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# Scorza-Dragoni's Theorem for Unbounded Set-Valued Functions and its Applications to Control Problems

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# SCORZA—DRAGONI'S THEOREM FOR UNBOUNDED SET-VALUED FUNCTIONS AND ITS APPLICATIONS TO CONTROL PROBLEMS*

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## 1. INTRODUCTION

Recently, several authors ([1])—([3]) extended Scorza—Dragoni's theorem ([4]) in different directions, partly for the purpose of existence problems of optimal controls. In this context, the extended versions of Scorza—Dragoni's theorem were used for the so-called Filippov's implicit function lemma, which plays an important role in the proofs of the existence of optimal controls in nonlinear systems.

In this note, we prove an analogue of Scorza—Dragoni's theorem (in a weaker form) for set-valued functions with unbounded values and we show that it can be used for existence theorems of optimal controls in a different direction.

## 2. SCORZA—DRAGONI'S THEOREM FOR SET-VALUED FUNCTIONS

### 2.1. Definitions

*By* $R^n$ *we denote the n-vector space, by* $|.|$ *the Euclidean norm,* $\varrho(X, x) = \inf_{x' \in X} |x' - x|$, $N(x, \delta) = \{x' \mid |x' - x| < \delta\}$, $N(X, \delta) = \{x \mid \varrho(X, x) < \delta\}$, $\alpha(X, Y) = \max \{\sup_{y \in Y} \varrho(X, y), \sup_{x \in X} \varrho(Y, x)\}$, $\operatorname{cl} X$, $\operatorname{co} X$, $\mathscr{F}(X)$, $\mathscr{C}(X)$ *the closure, convex hull, set of non-empty closed subsets, set of non-empty compact subsets of* $X$ *respectively, where* $x, y \in R^n$, $X$, $Y \subset R^n$. *For* $A \subset R^m \times R^n$ *and* $x \in R^m$, $y \in R^n$ *denote* $A^x = \{y \mid (x, y) \in X\}$, $A_y = \{x \mid (x, y) \in X\}$.

A mapping $F : D \to \mathscr{F}(R^n)$, $D \subset R^m$ will be called a set-valued function. $F$ will be called:

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—$\beta$-continuous [$\alpha$-continuous], if for every $x \in D$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for every $y \in N(x, \delta) \cap D$, $F(y) \subset N(F(x), \varepsilon)$ $[\alpha(F(x), F(y)) < \varepsilon]$.

—$\tilde{\beta}$-continuous, if for every $x \in D$, $F(x) = \bigcap_{\delta > 0} \operatorname{cl} F(N(x, \delta) \cap D)$ (or, equivalently, if from $x_k \to x \in D$, $y_k \to y$, $y_k \in F(x_k)$ it follows $y \in F(x)$).

—$\tilde{\alpha}$-continuous, if it is $\tilde{\beta}$-continuous and, moreover, to any $x \in D$, $y \in F(x)$, $x_k \to x$, $x_k \in D$ there are $y_k \in F(x_k)$, $y_k \to y$.

—(Borel)-measurable, if for every closed $Z \subset R^n$ the set $\{x|F(x) \subset Z\}$ is measurable (a Borel set).

2.2. Remarks.

1. $\beta$-continuity and $\tilde{\alpha}$-continuity imply $\tilde{\beta}$-continuity, $\alpha$-continuity implies $\tilde{\alpha}$-continuity.
2. $\tilde{\beta}$-continuity implies measurability.
3. $(\mathscr{C}(R^n), \alpha)$ is a separable metric space; if $F$ has compact values, it is $\alpha$-continuous if and only if $F$ is a continuous mapping of $D$ into the metric space $(\mathscr{C}(R^n), \alpha)$.
4. If all the values of $F$ are contained in a compact subset of $R^n$, then $\tilde{\alpha}$-continuity ($\tilde{\beta}$-continuity) is equivalent with $\alpha$-continuity ($\beta$-continuity).
5. If $D$ is measurable, $F$ is measurable if and only if one of the following properties are satisfied:
   (i) For every $Z$ closed $\{x|F(x) \cap Z \neq \emptyset\}$ is measurable
   (ii) For every $V$ open $\{x|F(x) \cap V \neq \emptyset\}$ is measurable
   (iii) For every $V$ open $\{x|F(x) \subset V\}$ is measurable. A similar statement is true for Borel-measurability. For the proofs, see [10].

2.3. **Lemma.** *Let $D \subset R^m$ and $F : D \to \mathscr{C}(R^n)$ be measurable. Then, for every $Y \in \mathscr{C}(R^n)$ and every $\eta > 0$ the set $E_\eta = \{x|\alpha(F(x), Y) < \eta\}$ is measurable.*

Proof. Let $D \subset \bigcup_{i=1}^{\infty} D_i$, $D_i \subset D$, $D_i \subset \mathscr{C}(R^m)$. Clearly, it suffices to prove that $E_\eta \cap D_i$ is measurable.

Denote by $F_i(x)$ the restriction of $F$ to $D_i$. Then $F_i(x)$ is measurable on a compact set $D_i$ and we can use [8] to prove that to every $\varepsilon > 0$ there is a closed set $D_{i,\varepsilon} \subset D_i$ such that $\mu(D_i - D_{i,\varepsilon}) < \varepsilon$ and $F$ is $\alpha$-continuous on $D_{i,\varepsilon}$. We have for any $x, y \in D_i$

$$\alpha(F_i(x), Y) + \alpha(F_i(x), F_i(y)) \geq \alpha(F_i(y), Y)$$
$$\alpha(F_i(y), Y) + \alpha(F_i(x), F_i(y)) \geq \alpha(F_i(x), Y),$$

from which it follows

$$|\alpha(F_i(x), Y) - \alpha(F_i(y), Y)| \leq \alpha(F_i(x), F_i(y))$$

and, consequently, $\alpha(F_i(x), Y)$ is continuous on $D_{i,\varepsilon}$. Since $\varepsilon > 0$ is arbitrary, $\alpha(F_i(x), Y)$ is measurable on $D_i$. Thus, $E_\eta \cap D_i$ is measurable.

2.4. **Proposition.** *A set-valued function $F : D \to \mathscr{C}(R^n)$ is measurable according to our definition if and only if it is measurable considered as a mapping into the metric space $\mathscr{C}(R^n)$ . $\alpha$).*

Proof. In one direction this proposition is a consequence of the fact that $\{E \mid E \subset Z\}$ for $Z$ closed is a closed subset of the metric space $(\mathscr{C}(R^n), \alpha)$; in the other direction it follows from Lemma 2.3, the separability of $(\mathscr{C}(R^n), \alpha)$ and Remark 2.2.5, (iii).

2.5. **Theorem.** *Let $D = A \times B$, $A \subset R^m$ measurable and bounded, $B \subset \subset \mathscr{F}(R^n)$. Let $F : D \to \mathscr{C}(R^p)$ be such that $F(.,y)$ is measurable for every $y \in B$ and $F(x,.)$ is $\alpha$-continuous for every $x \in A$. Then, for every $\varepsilon > 0$ there is a closed subset $A_\varepsilon \subset A$ such that $\mu(A - A_\varepsilon) < \varepsilon$ and $F \mid A_\varepsilon \times B$ is $\alpha$-continuous.*

Proof. For $A$ closed, this theorem follows from [2, Corollary 2.3] and Proposition 2.4. If $A$ is only measurable, there is a closed subset of $A$ with the measure arbitrarily close to $A$; the application of the above argument to it completes the proof.

This theorem would satisfy the needs of existence problems of optimal control for systems with compact control domains. The following theorem extends Scorza-Dragoni's theorem in a weaker form to set-valued functions with closed convex values. This extension will be shown useful for control problems with unbounded control domains in the following section.

2.6. **Theorem.** *Let $D = A \times B$, where $A \subset R^m$ is measurable and bounded, $B \in \mathscr{F}(R^n)$. Let $F : D \to \mathscr{F}(R^p)$ be such that*

(1) *$F(.,x)$ is measurable for every $x \in B$*

(ii) *$F(t,.)$ is $\tilde{\alpha}$-continuous on $B$ for every $t \in A$*

(iii) *$F(t,x)$ is convex for every $(t,x) \in A \times B$.*

*Then, for every $\varepsilon > 0$ there is a closed subset $A_\varepsilon \subset A$ such that $F \mid A_\varepsilon \times B$ is $\tilde{\beta}$-continuous and $\mu(A - A_\varepsilon) < \varepsilon$.*

Proof. Denote $K_j = \{y \in R^p \mid |y| \leq j\}$ and define $F_j(t,x) = F(t,x) \cap K_j$. $F_j$ is a set-valued function with convex compact values, defined on $D_j = \{(t, x) \mid F(t, x) \cap K_j \neq \emptyset\}$. Obviously $D_{j,x}$ are measurable. From the continuity of $F(t, .)$ there follows $F(t, B) = \text{cl} \bigcup_{j=1}^{\infty} F(t, r_j)$, where $\{r_j\}$ is a countable dense subset of $B$. Consequently, we have for any closed $Z \subset \mathscr{F}(R^n)$

$$\{t \mid F(t, B) \subset Z\} = \{t \mid \text{cl} \bigcup_j F(t, r_j) \subset Z\} = \bigcap_{j=1}^{\infty} \{t \mid F(t, r_j) \subset Z\},$$

which implies that $F(t, B)$ is measurable in $t$. Consequently, $T_j = \{t \mid F(t, B) \cap$

$\cap K_j \neq \emptyset\}$ is measurable. Since $\{t \mid D_j^t \cap Z \neq \emptyset\} = \{t \mid F(t, B) \cap K_j \cap Z \neq \emptyset\}$, it follows from this and Remark 2.2.5 that $D_j^t$ is a measurable set-valued function of $t$ on $T_j$. Finally, from Remark 2.2.4 it follows that $F_j$ is $\alpha$-continuous in $x$ and measurable in $t$ over $D_j$. We shall extend $F_j(t, x)$ to the set $T_j \times B$ so that it will be $\alpha$-continuous in $x$, measurable in $t$ with convex values from $\mathscr{C}(R^p)$.

For $(t, x) \in T_j \times B$, $x \in D_j^t$ denote $\varphi(t, x)$ the point of $F(t, x)$ which is closest to $K_j$. Since $F(t, x)$ is convex, $\varphi(t, x)$ is unique for every $x \in \notin D_j^t$.

If $x$ is a boundary point of $D_j^t$, $F_j(t, x)$ is a one-point set. To prove this, suppose the contrary. Then, $F_j(t, x)$ contains a segment, the center $y$ of which is an interior point of $K_j$. There is a sequence of points $x_k \to x$, $x_k \notin D_j^t$, which means $F(t, x_k) \cap K_j = \emptyset$. Thus, there is no sequence of points $y_k \in F(t, x_k)$ such that $y_k \to x$, which contradicts the $\tilde{\alpha}$-continuity of $F(t, .)$.

Thus, we can extend the definition of $\varphi$ to the set of boundary points of $D_j^t$ by defining $\varphi(t, x)$ as the unique point of $F_j(t, x)$.

Define

$$\tilde{\Phi}_j(t, x) = \begin{cases} F_j(t, x) & \text{for} \quad (t, x) \in D_j \\ \{\varphi(t, x)\} & \text{for} \quad (t, x) \in T_j \times B - D_j \end{cases}$$

$\tilde{\Phi}_j$ is a set-valued function $T_j \times B \to \mathscr{C}(R^p)$.

We prove that $\tilde{\Phi}_j(t, .)$ is $\alpha$-continuous for every $t \in T_j$ and $\tilde{\Phi}_j(., x)$ is mearusable for every $x \in B$. Since $F_j(t, .)$ is $\alpha$-continuous on $D_j^t$, for the continuity of $\tilde{\Phi}_j(t, .)$ on $D_j^t$ it suffices to prove that $\varphi(t, .)$ is continuous on cl $(B - D_j^t)$ for every $t \in T_j$. To prove this, suppose $x_k \in$ cl $(B - D_j^t)$, $x- \to x$. From $\tilde{\alpha}$-continuity of $F(t, .)$ it follows that there is a sequence of points $y_k \in F(t, x_k)$ such that $y_k \to \varphi(t, x)$. This implies $\varrho(K_j, \varphi(t, x)) \geq \limsup_{k\to\infty} \varrho(K_j, \varphi(t, x_k))$. On the other hand, if $z$ is a limit of any convergent subsequence of $\varphi(t, x_k)$, then $z \in F(t, x)$, which implies $\liminf_{k\to\infty} \varrho(K_j, \varphi(t, x_k)) \geq \varrho(K_j, \varphi(t, x))$. Thus, $\varrho(K_j, \varphi(t, x)) = \lim_{k\to\infty} \varrho(K_j, \varphi(t, x_k))$; the continuity of $\varphi(t, .)$ follows from the uniquenes of $\varphi(t, x)$ for every $x$.

For the proof of the measurability of $\tilde{\Phi}_j(., x)$ suppose that $Z$ is a given closed subset of $R^p$. For a given $x \in B$ we have $\{t \mid \tilde{\Phi}_j(t, x) \cap Z \neq \emptyset\} = \{t \mid F(t, x) \cap (Z \cap K_j) \neq \emptyset\} \cup \{t \mid \varphi(t, x) \in Z\}$. The measurability of the first set follows from the measurability of $F$. The measurability of the second one will be proved if we prove that $\varphi(., x)$ is measurable on its domain of definition $T_j - D_{j,x}$. To prove this note that there is a closed set $V_\varepsilon \subset T_j - D_{j,x}$ such that $\mu((T_j - D_{j,x}) - V_\varepsilon) < \varepsilon$ . $F(., x)$ is measurable on $V_\varepsilon$; thus, by [9, Lemma 1] there is a closed subset $W_\varepsilon \subset V_\varepsilon$ such that $F(., x)$ is $\tilde{\alpha}$-continuous on $W_\varepsilon$, and $\mu(V_\varepsilon - W_\varepsilon) < \varepsilon$. From the $\tilde{\alpha}$-continuity of $F(., x)$ on $W_\varepsilon$ it follows that $\varphi(., x)$ is continuous on $W_\varepsilon$. Since $\varepsilon$ is arbitrary and $\mu((T_j - D_{j,x}) - W_\varepsilon) < < 2_\varepsilon$, this proves the measurability of $\varphi(., x)$.

Now, we can apply Theorem 2.5 for the function $\tilde{\Phi}_j \mid T_j \times B$. Thus, for every $\varepsilon > 0$ there is a closed set $T_{j,\varepsilon}$ such that $\mu(T_j - T_{j,\varepsilon}) < 2^{-(j+1)}\varepsilon$ and $\tilde{\Phi}_j \mid T_{j,\varepsilon} \times B$ is continuous. Denote $A_\varepsilon' = A - \bigcup_{j=1}^{\infty} (T_j - T_{j,\varepsilon})$. We have $\mu(A - T_\varepsilon') \leq \sum_{j=1}^{\infty} \mu(T_j - T_{j,\varepsilon}) < \frac{1}{2}\varepsilon$. There is a closed subsst $A_\varepsilon \subset A_\varepsilon'$, such that $\mu(A_\varepsilon' - A_\varepsilon) < \frac{1}{2}\varepsilon$ and, consequently, $\mu(A - A_\varepsilon) < \varepsilon$. We shall prove that $F \mid A_\varepsilon \times B$ is $\tilde{\beta}$-continuous.

Let $(t_k, x_k) \in A_\varepsilon \times B$, $(t_k, x_k) \to (t, x)$, $y_k \in F(t_k, x_k)$, $y_k \to y$. Then $\{y_k\}$ is bounded and, therefore, there is a $j$ such that $|y_k| \leq j$, $|y| \leq j$. Thus, $t_k, t \in T_{j,\varepsilon}$ and from the $\alpha$-continuity of $\tilde{\Phi}_j$ on $T_{j,\varepsilon} \times B$ it follows $y \in F(t, x)$, which completes the proof.

It would be of some interest to prove Theorem 2.6 with the assumption (ii) replaced by some semicontinuity property. Namely, in the applications to control theory semicontinuity is frequently assumed rather than continuity.

The following example shows that this is not possible. It shows a real-valued function of two variables, semicontinuous in each variable which does not have semicontinuous restrictions as in Theorem 2.6. The modification of this example to set-valued functions is straightforward.

2.7. Example. Let $A = B = [0, 1]$, $f: A \times B \to R$ be defined as follows

$$f = \begin{cases} 1 & \text{on } M \\ 0 & \text{elsewhere,} \end{cases}$$

where $M$ is a subset of the diagonal in $A \times B$ such that its projection $M_1$ into $A$ is non-measurable. Obviously, for every $x \in [0, 1]$, $t \in [0, 1]$, $f(., x)$ and $f(t, .)$ are upper semicontinuous.

Assume that for every $\varepsilon > 0$ there is a measurable subset $A_\varepsilon \in A$ such that $\mu(A - A_\varepsilon) < \varepsilon$ and $f \mid A_\varepsilon \times B$ is upper semicontinuous. Then from the definition of $f$, $A_\varepsilon \cap M_1$ is closed and, consequently, $(\bigcup_k A_{1/k}) \cap M_1 = \bigcup_k (A_{1/k} \cap M_1)$ is measurable. Since $M_1 - \bigcup_k A_{1/k} \subset A - \bigcup_k A_{1/k}$ and $\mu(A - \bigcup_k A_{1/k}) = 0$, $M_1 - \bigcup_k A_{1/k}$ is measurable. Thus, $M_1 = [\bigcup_k A_{1/k} \cap M_1] \cup [M_1 - \bigcup_k A_{1/k}]$ is measurable, contrary to our assumption.

## 3. APPLICACION TO CONTROL PROBLEMS

A control problem $(f^0, f, U)$ is given by an equation

$$\dot{x} = f(t, x, u) \tag{1}$$

$x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^m)$, $f = (f^1, \ldots, f^n)$, a scalar cost function $f^0(t, x, u)$ and a control domain $U(t, x)$. Here $U : D - \mathscr{F}(R^m)$ is a set-valued function, $D \subset (R^{n+1})$ is a closed domain, $f$ and $f^0$ are defined on the set

$$\hat{D} = \{(t, x, u) \mid (t, x) \in D, u \in U(t, x)\}.$$

$(f, U)$ will be called control system.

A pair of functions $u(t) : [t_1, t_2] \to R^m$, $x(t) : [t_1, t_2] \to R^n$ will be called a control-trajectory pair ($CT$-pair), if $u(t)$ is measurable, $u(t) \in U(t, x(t))$ for $t \in [t_1, t_2]$, $x(t)$ is absolutely continuous and satisfies $\dot{x}(t) = f(t, x(t), u(t))$ for a.e. $t \in [t_1, t_2]$ . $u(t)$ is called control and $x(t)$ the corresponding trajectory.

Denote

$$\tilde{x} = (x^0, x) \in R^{n+1}, \ \tilde{f} = (f^0, f)$$

and

$$x^0(t) = \int_{t_1}^{t} f^0(s, x(s), u(s)) \, ds$$

for a CT-pair $\langle x(t), u(t) \rangle$.

The control problem $(f^0, f, U)$ will be called lower closed, if it has the following property:

Given a sequence $\langle x_k(t), u_k(t) \rangle$ of CT-pairs on a common interval $[t_1, t_2]$ such that $x_k(t)$ tend uniformly to an absolutely continuous function $x(t)$ and $x_k^0(t)$ tend pointwise to a function $x^0(t)$ of bounded variation, then there is a control $u(t)$ such that

$1^0$ $\langle x(t), u(t) \rangle$ is a CT-pair

$2^0$ $\int_{t_1}^{t} f^0(s, x(s), u(s)) \, ds \leq y^0(t)$, where $y^0(t)$ is the absolutely continuous part of $x^0(t)$.

This definition is a slight modification of the lower closure definition of [9]; it is more appropriate for the application to optimal control existence problems.

For the motivation of this definition and its connection with optimal control existence problem see [5—7, 9].

Denote

$$\tilde{Q}(t, x) = \{(y^0, y) \mid y^0 \geq f^0(t, x, u), \ y = f(t, x, u), \ u \in U(t, x)\}.$$

We show that Theorem 2.6 allows us to modify the continuity assumptions under which the lower closure of control problems is usually proved. This modification is close to the one of [1]; however, Theorem 2.6 allows us to separate more the assumptions on $Q$ from those on $f$ and $U$.

3.1. **Theorem.** *Let $D = T \times B$, $T \subset R^1$ compact, $B \subset R^n$ closed, and suppose that*

(i) *$U$ is Borel-measurable on $D$*

(ii) *$\tilde{f}(.,.,u)$ is Borel-measurable and $\tilde{f}(t,x,.)$ is continuous over $\hat{D}$.*

(iii) *$\tilde{Q}(t,x)$ is convex for every $(t,x) \in D$, $\tilde{Q}(t,.)$ is $\tilde{\alpha}$-continuous and $\tilde{Q}(.,x)$ measurable over $D$*

(iv) *for every compact $K$ there is a constant $\gamma$ such that $f^0(t,x,u) \geq \gamma$ for $(t,x) \in K$, $u \in U(t,x)$ and a nonnegative function $\varphi(\xi)$, $\xi \geq 0$ such that $\lim_{\varrho\to\infty} \xi^{-1}\varphi(\xi) = \infty$ and $f^0(t,x,u) \geq \varphi(|f(t,x,u)|)$ for $|f(t,x,u)|$ sufficiently large and $(t,x) \in K$.*

Then, the control problem $(f^0, f, U)$ is lower closed.

Proof. Let $\langle u_k(t), x_k(t)\rangle$ be a sequence of CT-pairs on $[t_1, t_2] \subset T$, $x_k(t) \to x(t)$ uniformly, $x_k^0(t) \to x^0(t)$ pointwise, $x(t)$ being absolutely continuous and $x^0(t)$ with bounded variation.

By Theorem 2.6, for an arbitrary given $\eta > 0$ there is a closed $T_\eta \subset T$ such that $\mu(T - T_\eta) < \eta$ and $\tilde{Q}(.,.)$ is $\tilde{\beta}$-continuous on $T_\eta \times B$. From this, (iv) and [6, § 2(i)] or [7, Prop. 3] it follows that $\tilde{Q}$ has the property $(Q)$ on $T_\eta \times B$, i.e. for every $t \in T_\eta$,

$$\tilde{Q}(t,x) = \bigcap_{\delta>0} \operatorname{cl} \operatorname{co} \tilde{Q}(N[(t,x),\delta)] \cap [T_\eta \times B]). \tag{2}$$

For $\eta > 0$ sufficiently small, $T_\eta$ is a set of positive measure and therefore almost every its point is its density point, i.e.

$$\lim_{h\to 0} h^{-1}\mu(I(t,h) \cap T_\eta^*) = 0 \tag{3}$$

for a.e. $t \in T_\eta$, where $I(t,h) = [t, t+h]$ and $T_\eta^* = T - T_\eta$.

Denote $T_\eta^0$ the set of points $t \in T_\eta$ for which $\dot{\tilde{x}}(t)$ exists and (3) is valid. We have $\mu(T_\eta - T_\eta^0) = 0$. Further, denote $\tilde{f}(t, x_k(t), u_k(t)) = \psi_k(t)$.

Let $t \in T_\eta^0$, $\alpha = 2\,|\dot{x}^0(t)|$. For $h > 0$ sufficiently small and $k > k'(h)$ sufficiently large we have

$$|x^0(t+h) - x^0(t)| < \tfrac{4}{3}\,|\dot{x}^0(t)\,|\,h = \tfrac{2}{3}\,\alpha h,$$

$$|x_k^0(t+h) - x^0(t+h)| < \tfrac{1}{6}\,\alpha h,\ |x_k^0(t) - x^0(t)| < \tfrac{1}{6}\,\alpha h,$$

which implies

$$|x_k^0(t+h) - x_k^0(t)| < \alpha h.$$

For every $\varepsilon < \frac{1}{2}$ there is a $\delta > 0$ such that for every $0 < h < \delta$ and $k > k(h) \geq k'(h)$ sufficiently large we have

$$\left|\frac{1}{h}[\tilde{x}(t+h) - \tilde{x}(t)] - \frac{1}{h}[\tilde{x}_k(t+h) - \tilde{x}_k(t)]\right| < \varepsilon \tag{4}$$

(5) $$\int_t^{t+h} \psi_k^0(s)\, \mathrm{d}s \leq |x_k^0(t+h) - x_k^0(t)| < \alpha . h$$

(6) $$|x_k(s) - x(t)| < \varepsilon \quad \text{for } s \in [t, t+h]$$

(7) $$\mu(T_\eta^* \cap I(t, h)) < \varepsilon\, .\, h\, .$$

Since the values of $x_k(t)$ and $x(t)$ are all contained in some compact set, from (iv) it follows that $\psi_k^0(s) \geq \gamma > -\infty$ for $s \in T$ and that there is a $\beta > 0$ such that

(8) $$|\psi_k(s)| \leq \varepsilon \psi_k^0(s)$$

for $s \in T$ and $|\psi_k(s)| \geq \beta$.

We have

(9) $$h^{-1}[\tilde{x}_k(t+h) - \tilde{x}_k(t)] = h^{-1} \int_t^{t+h} \tilde{\psi}_k(s)\, \mathrm{d}s = \mu(T_\eta \cap I(t, h))^{-1} .$$

$$\int_{T_\eta \cap I(t,h)} \tilde{\psi}_k(s)\, \mathrm{d}s - \mu(T_\eta^* \cap I(t, h))\, [\mu(T_\eta \cap I(t, h))\, .h]^{-1} .$$

$$\int_{T_\eta \cap I(t,h)} \tilde{\psi}_k(s)\, \mathrm{d}s + h^{-1} \int_{T^*_\eta \cap I(t,h)} \tilde{\psi}_k(s)\, \mathrm{d}s.$$

From (6) follows

(10) $$\mu(T_\eta \cap I(t, h))^{-1} \int_{T_\eta \cap I(t,h)} \tilde{\psi}_k(s)\, \mathrm{d}s \in \mathrm{cl\ co}\, \tilde{Q}(N[(t, x(t)), \varepsilon] \cap [T_\eta \times B]) .$$

From (5), (7), (8) and $\varepsilon < \frac{1}{2}$ follows

(11) $$|\mu(T_\eta^* \cap I(t, h))\, [\mu(T_\eta \cap I(t, h))h]^{-1} \int_{T_\eta \cap I(t,h)} \psi_k(s)\, \mathrm{d}s| \leq$$

$$\leq \mu(T_\eta^* \cap I(t, h))\, [\mu(T_\eta \cap I(t, h))h]^{-1} [\int_{T_\eta \cap I(t,h) \cap \{t \mid |\psi_k(t)| > \beta\}} |\psi_k(s)|\, \mathrm{d}s +$$

$$+ \int_{T_\eta \cap I(t,h) \cap \{t \mid |\psi_k(t)| \leq \beta\}} |\psi_k(s)|\, \mathrm{d}s] \leq 2\varepsilon h^{-1}[\beta h + (\alpha + |\gamma|)h] \leq 2\varepsilon[\beta + \alpha + |\gamma|]$$

(12) $$\mu(T_\eta^* \cap I(t, h))\, [\mu(T_\eta \cap I(t, h))h]^{-1} \int_{T_\eta \cap I(t,h)} \psi_k^0(s)\, \mathrm{d}s \geq 2\varepsilon\gamma .$$

From (5), (7), (8) further follows

(13) $$|h^{-1} \int_{T_\eta^* \cap I(t,h)} \psi_k(s)\, \mathrm{d}s| \leq h^{-1}\varepsilon h(\beta + \alpha + |\gamma|) = \varepsilon(\beta + \alpha + |\gamma|)$$

(14) $$h^{-1} \int_{T_\eta^* \cap I(t,h)} \psi_k^0(s)\, \mathrm{d}s \geq \gamma\varepsilon$$

From (9)—(14) follows

$$h^{-1}[\tilde{x}(t+h) - \tilde{x}(t)] \subset N[\mathrm{cl\ co}\, \tilde{Q}(N[(t, x(t)), \varepsilon] \cap [T_\eta \times B]), \varkappa\varepsilon] =$$

where $\varkappa = 3(\beta + \alpha + |\gamma|)$

By (2), for $h \to 0$ we get

$$\dot{\tilde{x}}(t) = \lim_{h \to 0} h^{-1}[\tilde{x}(t+h) - \tilde{x}(t)] \subset \tag{15}$$

$$\subset \bigcap_{\varepsilon > 0} N[\operatorname{cl} \operatorname{co} \tilde{Q}(N[(t, x), \varepsilon] \cap [T_\eta \times B]), \varkappa\varepsilon] = \tilde{Q}(t, x).$$

Thus, for $t \in T_\eta^0$ the set $P(t) = \{u \mid f(t, x, u) = \dot{x}(t),\ f^0(t, x, u) \leq \dot{x}^0(t)\}$ is non-empty; since $\eta > 0$ is arbitrary, this is true for a.e. $t \in T$.

Now it remains to prove that there is a measurable function $u(t)$ such that $u(t) \in P(t)$ for a.e. $t \in T$, since then by (15), $\langle u(t), x(t) \rangle$ form a CT-pair and

$$\int_{t_1}^{t} f^0(s, x(s), u(s))\, \mathrm{d}s \leq \int_{t_0}^{t} \dot{x}^0(s)\, \mathrm{d}s = y^0(t).$$

The existence of such a $u(t)$ can be proved almost exactly as in [9], the only difference being that for the application of the procedure of [5, pp. 386—385] one has to remove another t-set of arbitrarily small measure so that on the remaining set $\tilde{Q}(.,.)$ is $\tilde{\beta}$-continuous.

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