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A NOTE ON THE EXTENSION OF MEASURES ON LATTICES

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Any measure γ defined on a subalgebra R of a σ-complete Boolean algebra H can be extended to a measure γ on the σ-algebra S generated by R. In paper [3] we generalized this theorem for a type of not necessarily distributive lattices (σ-continuous, orthocomplemented, modular). In the present note we prove with the help of some results of [3] an extension theorem for another type of lattices: σ-continuous, complemented, modular and satisfying the following condition[1]:

(H) To any $x, y, z \in H$ such that $x \leqq y \leqq z$ and any complements x' of x and z' of z there is a complement y' of y such that $x' \geqq y' \geqq z'$.

In the second part of the paper we shall try among other facts to extend the measure with the help of the known method of the induced outer measure and the measurable elements (see [1]). We shall show that this method cannot be used succesfully on certain types of not distributive lattices.

1

We start with some notations and definitions. By $\bigvee_{t \in T} x_t$ we denote the least upper bound of a system $\{x_t\}_{t \in T}$ of elements of H. For a sequence $\{x_n\}_{n=1}^{\infty}$ we write also $\bigvee_{n=1}^{\infty} x_n$, for a finite sequence $\{x_1, \ldots, x_k\}$ also $x_1 \cup \ldots \cup x_k$. Similarly we denote the greatest lower bound.

If $\{x_n\}$ is a non decreasing sequence and $x = \vee x_n$, we write $x_n \nearrow x$. Analoguously $x_n \searrow x$. A σ-complete lattice H is said to be σ-continuous if $x_n \nearrow x$ (resp. $x_n \searrow x$) implies $x_n \cap y \nearrow x \cap y$ (resp. $x_n \cup y \searrow x \cup y$).

A sublattice R of a complemented lattice H is called a ring (only in this paper) if $x \cap y' \in R$ for all $x, y \in R$ and all complements y' of y. A σ-ring is a σ-complete ring in our terminology. A real — valued function γ on a ring R is said to be a measure if the following properties are satisfied:

[1] The problem is open whether any complemented, modular lattice fulfils (H).

(i) If $x_n \nearrow x$, $x_n \in R$ $(n = 1, 2, \ldots)$, $x \in R$, then $\lim \gamma(x_n) = \gamma(x)$.
(ii) $\gamma(x \cup y) + \gamma(x \cap y) = \gamma(x) + \gamma(y)$ for any $x, y \in R$.
(iii) γ is non — negative, $\gamma(0) = 0$.

In [4] we proved (Theorem 4) that the just introduced definition is equivalent (e. g. in a σ-complete, modular, complemented lattice) to the usual definition of a measure as a σ-additive function (see also Part 2).

Lemma. *Let H be a σ-continuous, modular, complemented lattice fulfilling the condition (H), R be a ring, $S(R)$ the σ-ring generated by R and $M(R)$ the monotone set generated by R.*[2] *Then $S(R) = M(R)$.*

Proof. Write $S = S(R)$, $M = M(R)$. $M \subset S$, since S is monotone. In order to prove the opposite inclusion it suffices to prove that M is a ring. Let $x \in R$ be an arbitrary but fixed element. Put $G = \{y \in M\colon x \cap y' \in M$ for each complement y' of $y\}$. Evidently $G \supset R$. We prove that G is monotone.

Let $y_n \in M$ $(n = 1, 2, \ldots)$, $y_n \nearrow y$, y' be a complement of y. According to the condition (H), there is a non increasing sequence $\{y_n'\}$ of complements of elements y_n such that $y_n' \geqq y'$. Put $z = \bigwedge_{n=1}^{\infty} y_n'$. $z \cup y = 1$, since $z \geqq y'$. On the other hand $z \cap y_n \leqq y_n' \cap y_n = 0$. Further $z \cap y = z \cap \bigvee y_n = \bigvee (z \cap y_n) = 0$ since M is σ-continuous. Since M is modular, we get $z = y'$, hence $y' = \bigwedge_{n=1}^{\infty} y_n'$. Since $x \cap y_n' \in M$ $(n = 1, 2, \ldots)$ and $\{x \cap y_n'\}$ is a non increasing sequence, we have $x \cap y' = x \cap \bigwedge y_n' = \bigwedge x \cap y_n' \in M$. Hence we proved that for each complement y' of y we have $x \cap y' \in M$ i. e. $y \in G$. In a similar way it can be proved that G is closed under the limits of non increasing sequences.

Since G is monotone and $G \supset R$, we get $G \supset M$ i. e. $x \cap y' \in M$ for any $x \in R$ and $y \in M$ and any complement y' of y. Take $y \in M$ and put $F = \{x \in M : x \cap y' \in M$ for each complement y' of $y\}$. By the preceding we have $F \supset R$. It can be easily proved that F is monotone, therefore $F \supset M$. Hence for each $x, y \in M$ and each complement y' of y we have $x \cap y' \in M$. Similar arguments show that M is closed under the lattice operations.

Theorem. *Let H be a σ-continuous, complemented, modular lattice satisfying the condition (H). Let $R \subset H$ be a ring, γ be a σ-finite measure on R, S be the σ-ring generated by R. Then there is a σ-finite measure $\bar{\gamma}$ on S that is an extension of γ. The measure $\bar{\gamma}$ is determined uniquely.*

Proof. Suppose first that γ is a finite measure on a ring A. The following assertion follows from Theorem 1 of [4]. There are a sublattice N of H, $N \supset A$

[2] A set M is monotone if it contains the limits of all monotone sequences of elements from M.

and a real function γ^* on N with the following properties: γ^* is an extension of γ, γ^* is finite, non-negative non decreasing; $\gamma^*(x) + \gamma^*(x \cup y) + \gamma^*(x \cap y)$ for all $x, y \in N$. Besides if $x_n \in N$, $x_n \nearrow x$ (resp. $x_n \searrow x$) and $\{\gamma^*(x\text{-})\}$ is bounded, then $x \in N$ and $\gamma^*(x) = \lim \gamma^*(x_n)$.

Let F be the least set over A with the following property:

(α) If $x_n \in F$ ($n = 1, 2, \ldots$), $x_n \nearrow x$ (resp. $x_n \searrow x$) and $\{\gamma^*(x_n)\}$ is bounded, then $x \in F$ and $\gamma^*(x) = \lim \gamma^*(x_n)$.

Evidently $F \subset S(A)$. Put now $\bar{\gamma}(x) = \gamma^*(x)$ for $x \in F$ and $\bar{\gamma}(x) = \infty$ for $x \in S(A) - F$. $\bar{\gamma}$ is non-negative, $\bar{\gamma}(O) = 0$, $\bar{\gamma}$ is an extension of γ. Now we shall prove the following assertion:

$$(*) \quad \text{If } x \leqq y \text{ and } \bar{\gamma}(y) < \infty, \text{ then } \bar{\gamma}(x) < \infty.$$

In fact, put $P = \{z \in M(A) : z \cap y \in F\}$. It can be easily found that P is monotone and $P \supset A$, hence $P \supset M(A) = S(A)$. Therefore $x \in P$, hence $x = x \cap y \in F$ and $\bar{\gamma}(x) = \gamma^*(x) < \infty$.

We get from (*) that γ is non decreasing. Therefore, if $x \in S(A)$, $x_n \nearrow x$, then $\bar{\gamma}(x) \geqq \lim \bar{\gamma}(x_n)$. The equality is evident, if $\lim \bar{\gamma}(x_n) = \infty$ and it follows from the definitions of F and $\bar{\gamma}$ in the reverse case. Similarly the equality $\bar{\gamma}(x \cup y) + \bar{\gamma}(x \cap y) = \bar{\gamma}(x) + \bar{\gamma}(y)$ for all $x, y \in S(A)$ can be proved.

The case of a σ-finite measure can be studied as well as in [3]. Let γ be a σ-finite measure on R. Put $A = \{x \in R : \gamma(x) < \infty\}$, A is a ring. By the preceding we can extend γ to a measure $\bar{\gamma}$ defined on S(A). But $S(A) = S(R)$, which follows from the σ-finitness of γ. (To any $x \in R$ there is a sequence $\{x_n\}$ of elements of A such that $x_n \nearrow x$.) The measure $\bar{\gamma}$ is σ-finite, since the set $P = \{d \in S : d \leqq \bigvee a_n, a_n \in A\}$ is monotone and contains R.

Finally, let τ be any measure on S that is an extension of γ. Since the set $Q = \{x \in S : \tau(x) = \bar{\gamma}(x)\}$ satisfies the property (α) and contains A, we have $Q \supset F$, hence $\bar{\gamma} = \tau$ on F. To any $x \in S$ there is a sequence $\{x_n\}$ of elements of F such that $x_n \nearrow x$. Therefore $\tau(x) = \lim \tau(x_n) = \lim \bar{\gamma}(x_n) = \bar{\gamma}(x)$.

2

First some remarks on additivity. A measure γ is additive if and only if $\gamma(\bigvee_{i=1}^{n} a_i) = \sum_{i=1}^{n} \gamma(a_i)$ for any disjoint sequence $\{a_i\}$. It is natural to say that $\{a_n\}$ is a disjoint sequence if $a_i \cap a_j = O$ for $i \neq j$. In a distributive lattice $a \cap b = O$, $a \cap c = O$ implies $a \cap (b \cup c) = O$. In non distributive lattices this assertion need not hold. This is a reason why we defined in [3] a disjoint sequence as a sequence $\{a_i\}$ for which $\bigvee_{i \in \alpha} a_i \cap \bigvee_{j \in \beta} a_j = O$ whenever α, β are finite disjoint sets of indices.

Of course, there is a modular, non distributive lattice S in which

$$(1)\qquad a \cap b = O,\ a \cap c = O \quad \Rightarrow \quad a \cap (b \cup c) = O,$$

(E. g. put $S = \{O, 1, a, b, c, d\}$, where $O \leqq x \leqq 1$ for all $x \in S$, $d \leqq a, b, c \leqq 1$.) But this is impossible if moreover S is complemented.

Proposition 1. *Any complemented modular lattice S with* (1) *is distributive.*

Proof. It suffices to prove that any $x \in S$ has the unique complement (see [4]). Let b, c be complements of an element a, hence $a \cap b = O$, $a \cap c = O$. Then $b \cup c$ is a complement of a, since $a \cap (b \cup c) = O$. As S is modular, we get $b \cup c = b$. Similarly $b \cup c = c$, hence $b = c$.

In modular lattices we can work very well with a valuation, i. e. with a function v, for which

$$(2)\qquad v(a \cup b) + v(a \cap b) = v(a) + v(b).$$

If $a \cap b = O$ and $v(O) = 0$, then the additivity follows from (2), but only for two elements. The additivity, e. g., for three elements connects with the following property:

$$(3)\qquad v(a \cup b \cup c) = v(a) + v(b) + v(c) - v(a \cap b) - v(a \cap c) - \\ - v(b \cap c) + v(a \cap b \cap c).$$

Proposition 2.[3] *If S is a distributive lattice and v is a valuation on S, then* (3) *holds for any $a, b, c \in S$.*

Proof. $v((a \cup b) \cup c) = v(a \cup b) + v(c) - v((a \cup b) \cap c) =$
$= v(a) + v(b) - v(a \cap b) + v(c) - v((a \mathscr{X} c) \cup (b \cap c)) =$
$= v(a) + v(b) + v(c) - v(a \cap b) - v(a \cap c) - v(b \cap c) +$
$+ v(a \cap b \cap c)$.

Proposition 3.[4] *If S is a lattice, if v satisfies* (3) *for any a, b, c and v is a positive valuation* (i. e. $a < b \Rightarrow v(a) < v(b)$), *then S is a distributive lattice.*

Proof. Evidently, v satisfies also (2) (put $a = c$). Hence, applying (3) and then (2) (twice), we get

$$v(a \cup b \cup c) = v(a \cup b) + v(c) - v((a \cap c) \cup (b \cap c)).$$

On the other hand

$$v(a \cup b \cup c) = v(a \cup b) + v(c) - v((a \cup b) \cap c).$$

From these two relations we have

$$v((a \cap c) \cup (b \cap c)) = v((a \cup b) \cap c).$$

[3] Of course, formula (3) can be easily generalised for any finite number of elements.
[4] See also [4].

Since $(a \cap c) \cup (b \cap c) \leqq (a \cup b) \cap c$ and v is positive, there is $(a \cup b) \cap c = (a \cap c) \cup (b \cap c)$ for any $a, b, c \in S$.

We have just been studying two examples in which the distributive law plays a central role. It seems that a similar situation exists also in our main problem.

We start with a lattice S and a map $x \to x^*$ of S into S. We present two formulations of measurability. Let γ be an arbitrary real — valued function on S. By M_1 denote the set of all elements a such that

$$\gamma(e) = \gamma(e \cap a) + \gamma(e \cap a^*) \tag{4}$$

for any $e \in S$. By M_2 denote the set of all elements a with the following property:

$$\gamma(p \cup q) = \gamma(p) + \gamma(q) \tag{5}$$

as soon as $p, q \in S$, $p \leqq a$, $q \leqq a^*$. First we compare these two concepts.

Proposition 4. *If S is a modular lattice with the least element O and $a \cap a^* = O$ for any $a \in S$, then $M_1 \subset M_2$. If S is an arbitrary lattice in which*

$$e = (e \cap a) \cup (e \cap a^*) \quad \textit{for any} \quad e, a \in S, \tag{6}$$

then $M_2 \subset M_1$.

Proof. In the first case take $a \in M_1$, $p \leqq a$, $q \leqq a^*$. Then $(p \cup q) \cap a = p \cup (q \cap a) \leqq p \cup (a^* \cap a) = p$. Similarly $(p \cup q) \cap a^* = q$. If we put $e = p \cup q$ into (4) we obtain (5). In the second case it suffices to put $p = e \cap a$, $q = e \cap a^*$ into (5) and to notice that $e = p \cup q$.

Proposition 5. *Let S be a lattice with a map $x \to x^*$ having the following properties*: (6),

$$a \cap (a \cap b)^* \leqq b^*, \tag{7}$$

$$a \leqq b \quad \Rightarrow \quad a^* \geqq b^*. \tag{8}$$

Then M_2 is a sublattice of S.

Proof. Notice first that for any $a, b \in S$ we have by (6)

$$(a \cup b) \cap b^* \leqq (a \cup b) \cap (a \cup b^*) = a. \tag{9}$$

Take $a, b \in M_2$ and $p \leqq a \cup b$, $q \leqq (a \cup b)^*$. By (6) we obtain $p \cup q = (p \cap b) \cup [(p \cap b^*) \cup q]$. Since $p \cap b \leqq b$, $(p \cap b^*) \cup q \leqq b^* \cup q \leqq b^* \cup (a \cup b^* \leqq b)^* \cup b^* = b^*$ (by (8)) and $p \cap b^* \leqq a$ (by (9)) $q \leqq a^*$ (by (8)), we have

$$\gamma(p \cup q) = \gamma(p \cap b) + \gamma((p \cap b^*) \cup q) = \gamma(p \cap b) + \gamma(p \cap b^*) + \gamma(q) = \\ = \gamma((p \cap b) \cup (p \cap b^*)) + \gamma(q),$$

hence $a \cup b \in M_2$.

Take now any $r \leqq a \cap b$, $s \leqq (a \cap b)^*$. Hence $r \cup s = [r \cup (s \cap a)] \cup$

$\cup (s \cap a^*)$. As $r \cup (s \cap a) \leqq a$, $s \cap a^* \leqq a^*$, $s \cap a \leqq a \cap (a \cap b)^* \leqq b^*$ (by (7)) and $r \leqq b$, we have

$$\gamma(r \cup s) = \gamma(r) + \gamma(s \cap a) + \gamma(s \cap a^*) = \gamma(r) + \gamma(s),$$

hence also $a \cap b \in M_2$.

Unfortunately, we cannot continue in our considerations, because we do not know any example of a non distributive lattice satisfying all the assumptions of Proposition 5. E. g., if S is a modular, orthocomplemented lattice, and $x^* = x^{\top}$ (the orthocomplement, see [2]) then the conditions (7) and (8) are satisfied. But any modular, orthocomplemented lattice in which (6) holds, is distributive ([2], p. 227, Note 1,1).

The purpose of this theory is to obtain an additive function γ on measurable elements. The following two properties are interesting in this connection. Although in Part 1 and in [3] we obtained some extension theorems in complemented, resp. orthocomplemented lattices, the following propositions show that the corresponding results cannot be obtained by extending γ to the induced outer measure $\bar{\gamma}$ and then by restricting $\bar{\gamma}$ to the measurable elements.

Proposition 6. *Let S be a modular, orthocomplemented lattice. Let M_1 be a sublattice and $a, b \in M_1 \Rightarrow a \cap b^{\perp} \in M_1$. Let γ be additive and increasing on M_1. Then M_1 is a distributive lattice.*

Proof. As $a, b \in M_1$, we have $\gamma(a) = \gamma(a \cap b) + \gamma(a \cap b^{\perp})$. By the additivity of γ we get $\gamma(a) = \gamma((a \cap b) \cup (a \cap b^{\perp}))$, hence $a = (a \cap b) \cup (a \cap b^{\perp})$. Also $b^{\perp} \in M_1$, since $(b^{\perp})^{\perp} = b$. The distributivity follows now from the known results ([2]).

Proposition 7. *Let S be a modular, complemented lattice. Let $M_3 = \{b \in S : \gamma(a) = \gamma(a \cap b) + \gamma(a \cap b') \text{ for all complements } b' \text{ of } b\}$. Let M_3 be a sublattice of S and $b \in M_3 \Rightarrow b' \in M_3$ for all complements b' of b. Let γ be additive and increasing on M_3. Then M_3 is a distributive lattice.*

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