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ON THE AVERAGE ORDER OF AN ARITHMETICAL FUNCTION

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In paper [1] the following conjecture is stated: Let the system of the arithmetical progressions

$$\ldots, a_i - n_i, a_i, a_i + n_i, \ldots \; i = 1, 2, \ldots, k$$

have the property that every integer belongs to exactly one of these progressions. If

$$n_i = \prod_{t=1}^{r} p_t^{\lambda_t} \text{ (the standard form of } n_i\text{)},$$

then

$$k \geqq 1 + \sum_{t=1}^{r} \lambda_t (p_t - 1).$$

This conjecture has been proved in [5].

In solving this problem we have been led to the following arithmetical function. For

$$N = \prod_{t=1}^{r} p_t^{\lambda_t} \quad \text{put} \quad f(N) = \sum_{t=1}^{r} {}^{\lambda_t}(p_t - 1).$$

It is easy to see that f has the property: $f(N_1 \cdot N_2) = f(N_1) + f(N_2)$ for arbitrary integres N_1, N_2. Further, we have the trivial estimations:

$$\log_2 N \leqq f(N) \leqq N - 1.$$

The present paper is devoted to the study of the average order of this function. We prove the following theorem:

Theorem. $\lim_{n=\infty} \frac{1}{N} [f(1) + f(2) + \ldots + f(N)] = \frac{\pi^2}{12} \frac{N}{\log N}.$

Proof. It is easy to check the following relation:

$$S(f, N) = f(1) + f(2) + \ldots + f(N) = \sum_{p \leqq N} (p - 1) \left(\left[\frac{N}{p} \right] + \left[\frac{N}{p^2} \right] + \ldots \right).$$

Obviously $S(f, N) = S^{(1)}(N) + S^{(2)}(N)$, where

$$S^{(1)}(N) = \sum_{p \leq \sqrt{N}} (p-1)\left(\left[\frac{N}{p^2}\right] + \left[\frac{N}{p^3}\right] + \dots\right),$$

$$S^{(2)}(N) = \sum_{p \leq N} (p-1)\left[\frac{N}{p}\right].$$

For $S^{(1)}(N)$ we have the estimation:

$$S^{(1)}(N) \leqq \sum_{p \leq \sqrt{N}} (p-1)\left(\frac{N}{p^2} + \frac{N}{p^3} + \dots\right) = N \sum_{p \leq \sqrt{N}} \frac{1}{p} \leqq \frac{1}{2} N\pi(\sqrt{N}),$$

where $\pi(N)$ is the prime number function.
Hence we get (by prime number theorem):

$$S^{(1)}(N) = o(N^2/\log N). \tag{1}$$

Further, $S^{(2)}(N) = P(N) - Q(N)$, where

$$P(N) = \sum_{p \leq N} p\left[\frac{N}{p}\right], \quad Q(N) = \sum_{p \leq N} \left[\frac{N}{p}\right].$$

Obviously $Q(N) \leqq N \sum_{p \leq N} \frac{1}{p}$ and since $\sum_{p \leq N} \frac{1}{p} = O(\log\log N)$ (see [2], p. 28), we obtain

$$Q(N) = o(N^2/\log N). \tag{2}$$

By (1) and (2) we have

$$\frac{S(f, N)}{N} = o\left(\frac{N}{\log N}\right) + \frac{P(N)}{N},$$

so that in order to prove our theorem it is sufficient to show

$$\liminf_{N\to\infty} \frac{P(N)\log N}{N^2} \geqq \frac{\pi^2}{12}, \tag{a}$$

$$\limsup_{N\to\infty} \frac{P(N)\log N}{N^2} \leqq \frac{\pi^2}{12}. \tag{b}$$

Let ε be an arbitrary number of the interval $(0, 1)$. Let

$$p_1 < p_2 < \ldots < p_k < \ldots$$

be the increasing sequence of all primes. Since $p_k \sim k \log k$ $(k \to \infty)$* (see [3] p. 153), there exists an integer $N_0 = N_0(\varepsilon) > 1$ so that for any natural $k > > \pi(N_0)$

$$(1 - \varepsilon)k \log k < p_k < (1 + \varepsilon)k \log k \tag{3}$$

holds. Choose an integer j such that

$$\sum_{k=1}^{j} \frac{1}{k^2} - \frac{j}{(j+1)^2} > \frac{\pi^2}{6} - \frac{\varepsilon}{2}, \tag{4}$$

$$\sum_{k=1}^{j} \frac{1}{k^2} + \frac{j+2}{(j+1)^2} < \frac{\pi^2}{6} + \frac{\varepsilon}{2}.$$

holds. This is clearly possible. In the following we suppose that $N > > N_0(\varepsilon)(j+1)$. $P(N)$ can be written in the form

$$P(N) = P_1(N) + \ldots + P_j(N) + P_j^*(N), \tag{5}$$

where

$$P_k(N) = \sum_{\frac{N}{k+1} < p \leqq \frac{N}{k}} p \left[\frac{N}{p}\right] \qquad (k = 1, 2, \ldots, j)$$

and

$$P_k(N) = \sum_{1 < p \leqq \frac{N}{j+1}} p \left[\frac{N}{p}\right].$$

Obviously

$$P_j^*(N) \leqq \sum_{1 < p \leqq \frac{N}{j+1}} N = N\pi\left(\frac{N}{j+1}\right) = \pi\left(\frac{N}{j+1}\right) \frac{\log \frac{N}{j+1}}{\frac{N}{j+1}} \cdot \frac{N^2}{\log \frac{N}{j+1}} \cdot \frac{1}{j+1}.$$

and hence (by prime number theorem) we get

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* $h(k) \sim g(k)$ denotes that $\lim_{k \to \infty} \frac{h(k)}{g(k)} = 1$.

$$(6) \qquad 0 \leqq \liminf_{N\to\infty} \frac{P_j^*(N)\log N}{N^2} \leqq \limsup_{N\to\infty} \frac{P_j^*(N)\log N}{N^2} \leqq \frac{1}{j+1}.$$

If $\frac{N}{k+1} < p \leqq \frac{N}{k}$, then $\left[\frac{N}{p}\right] = k$, hence

$$P_k(N) = k \,.\, \sum_{\frac{N}{k+1} < p \leqq \frac{N}{k}} p = k \,.\, \sum_{\pi\left(\frac{N}{k+1}\right) < r \leqq \pi\left(\frac{N}{k}\right)} p_r.$$

This together with (3) implies the inequalities:

$$(7) \quad (1-\varepsilon)k \sum_{\pi\left(\frac{N}{k+1}\right) < r \leqq \pi\left(\frac{N}{k}\right)} r \log r < P_k(N) < (1+\varepsilon)k \sum_{\pi\left(\frac{N}{k+1}\right) < r \leqq \pi\left(\frac{N}{k}\right)} r \log r.$$

The function $\psi(t) = t \log t$ is a non-negative and increasing function in the interval (e, ∞), which satisfies $\psi(x+1) = O(\psi(x))$ $(x \to \infty)$. By theorem 4 from [4] (p. 8) we have

$$(8) \qquad \sum_{\pi\left(\frac{N}{k+1}\right) < r \leqq \pi\left(\frac{N}{k}\right)} r \log r = \int_{\pi\left(\frac{N}{k+1}\right)}^{\pi\left(\frac{N}{k}\right)} t \log t \, \mathrm{d}t + O\left(\pi\left(\frac{N}{k}\right) \log \pi\left(\frac{N}{k}\right)\right),$$

$$(9) \qquad \int_{\pi\left(\frac{N}{k+1}\right)}^{\pi\left(\frac{N}{k}\right)} t \log t \, \mathrm{d}t = \frac{1}{2}\left[\pi^2\left(\frac{N}{k}\right) \log \pi\left(\frac{N}{k}\right) - \pi^2\left(\frac{N}{k+1}\right) \log \pi\left(\frac{N}{k+1}\right)\right] - \frac{1}{4}\left[\pi^2\left(\frac{N}{k}\right) - \pi^2\left(\frac{N}{k+1}\right)\right].$$

By an easy calculation we get from (7), (8) and (9):

$$(10) \qquad (1-\varepsilon)k \,.\, F(N, k) < P_k(N) < (1+\varepsilon)k \,.\, F(N, k),$$

where

$$F(N, k) = \frac{1}{2}\left[\pi^2\left(\frac{N}{k}\right) \log \pi\left(\frac{N}{k}\right) - \pi^2\left(\frac{N}{k+1}\right) \log \pi\left(\frac{N}{k+1}\right)\right] - \frac{1}{4}\left[\pi^2\left(\frac{N}{k}\right) - \pi^2\left(\frac{N}{k+1}\right)\right] + O\left[\pi\left(\frac{N}{k}\right) \log \pi\left(\frac{N}{k}\right)\right].$$

Considering the prime number theorem it is easy to check that for every fixed $v > 0$ the relations

$$\pi^2\left(\frac{N}{v}\right)\log\pi\left(\frac{N}{v}\right) \sim \frac{1}{v^2}\frac{N^2}{\log N}\ (N \to \infty) \tag{11}$$

and

$$\pi^2\left(\frac{N}{v}\right) = o\left(\frac{N^2}{\log N}\right)\ (N \to \infty) \text{ hold.} \tag{12}$$

Further, we can see in the same way that for any fixed $k > 0$

$$O\left(\pi\left(\frac{N}{k}\right)\log\pi\left(\frac{N}{k}\right)\right) = o\left(\frac{N^2}{\log N}\right)\ (N \to \infty) \text{ holds.} \tag{13}$$

On account of (11), (12) and (13) we obtain for the function $F(N, k)$ the following relation (k is fixed)

$$\lim_{N\to\infty}\frac{F(N,k)\log N}{N^2} = \frac{1}{2}\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) \tag{14}$$

According to (4), (5), (6), (13) and (14) we have

$$\liminf_{N\to\infty}\frac{P(N)\log N}{N^2} \geqq \sum_{k=1}^{j}\liminf_{N\to\infty}\frac{P_k(N)\log N}{N^2} \geqq$$

$$\frac{1-\varepsilon}{2}\sum_{k=1}^{j} k\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) = \frac{1-\varepsilon}{2}\left(\sum_{k=1}^{j}\frac{1}{k^2} - \frac{j}{(j+1)^2}\right) >$$

$$> (1-\varepsilon)\left(\frac{\pi^2}{12} - \frac{\varepsilon}{4}\right);$$

$$\limsup_{N\to\infty}\frac{P(N)\log N}{N^2} \leqq \limsup\frac{P(N)\log N}{N^2} + \frac{1}{j+1} \leqq$$

$$\leqq \frac{1+\varepsilon}{2}\sum_{k=1}^{j} k\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) + \frac{1}{j+1} = \frac{1+\varepsilon}{2}\left[\sum_{k=1}^{j}\frac{1}{k^2} -\right.$$

$$\left. - \frac{j}{(j+1)^2} + \frac{2}{1+\varepsilon}\frac{1}{j+1}\right] < \frac{1+\varepsilon}{2}\left[\sum_{k=1}^{j}\frac{1}{k^2} - \frac{j}{(j+1)^2} + \frac{2}{j+1}\right] =$$

$$= \frac{1+\varepsilon}{2}\left[\sum_{k=1}^{j} \frac{1}{k^2} + \frac{j+2}{(j+1)^2}\right] < (1+\varepsilon)\left(\frac{\pi^2}{12} + \frac{\varepsilon}{4}\right).$$

Since the last inequalities are valid for arbitrary $\varepsilon \in (0, 1)$, we have proved the inequalities (a) and (b). Hence we have

$$\lim_{N\to\infty} \frac{S(f, N)}{N} \cdot \frac{\log N}{N} = \frac{\pi^2}{12},$$

which completes the proof of our theorem.

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