 1, 2, \dots, p$. Analogously we find out that the vertices b_0, c_0 cannost be reduced.

For $i = 1, 2, \dots, p$ we cannot μ-reduce the vertex

a_i, because then $\varrho(a_0, b_i) = 3$,

b_i, because then $\varrho(b_0, c_i) = 3$ and also $\varrho(b_0, a_i) = 3$,

c_i, because then $\varrho(c_0, b_i) = 3$.

Hence, this graph is μ-irreducible, the minimum degree of the vertices is 3 and it is sufficient to put $p > \frac{N-4}{3}$.

Corollary 1. *For every natural number $n = 3k + 4$, where $k \geqslant 2$ is natural there exists a μ-irreducible δ_2-graph with n vertices. Hence there exists an infinite number of μ-irreducible δ_2-graphs, and hence also of η-irreducible δ_2-graphs.*

Theorem 5. *Every μ-irreducible δ_2-graph with n vertices, where $n \leqslant 10$ is isomorphic with one of graphs shown in Figures 4.*

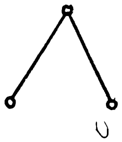

Fig. 4.1.

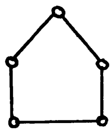

Fig. 4.2.

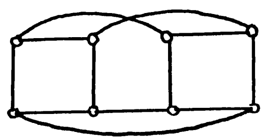

Fig. 4.3.

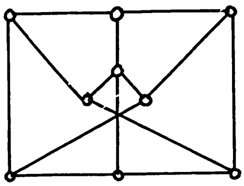

Fig. 4.5.

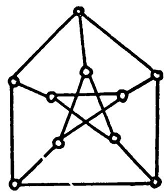

Fig. 4.4.

Proof. By Theorem 2 every μ-irreducible δ_2-graph with at least 4 vertices may be considered as the union $F_k \oplus R$, $R = (V, E)$ through the system of bases $\mathscr{X}$, where R fulfills the conditions 2., 3. of this Theorem and $\mathscr{X}$ is an α_1-covering. By Assertion 3 every η-irreducible δ_2-graph with minimum degree $s < 3$ is isomorphic with a graph in Figure 4.1 or 4.2. Hence it is sufficient to consider minimum degree $s \geqslant 3$. From Theorem 2 it follows that $|V| \leqslant 6$. According to Remark 5, $|\mathscr{X}| \leqslant 4$. By Assertion 1 it is sufficient to take only those $\mathscr{X}$ which are α_2-coverings. From graphs constructed in this way we exclude the μ-reducible and isomorphic graphs.

Theorem 6. *For every natural number n, $n \geqslant 8$ there exists an η-irreducible δ_2-graph with n vertices and minimum degree $s = 3$.*

Proof. For $n = 8$ and 9 there are such graphs on Figure 4.3 and 4.4. In the proof of Theorem 4 we constructed μ-irreducible δ_2-graphs for $n = 3p + 4$, where $p \geqslant 2$. By a μ-extension through the system $A \cup \{b_0, c_0\}$ we obtain η-irreducible δ_2-graphs with $3p + 5$ vertices and by a μ-extension of these graphs through the set $B \cup \{a_0, c_0\}$ there arise η-irreducible δ_2-graphs with $3p + 6$ vertices. Hence we are able to construct an η-irreducible δ_2-graph for all $n \geqslant 8$. It is obvious that the degree of the vertex v (see Figure 3) remains 3.

Corollary 2. *Let $R = (V, E)$ be a graph with n vertices without triangles, fulfilling the conditions 2., 3. from Theorem 2. Let $\mathscr{X}$ be an α_1-covering of the graph R. Then for all $n \geqslant 4$:*

a) $|\mathscr{X}| \geqslant 3$,
b) *there exist R such that $|\mathscr{X}| = 3$.*

Corollary 3.1. *Let G be an η-irreducible δ_2-graph with n vertices and $M \in \mu(G)$. Then $|M| \geqslant 3$ for $n > 5$.*

Corollary 3.2. *For all $n \geqslant 8$ there exists an η-irreducible δ_2-graph with n vertices and $M \in \mu(G)$ such that $|M| = 3$.*

Proof of Corollary 3.1. Let $M \in \mu(G)$ exist such that $|M| = 2$. It follows from Assertion 3 for $n > 5$ that M cannot be the neighbourhood of any vertex. If we μ-extend the graph G through M, we get an η-irreducible δ_2-graph with $n + 1$ vertices $(n + 1 > 6)$ and with minimum degree 2. This is a contradiction with Assertion 3.

Theorem 7. *Let $G = (U, H)$ be a δ_2-graph. Let $|U| = n$, $|H| = m$.*

A. *If $k = \max\limits_{x \in U} |\Omega(x)|$ then $m \leqslant k(n - k)$.*

B. *If $p = \max\limits_{M \in \mu(G)} |M|$ then $m \leqslant p(n - p)$.*

Proof. A. Let for $a \in U$ be $|\Omega(a)| = k$. Since for every two vertices $y_1, y_2 \in \in \Omega(a)$ we have $(y_1, y_2) \notin H$, hence for every $y \in \Omega(a)$ we have $|\Omega(y) \leqslant n - k$ and for the remaining vertices $z \in U$ it is obvious that $|\Omega(z)| \leqslant k$ holds. Thus we may write: $2m = \sum\limits_{x \in U} |\Omega(x)| \leqslant k(n - k) + (n - k)k$, and hence $m \leqslant k(n - k)$.

B. 1. If $p \leqslant \left[\dfrac{n}{2}\right]$ then $k \leqslant p \leqslant \left[\dfrac{n}{2}\right]$ and hence $m \leqslant k(n - k) \leqslant p(n - p)$.

2. We shall prove by induction the assertion for $p \geqslant \left[\dfrac{n+1}{2}\right]$. Our theorem obviously holds for graphs with $|U| = 3$. Let us suppose that the assertion holds for graphs with at most n vertices. Let us consider a graph $G = (U, H)$ where $|U| = n + 1$, $\max\limits_{M \in \mu(G)} |M| = p = |M_1|$, $M_1 \in \mu(G)$. Let y be an arbitrary vertex from M_1. Then there exists at most one vertex $x \in \Omega(y)$ such that the set $(M_1 - \{y\}) \cup \{x\}$ is a μ-set of the graph G. In the reverse case there

exist at least two such vertices x_1, x_2 and one may form a μ-set $N = (M - \{y\}) \cup \cup \{x_1, x_2\}$ with $|N| > p$ which is a contradiction with the assumption.

By omitting the vertex y and the edges incident with it we get the graph G' which we may complete to a δ_2-graph G'' by adding to it some edges. For G'' it is obvious that either $\max_{M \in \mu(G'')} |M| = p - 1$ and because of $|\Omega(y)| \leqslant n + 1 - p$ we have $m \leqslant (p - 1)[n - (p - 1)] + n + 1 - p = p(n + 1 - p)$ or $\max_{M \in \mu(G'')} |M| = p$ and we have $m \leqslant p(n - p) + n + 1 - p = p(n + 1 - p) + (n + 1 - 2p) \leqslant p(n + 1 - p)$, since $n + 1 - 2p \leqslant 0$ for $p > \left[\frac{n}{2}\right]$.

Remark 8. Assertion A from Theorem 7 may be evidently *sharpened* as follows: $m \leqslant k\frac{n}{2}$ if $k \leqslant \frac{n}{2}$. In the proof of Assertion A $|\Omega(y)| \leqslant \min(k, n-k)$ for every $y \in \Omega(a)$; thus $|\Omega(y)| \leqslant k$ for $k \leqslant \frac{n}{2}$. Hence $2m \leqslant k \cdot k + (n - k)k = nk$, i. e. $m \leqslant k\frac{n}{2}$.

Corollary 4. *Let $G = (U, H)$ be a graph without triangles. Let $|U| = n$ $|H| = m$. Then $m \leqslant \frac{n^2}{4}$.* (See also [2]).

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