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NOTE ON THE BAIRE MEASURE

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In the paper we compare two different definitions of the Baire measures from [1] and from [2]. We prove a general theorem on the regularity of a measure (Theorem 3) and as its corollary a theorem on the regularity of a certain type of Baire measures (Theorem 3,2). The paper contains also a theorem on the measurability of Baire sets with respect to a Carathéodory outer measure in general topological spaces (Theorem 2,4).

Let X be a Hausdorff topological space, $\boldsymbol{C}_0$ be the family of all compact G_δ subsets of X, $\boldsymbol{U}_0$ be the family of all open sets belonging to the σ-ring $\boldsymbol{S}(\boldsymbol{C}_0)$ generated by $\boldsymbol{C}_0$, $\boldsymbol{Z}$ be the family of all sets of the form $f^{-1}\{0\}$, where f is a real-valued continuous function on X, $\boldsymbol{A}(\boldsymbol{Z})$ be the σ-algebra generated by $\boldsymbol{Z}$, $\boldsymbol{U}$ be the family of all open sets belonging to $\boldsymbol{A}(\boldsymbol{Z})$, $\boldsymbol{C}$ be the family of all compact sets belonging to $\boldsymbol{A}(\boldsymbol{Z})$. Further, for any family $\boldsymbol{D}$ of subsets of X denote by $\boldsymbol{H}(\boldsymbol{D})$ the hereditary σ-ring generated by $\boldsymbol{D}$.

1. $\boldsymbol{C} = \boldsymbol{C}_0$ in any locally compact Hausdorff topological space X.

According to Theorem 1.2, [2], p. 152 we have $\boldsymbol{C} \subset \boldsymbol{Z}$. Each set of $\boldsymbol{Z}$ is G_δ, hence $\boldsymbol{C} \subset \boldsymbol{C}_0$. From Theorem C, [1], p. 217 it follows that $\boldsymbol{C}_0 \subset \boldsymbol{Z}$ and hence $\boldsymbol{C}_0 \subset \boldsymbol{C}$.

2. In P. R. Halmos' book [1] any measure in a locally compact Hausdorff topological space X defined on $\boldsymbol{S}(\boldsymbol{C}_0)$ and finite on $\boldsymbol{C}_0$ is called to be a Baire measure. The set of all such measures denote by **A**. Each measure from **A** is regular (Theorem G, [1], p. 228).

In paper [2] a Baire measure is understood to be any measure in a topological space X defined on $\boldsymbol{A}(\boldsymbol{Z})$ and finite on $\boldsymbol{C}$. The family of all such measures denote by **B**. If X is a paracompact, locally compact Hausdorff topological space and card X is less than the first unaccessible cardinal number, then each measure of **B** is regular (Theorem 2.1, [2], p. 152).

2,1. These two definitions of the Baire measure coincide (i. e. the domains and the conditions of finiteness for **A** and **B** coincide) if and only if X is a σ-compact, locally compact Hausdorff topological space.

The sufficient condition follows from Theorem 1.1, [2], p. 151 and the above condition for the families $\mathbf{C}$ and $\mathbf{C}_0$. The necessary condition follows evidently from the relations $X \in \mathbf{A}(\mathbf{Z}) = \mathbf{S}(\mathbf{C}_0)$.

2,2. If X is a locally compact Hausdorff topological space and $\mu \in \mathbf{B}$, then the partial function of μ on $\mathbf{S}(\mathbf{C}_0)$ belongs to $\mathbf{A}$.

The assertion follows from the relations $\mathbf{C} = \mathbf{C}_0$ and $\mathbf{S}(\mathbf{C}_0) \subset \mathbf{A}(\mathbf{Z})$.

The reverse of 2,2 holds also in some sense.

2,3. If X is a locally compact Hausdorff topological space, then any measure $\mu \in \mathbf{A}$ can be extended to a measure $\mu_1 \in \mathbf{B}$.

First we prove the following theorem:

2,4. Theorem. (1) *Let X be any topological space, γ be an outer measure defined on the family of all subsets of X with the tollowing property*: $\gamma(A \cup B) = \gamma(A) + \gamma(B)$ *for any* $A, B \subset X$ *such that there are open sets* U, V *with* $\bar{A} \subset U$, $\bar{B} \subset V$ *and* $U \cap V = \emptyset$.

Then each set of $\mathbf{A}(\mathbf{Z})$ *is* γ*-measurable.*

Proof. Define a relation $\mathbf{R}$ on the family of all subsets of X in the following way: $A\mathbf{R}B \Leftrightarrow$ there are open sets U, V such that $\bar{A} \subset U, \bar{B} \subset V, U \cap V = \emptyset$. Evidently $\mathbf{R}$ is a symmetric relation with the following property: If $A\mathbf{R}B$ and $A_1 \subset A$, $B_1 \subset B$, then $A_1\mathbf{R}B_1$. Further evidently $A, B \subset X$, $A\mathbf{R}B \Rightarrow \gamma(A \cup B) = \gamma(A) + \gamma(B)$.

Let $Z \in \mathbf{Z}$. Then according to Theorem 1, [4] (also [7], Theorem 1) the set Z is measurable with respect to γ, if we prove that

$$Z = \bigcap_{n\ 1}^{\infty} V_n,\ V_{n+1} \subset V_n,\ V_1 \subset X,\ Z\mathbf{R}(X - V_n),\ (V_n - V_{n+1})\mathbf{R}V_{n+2}\ n = 1,2, \ldots$$

By the definition of $\mathbf{Z}$ there follows the existence of a non-negative continuous function g on X such that $Z = g^{-1}\{0\} = \bigcap_{n=1}^{\infty} g^{-1}(-\infty, 1/n)$.

Put $V_n = g^{-1}(-\infty, 1/n)$ $n = 1,2, \ldots$ Then $V_{n+1} \subset V_n$ $n = 1,2, \ldots$ Further $\bar{Z} = Z \subset V_{n+1}$, $\overline{X - V_n} = X - V_n \subset \{x : g(x) > 1/(n + 1/2)\} = V$, $V_{n+1} \cap V = \emptyset$, where the sets V_{n+1}, V are open, hence $Z\mathbf{R}V_n$.

For $n = 1,2, \ldots$ we have

$$V_n - V_{n+1} = \{x: 1/(n+1) \leqq g(x) < 1/n\} \subset \{x: 1/(n+1) \leqq g(x) \leqq \leqq 1/n\} \subset \{x: g(x) > 1/(n + 3/2)\} = U.$$

$$V_{n+2} = \{x: g(x) < 1/(n+2)\} \subset \{x: g(x) < 1/(n + 7/4)\} = W.$$

Hence we get $\overline{V_n - V_{n+1}} \subset U$, $\bar{V}_{n+2} \subset W$, $U \cap W = \emptyset$, where U, W are open sets, and hence $(V_n - V_{n+1})\mathbf{R}V_{n+2}$ for all n.

(1) A weaker form of Theorem 2,4 was proved by E. Futaš in his dissertation.

We proved that all sets of $\boldsymbol{Z}$ are γ-measurable and hence all sets of $\boldsymbol{S}(\boldsymbol{Z}) = \boldsymbol{A}(\boldsymbol{Z})$ are γ-measurable too.

Proof of 2,3. Let X be a locally compact Hausdorff topological space and $\mu \in \mathbf{A}$. Let γ be the outer measure on $\boldsymbol{H}(\boldsymbol{S}(\boldsymbol{C}_0)) = \boldsymbol{H}(\boldsymbol{C}_0)$ induced by μ. Then each set of $\boldsymbol{C}_0$ is γ-measurable and by the example 2 from [3], γ is a Carathéodory outer measure on $\boldsymbol{H}(\boldsymbol{C}_0)$, i. e. $\gamma(A \cup B) = \gamma(A) + \gamma(B)$ for any $A, B \in \boldsymbol{H}(\boldsymbol{C}_0)$ such that there are open sets U, V with $\bar{A} \subset U$, $\bar{B} \subset V$ and $U \cap V = \emptyset$.

For $E \subset X$ put

$$\gamma_1(E) = \inf \{\gamma(A) : E \subset A \in \boldsymbol{H}(\boldsymbol{C}_0)\}.$$

Evidently γ_1 is an outer measure on all subsets of X. Let $A, B \subset X$ be such that there are open U, V with $\bar{A} \subset U$, $\bar{B} \subset V$, $U \cap V = \emptyset$. If there is a set $E \in \boldsymbol{H}(\boldsymbol{C}_0)$, $E \supset A \cup B$, then the sets $A, B, A \cup B \in \boldsymbol{H}(\boldsymbol{C}_0)$ and $\gamma_1(A \cup B) = \gamma(A \cup B) = \gamma(A) + \gamma(B) = \gamma_1(A) + \gamma_1(B)$. If there exists no set $E \in \boldsymbol{H}(\boldsymbol{C}_0)$ for which $A \cup B \subset E$, then either there exists no set of $\boldsymbol{H}(\boldsymbol{C}_0)$ containing A, or there exists no set of $\boldsymbol{H}(\boldsymbol{C}_0)$ containing B. In both cases $\gamma_1(A \cup B) = \gamma_1(A) + \gamma_1(B) = \infty$. We have just proved that γ_1 is a Carathéodory outer measure on all subsets of X and by Theorem 2,4 all sets of $\boldsymbol{A}(\boldsymbol{Z})$ are measurable with respect to γ_1. For $E \in \boldsymbol{A}(\boldsymbol{Z})$ put $\mu_1(E) = \gamma_1(E)$. Evidently $\mu_1 \in \mathbf{B}$ and $\mu_1(E) = \mu(E)$ for $E \in \boldsymbol{S}(\boldsymbol{C}_0)$.

2,5. Note. Although 2,3 holds, Theorem G, [1], p. 228 does not follow from Theorem 2,1, [2], since Theorem 2,1 does not hold in general locally compact topological spaces, as it is shown in [2].

2,6. If X is any locally compact Hausdorff topological space, then the extension of $\mu \in \mathbf{A}$ to a measure $\mu_1 \in \mathbf{B}$ need not be unique. Some sufficient conditions for the uniqueness of the extension to a σ-finite measure follow from Theorem 2,1 of paper [2].

Example. Let X be a set of all ordinal numbers less than the first uncountable ordinal number Ω. The set X with the order topology is a locally compact Hausdorff space ([5], example E (e), p. 163). For $E \in \boldsymbol{A}(\boldsymbol{Z})$ put $\mu(E) = 1$, if E contains an uncountable closed set, $\mu(E) = 0$ in the reverse case ([2], example 3,6, p. 159). Denote by ν the restriction of μ on $S(C_0)$. Let γ be the outer measure on $\boldsymbol{H}(\boldsymbol{C}_0)$ induced by ν. For any $E \in \boldsymbol{A}(\boldsymbol{Z})$ put

$$\mu_1(E) = \inf \{\gamma(A) : E \subset A \in \boldsymbol{H}(\boldsymbol{C}_0)\}.$$

Evidently μ and μ_1 are two distinct extensions of ν on $\boldsymbol{A}(\boldsymbol{Z})$, since $\mu(X) = 1$, but $\mu_1(X) = \infty$, as X is not σ-compact.

2,7. Any measure $\mu \in \mathbf{A}$ can be extended to a measure belonging to $\mathbf{B}$ also in another way. We construct the extension μ_λ of the measure μ defined on $[\boldsymbol{S}(\boldsymbol{C}_0)]_\lambda$ in example 1, [6], p. 53. It can be shown that $\boldsymbol{A}(\boldsymbol{Z}) \subset [\boldsymbol{S}(\boldsymbol{C}_0)]_\lambda$

and hence the restriction of μ_λ to $\boldsymbol{A}(\boldsymbol{Z})$ is a measure from $\mathbf{B}$ (μ_λ is finite on $\boldsymbol{C} = \boldsymbol{C}_0$).

As Example 2,6 shows, the extension constructed in the proof of 2,3 and the above extension need not coincide.

3. Theorem. *Let X be an arbitrary set of elements. Let $\boldsymbol{C}_1$ and $\boldsymbol{U}_1$ be families of subsets of X with the following properties:*

V_1 $\emptyset \in \boldsymbol{C}_1, \emptyset \in \boldsymbol{U}_1$.

V_2 *If* $U_n \in \boldsymbol{U}_1$ *for* $n = 1,2, \ldots$, *then also* $\bigcup_{n=1}^{\infty} U_n \in \boldsymbol{U}_1$.

V_3 *If* $C_1, C_2 \in \boldsymbol{C}_1$, *then* $C_1 \cup C_2 \in \boldsymbol{C}_1$.

V_4 $U - C \in \boldsymbol{U}_1$, $C - U \in \boldsymbol{C}_1$ *for any* $U \in \boldsymbol{U}_1$, $C \in \boldsymbol{C}_1$.

V_5 *To any* $C \in \boldsymbol{C}_1$ *there are* $U \in \boldsymbol{U}_1$, $D \in \boldsymbol{C}_1$ *such that* $C \subset U \subset D$.

V_6 $\boldsymbol{U}_1 \subset \boldsymbol{S}(\boldsymbol{C}_1)$, *where* $\boldsymbol{S}(\boldsymbol{C}_1)$ *is the σ-ring generated by* $\boldsymbol{C}_1$.

V_7 *If* $C \in \boldsymbol{C}_1$, *then* $C = \bigcap_{n=1}^{\infty} U_n$, $U_n \in \boldsymbol{U}_1$, $n = 1,2, \ldots$

Let μ be a measure on $\boldsymbol{S}(\boldsymbol{C}_1)$ such that to any $C \in \boldsymbol{C}_1$ there is a sequence $\{U_n\}_{n=1}^{\infty}$ of sets from $\boldsymbol{U}_1$, with $\mu(U_n) < \infty$, $n = 1,2, \ldots$ $\bigcup_{n=1}^{\infty} U_n \supset C$.

Then μ is a regular measure, i. e.

$$\mu(E) = \sup\{\mu(C) : E \supset C \in \boldsymbol{C}_1\} = \inf\{\mu(U) : E \subset U \in \boldsymbol{U}_1\},$$

for all $E \in \boldsymbol{S}(\boldsymbol{C}_1)$.

Proof. Let γ be the outer measure on $\boldsymbol{H}(\boldsymbol{S}(\boldsymbol{C}_1)) = \boldsymbol{H}(\boldsymbol{C}_1)$ induced by μ. Then all sets of $\boldsymbol{C}_1$ are γ-measurable and by Theorem 6 from [3]

$$\gamma(E) = \sup\{\gamma(C) : E \supset C \in \boldsymbol{C}_1\} = \inf\{\gamma(U) : E \subset U \in \boldsymbol{U}_1\}$$

for each $E \in \boldsymbol{S}(\boldsymbol{C}_1)$. Besides $\mu(E) = \gamma(E)$ for $E \in \boldsymbol{S}(\boldsymbol{C}_1)$, from which the assertion of the Theorem follows.

Corollary. *If X, $\boldsymbol{C}_1$, $\boldsymbol{U}_1$ satisfy the assumptions of Theorem 3 and μ is a measure on $\boldsymbol{S}(\boldsymbol{C}_1)$, finite on C_1, then*

$$\mu(E) = \sup\{\mu(C) : E \supset C \in \boldsymbol{C}_1\} = \inf\{\mu(U) : E \subset U \in \boldsymbol{U}_1\}.$$

Proof. If $C \in \boldsymbol{C}_1$, then by V_5 there are $U \in \boldsymbol{U}_1$, $D \in \boldsymbol{C}_1$ such that $C \subset U \subset D$ and hence $\mu(U) \leqq \mu(D) < \infty$.

3,1. Note. From the Corollary we get the assertion of Theorem G, [1]' p. 228, if X is a locally compact Hausdorff topological space, $\boldsymbol{C}_1 = \boldsymbol{C}_0$ and $\boldsymbol{U}_1 = \boldsymbol{U}_0$.

3,2. In paracompact, locally compact Hausdorff topological spaces we get the following theorem on $\mu \in \mathbf{B}$ as a corollary of Theorem 3.

Theorem.(2) *Let X be a paracompact, locally compact Hausdorff topological space. Let* $\mu \in \mathbf{B}$, *and let there exist a sequence* $\{U_n\}_{n=1}^{\infty}$ *of sets from* $\boldsymbol{U}$ *with*

$\mu(U_n) < \infty$ $n = 1,2, \ldots, \bigcup_{n=1}^{\infty} U_n = X$. *Then* μ *is regular, i. e.*

$$\mu(E) = \sup \{\mu(Z) : E \supset Z \in \boldsymbol{Z}\} = \inf \{\mu(U) : E \subset U \in \boldsymbol{U}\}$$

for each $E \in \boldsymbol{A}(\boldsymbol{Z})$.

Proof. Put $\boldsymbol{C}_1 = \boldsymbol{Z}$, $\boldsymbol{U}_1 = \boldsymbol{U}$ and show that $\boldsymbol{C}_1$ and $\boldsymbol{U}_1$ satisfy the properties $V_1 - V_7$.

V_1, V_2, V_5, V_6 evidently hold. Let $Z_1, Z_2 \in \boldsymbol{Z}$, then $Z_1 = f^{-1}\{0\}$, $Z_2 = g^{-1}\{0\}$, where f and g are continuous, real-valued functions on X. Put $h = \min(|f|, |g|)$. Then $h$is a continuous function on X and $Z_1 \cup Z_2 = h^{-1}\{0\}$, i. e. $Z_1 \cup Z_2 \in \boldsymbol{Z}$. We have just proved V_3. If $Z \in \boldsymbol{Z}$, $U \in \boldsymbol{U}$, then $Z - U$ is a closed set of $\boldsymbol{A}(\boldsymbol{Z})$ and $Z - U \in \boldsymbol{Z}$ by Theorem 1.2, [2]. Further $U - Z \in \boldsymbol{A}(\boldsymbol{Z})$, $U - Z$ is an open set and hence $U - Z \in \boldsymbol{U}$. Therefore V_4 holds. It is known that the property V_7 is satisfied too.

REFERENCES

[1] Halmos P. R., *Measure Theory*, New York 1950.

[2] Roos K., Stromberg K., *Baire sets and Baire measures*, Arkiv mat. *6* (1965), 151—160.

[3] Riečanová Z., *Note on absolutely mesaurable sets*, Mat. časop. *18* (1968), 339—345.

[4] Riečan B., *O merateľných množinách v topologických priestoroch*, Čas. pěst. mat. 93 (1968), 1—7.

[5] Kelley J. L., *General Topology*, New York 1955.

[6] Berberian S. K., *Measure and Integration*, New York 1965.

[7] Riečan B., *On measurable sets in topological spaces*, Proc. of the Second Prague Topological Symposium, 1966, 295—296.

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(2) It can be easily shown that a space X and a measure μ satisfying the assumptions of Theorem 2,1 from [2], satisfy the assumptions of Theorem 3,2 too. We cannot show the reverse is false, because K. Kuratowski proved that the existence of an unaccessible cardinal number cannot be proved. The problem of the relation between Theorem 2,1 of [2] and Theorem 3,2 is open.