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LINEAR CONGRUENCES IN DISTRIBUTIVE LATTICES

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The structure of linear congruences in distributive lattices is studied in this paper. The problem of equality of the local and lattice dimensions of a distributive lattice is solved affirmatively by Theorem 5.7 in the case when the dimensions (at least one of them) are finite. (The negative answer in the infinite case has been shown already in [1]).

0. Preliminary

We shall use the denotation and the terminology introduced in [1] with the following additions.

By a relation we mean a set of ordered pairs; if R is a relation, then $\mathrm{D}(R)$ is the set of all first elements of pairs from R. If for any $x \in \mathrm{D}(R)$ there is at most one element y such that $\langle xy \rangle \in R$, then R is a function. A congruence Q on a lattice L is called linear, if $\langle xy \rangle \in Q$ implies that x, y are comparable. Two congruences on L are orthogonal, if their intersection is the identity on L. The (only) maximal congruence orthogonal to a congruence Q on L will be denoted $Q^{\perp}$. By $+$, $\sum$ we shall denote the join of congruences in the distributive lattice of all congruences on L, while $\cup$, $\bigcup$ will denote the set-theoretical join. We say that an ordered set A satisfies the condition of maximality, if every chain in A has an upper bound in A.

In the lattice L, the complement $L - X$ of a subset X of L will be denoted by $-X$, the set of all lower (upper) bounds of X is denoted by $\mathscr{L}(X)(\mathscr{U}(X))$. Convex subsets r, p of L are called projective, if any interval $[xy] \subseteq r$ is projective with some $[uv] \subseteq p$ and conversely. If a, b are subsets of L and for any $y \in a$, $z \in b$ we have $y \wedge z = x$, then we say that a, b are orthogonal over x. A subset b of L is called an independent system over x, if $x \notin b$ and if for any $y \in b$ the sets $\{y\}$, $b - \{y\}$ are orthogonal over x. Then, denoting $\operatorname{card} b = \mathfrak{k}$ ($\mathfrak{k}$ need not be finite), we say also that b is a $\mathfrak{k}$-system over x, or a lower $\mathfrak{k}$-system (in L) and x is a lower $\mathfrak{k}$-element (in L). If r is a convex chain in L and if every $x \in r$, which is not the greatest element in r, is a lower $\mathfrak{k}$-element in L, then r is called a lower $\mathfrak{k}$-chain (in L). The notions of the upper $\mathfrak{k}$-element and the upper $\mathfrak{k}$-chain are defined dually. Thus, according to [1], lodim $L = \sup\{\mathfrak{k};$ there is a lower or upper $\mathfrak{k}$-element in $L\}$ and if lodim L is finite, lodim $L = \sup\{\mathfrak{k};$ there is a lower $\mathfrak{k}$-element in $L\}$. We remember also that the lattice dimension of L, ldim L, is a cardinal $\mathfrak{k}$ if L is a subdirect product of $\mathfrak{k}$ chains but not a subdirect product of less than $\mathfrak{k}$ chains, that lodim $L \leqslant$ ldim L and that if lodim L is finite, then lodim $L \leqslant$ $\leqslant$ dim $L \leqslant$ ldim L holds.

1. Convex chains

1.0. In this section let r be a convex chain in a distributive lattice L. We define two congruences, one of them „parallel" to r, the other congruence orthogonal to the first one.

1.1. Let us define a relation $\bar{r}$ between elements of the lattice L and of the chain r as follows: $\langle ty\rangle \in \bar{r}$ if

(i) $x \vee (y \wedge t) = y$ for every $x \in r,\ x \leqslant y$,

(ii) $z \wedge (y \vee t) = y$ for every $z \in r,\ y \leqslant z$.

1.2. *The condition* 1.1(i) *is satisfied if and only if*

(i) *there is no* $x \in r,\ x < y$, *or*

(ii) *there is* $x \in r,\ x < y,\ x \vee (y \wedge t) = y$.

Proof. Evidently (i) implies 1.1(i). Let 1.2(ii) hold and let be $x_1 \in r,\ x_1 \leqslant y$. We cannot have $x \geqslant x_1 \vee (y \wedge t)$, because it would give $x \geqslant y \wedge t,\ x \vee$ $\vee (y \wedge t) = x \neq y$. As $x_1 \leqslant x_1 \vee (y \wedge t) \leqslant y$ and therefore $x_1 \vee (y \wedge t) \in r$ holds, we have $x \leqslant x_1 \vee (y \wedge t)$ and $x_1 \vee (y \wedge t) = x \vee (x_1 \vee (y \wedge t)) =$ $= x_1 \vee (x \vee (y \wedge t)) = x_1 \vee y = y$. Thus the condition 1.1(i) is fulfilled. Conversely, 1.1(i) evidently implies (i) or (ii).

1.3. *The relation $\bar{r}$ is a homomorphism of the convex sublattice* $\mathrm{D}(\bar{r})$ *of L onto the chain r.*

Proof. Let $\langle ty\rangle, \langle ty_1\rangle \in \bar{r}$. The elements y, y_1 are comparable, let, e. g., $y \leqslant y_1$. Then $y = y_1 \wedge (y \vee t) = y \vee (y_1 \wedge t) = y_1$ holds, therefore $\bar{r}$ is a function.

Now let $\langle ty\rangle, \langle t_1y_1\rangle \in \bar{r}$. Again let, e. g., $y \leqslant y_1$. For every $x \in r,\ x \leqslant y$ we have $x \vee (y \wedge (t \wedge t_1)) = x \vee ((y \wedge y_1) \wedge (t \wedge t_1)) = x \vee ((y \wedge t) \wedge$ $\wedge (y_1 \wedge t_1)) = (x \vee (y \wedge t)) \wedge (x \vee (y_1 \wedge t_1)) = y \wedge y_1 = y$, for every $z \in r$, $y \leqslant z$ we have $z \wedge (y \vee (t \wedge t_1)) = z \wedge ((y \vee t) \wedge (y \vee t_1)) = (z \wedge (y \vee t)) \wedge$ $\wedge (y \vee t_1) = y \wedge (y \vee t_1) = y$. Dually $\langle t \vee t_1, y_1\rangle \in \bar{r}$ is shown. Thus $\bar{r}$ is a homomorphism and $\mathrm{D}(\bar{r})$ is a sublattice of L.

Let $\langle t_0\ y_0\rangle, \langle t_1y_1\rangle \in \bar{r},\ t_0 \leqslant t \leqslant t_1$. Then $y_0 \leqslant y_1$ holds, because $\bar{r}$ is a homomorphism. We denote $y = (y_0 \vee t) \wedge y_1 = y_0 \vee (t \wedge y_1)$ and have $y_0 \leqslant y \leqslant y_1$.

We have also $y_0 \vee (y \wedge t) = y \wedge (y_0 \vee t) = (y_0 \vee (t \wedge y_1)) \wedge (y_0 \vee t) = y_0 \vee (t \wedge y_1) = y$. If $y_0 = y$, then for every $x \leqslant y_0 = y$ we have $y = y_0 = x \vee (y_0 \wedge t_0) \leqslant x \vee (y \wedge t) \leqslant y$. Thus by 1.2 the condition 1.1(i) is satisfied. The condition 1.1(ii) is verified dually. Therefore $\langle ty \rangle \in \bar{r}$, $t \in \mathrm{D}(\bar{r})$ and $\mathrm{D}(\bar{r})$ is convex in L.

1.4. *If $x \neq y$ and the interval $[xy] \subseteq r$ is projective with $[st]$, then $\langle sx \rangle, \langle ty \rangle \in \bar{r}$.*

Proof. By 2.2 [1] the projectivity of $[xy]$, $[st]$ implies $x \vee (y \wedge t) = y$. Hence 1.2(ii) and therefore also 1.1(i) are fulfilled. Now let $z \in r$, $y \leqslant z$. We denote $a = (x \vee s) \wedge z = x \vee (s \wedge z)$ and have $x \leqslant a \leqslant z$, hence $a \in r$. Further we have $a \wedge y = (x \vee s) \wedge z \wedge y = (x \vee s) \wedge y = x$ (again using 2.2 [1]), therefore the comparability of a, y and the condition $x \neq y$ imply $y = a \vee y = (z \wedge s) \vee x \vee y = (z \wedge s) \vee y = z \wedge (s \vee y) = z \wedge (t \vee y)$. Thus 1.1(ii) and $\langle ty \rangle \in \bar{r}$ hold. The second part $\langle sx \rangle \in \bar{r}$ is proved dually.

1.5. *Let us define $R_r^0 = \{t;\ (\forall\, x, y \in r)[t \wedge x = t \wedge y]\}$, $R_r^1 = \{t; (\forall x, y \in r)[t \vee x = t \vee y]\}$. Then we have*

(i) $R_r^0 \cup R_r^1 \cup \mathrm{D}(\bar{r}) = L$,
(ii) $R_r^0 \cap R_r^1 = 0$, *if* $\operatorname{card} r > 1$,
(iii) $R_r^0 \cap \mathrm{D}(\bar{r}) = 0$ *if and only if there is no least element in r,*
(iv) *if y is the least element in r, then $R_r^0 = \{t; \langle ty \rangle \in \bar{r}\}$,*
(v) *if r is an interval, then* $\mathrm{D}(\bar{r}) = L$.

Proof. To prove (i) let us suppose $t \notin R_r^0 \cup R_r^1 \cup \mathrm{D}(\bar{r})$. Then first there are $x, y \in r$ such that $x < y$ and $t \wedge x < t \wedge y$. If we denote $y_0 = x \vee (t \wedge y) = (x \vee t) \wedge y$, then according to $x \wedge (t \wedge y) = t \wedge (x \wedge y) = t \wedge x$ we have $[t \wedge x, t \wedge y] \sim [xy_0]$ and $x < y_0$. By 1.2 the assumption $y_0 < y$ gives $\langle ty_0 \rangle \in \bar{r}$, $t \in \mathrm{D}(\bar{r})$, because $y \wedge (y_0 \vee t) = y \wedge (x \vee (t \wedge y) \vee t) = y \wedge (x \vee t) = y_0$ and dually $x \vee (y_0 \wedge t) = y_0$ hold. Thus we have $y_0 = y$ and $y \vee t = x \vee (t \wedge y) \vee t = x \vee t$, therefore $t \vee x = t \vee z = t \vee y$ for $x \leqslant z \leqslant y$.

By dual reasoning we get $z = x \wedge (t \vee z) = (x \wedge t) \vee z$, $t \wedge x = z \wedge x$ for $z < x$, $t \vee z < t \vee x$. Then the element $a = (y \wedge t) \vee z = y \wedge (t \vee z)$ belongs to r and fulfils $a \wedge x = y \wedge (t \vee z) \wedge x = x \wedge (t \vee z) = z \neq x$, $a \vee x = (y \wedge t) \vee z \vee x = (y \wedge t) \vee x = y \neq x$, which is impossible as a, x are comparable.

We see that $t \vee z = t \vee y$ holds for $z \leqslant y$. Hence there exists $z \in r$, $y < z$, $t \vee y < t \vee z$. Again we show analogously that $y = z \wedge (t \vee y)$ and by 1.2 we get $\langle ty \rangle \in \bar{r}$, $t \in \mathrm{D}(\bar{r})$, which is a contradiction and (i) has been proved.

The modularity of L gives immediately (ii).

Now let y be the least element in r. If $t \wedge y = t \wedge z$ holds for any $z \in r$, then $z \wedge (t \vee y) = (z \wedge t) \vee y = (y \wedge t) \vee y = y$, i. e. 1.1(ii) is fulfilled. As 1.1(i) follows from 1.2, we have $\langle ty \rangle \in \bar{r}$. Conversely, if we have $\langle ty \rangle \in \bar{r}$, then $y = z \wedge (t \vee y)$ and $y \wedge t = z \wedge (t \vee y) \wedge t = z \wedge t$ for any $z \in r$. We have proved (iv) and one implication from (iii).

Finally, suppose that there is no least element in r, let $t \in \mathrm{D}(\bar{r})$, then by 1.2 there is $x \in r$, $x < y$, $x \vee (y \wedge t) = y$. We further have $x \wedge (y \wedge t) = (x \wedge y) \wedge t = x \wedge t$, so that $[x \wedge t, y \wedge t] \sim [xy]$. That means $x \wedge t < y \wedge t$, i. e. $t \notin R_r^0$. The proof of (iii) is complete. The statement (v) follows from (i) and (iv).

1.6. (i) *R_r^0 is an ideal in L,*
(ii) *if $s \in R_r^0$, $\langle ty \rangle \in \bar{r}$, then $\langle s \vee t, y \rangle \in \bar{r}$.*

Proof. Let $s, t \in R_r^0$, $x, y \in r$. Then $(s \vee t) \wedge x = (s \wedge x) \vee (t \wedge x) = (s \wedge y) \vee (t \wedge y) = (s \vee t) \wedge y$, so that $s \vee t \in R_r^0$.

Let $s \leqslant t \in R_r^0$, $x, y \in r$. Then $s \wedge x = (s \wedge t) \wedge x = s \wedge (t \wedge x) = s \wedge (t \wedge y) = (s \wedge t) \wedge y = s \wedge y$, so that $s \in R_r^0$. We have proved (i).

To prove (ii) let $s \in R_r^0$, $\langle ty \rangle \in \bar{r}$. Then for any $x \in r$, $x \leqslant y$ we have $x \vee (y \wedge (s \vee t)) = x \vee (y \wedge s) \vee (y \wedge t) = (y \wedge s) \vee (x \vee (y \wedge t)) = (y \wedge s) \vee y = y$ and for any $z \in r$, $y \leqslant z$ we have $z \wedge (y \vee (s \vee t)) = z \wedge (s \vee (y \vee t)) = (z \wedge s) \vee (z \wedge (y \vee t)) = (z \wedge s) \vee y = (y \wedge s) \vee y = y$.

1.7. *Let us define a relation R_r on L as follows*
(i) *if $s, t \in R_r^0$, then $\langle st \rangle \in R_r$,*
(ii) *if $s, t \in R_r^1$, then $\langle st \rangle \in R_r$,*
(iii) *if $\langle sy \rangle$, $\langle ty \rangle \in \bar{r}$, then $\langle st \rangle \in R_r$.*

Then the relation R_r is a congruence on L and L/R_r is a chain, which is isomorphic to r if $\mathrm{D}(\bar{r}) = L$.

Proof. The symmetry of R_r follows immediately from the definition, the reflexivity follows from 1.5(i). As for card $r = 0,1$, $R_r^0 = R_r^1 = L$ holds, the transitivity of R_r is a consequence of 1.5(ii), (iii), (iv). Finally the compatibility of R_r with the lattice operations in L and the linearity of L/R_r follow from 1.3, 1.6 and statements dual to 1.6. If $\mathrm{D}(\bar{r}) = L$, R_r is completely defined by (iii), thus L/R_r is evidently isomorphic to r.

1.8. *Let us define a relation Q_r on L as follows: $\langle uv \rangle \in Q_r$ if there is an interval $[xy] \subseteq r$ projective with $[uv]$ or with $[vu]$. If $r \neq 0$, then Q_r is a congruence in L.*

Proof. The assumption $r \neq 0$ gives immediately the reflexivity of Q_r. The symmetry follows from the definition.

Let $\langle ps \rangle$, $\langle st \rangle \in Q_r$ and let $[xy] \subseteq r$ projective with $[ps]$, $[vz] \subseteq r$ projective with $[st]$. According to 1.4, we have $\langle sy \rangle$, $\langle sv \rangle \in \bar{r}$, thus by 1.3 we have $y = v$. From 2.2 [1] it follows that $x \wedge t = x \wedge y \wedge t = x \wedge y \wedge s = x \wedge s = x \wedge p$. Analogously we get $z \wedge p = x \wedge p$ and dually $x \vee t = z \vee p = z \vee t$. Therefore $[xz]$, $[pt]$ are projective (using 2.1, 2.2 [1]) and $\langle pt \rangle \in Q_r$.

Now let $[xy] \subseteq r$ projective with $[ps]$, $[vz] \subseteq r$ projective with $[ts]$. Again we get $y = z$, using 1.3, 1.4. The elements x, v are comparable, let e. g., $x \leqslant v$. From the properties of transposed intervals it follows that $[vy]$, $[(v \vee p) \wedge s, s]$ are projective. As $[vy]$, $[ts]$ are also projective, by 2.3(i) [1] we get $(v \vee p) \wedge \wedge s = t$. The interval $[xv]$ is projective with $[p, (v \vee p) \wedge s] = [pt]$, hence $\langle pt \rangle \in Q_r$. We have proved the transitivity of Q_r (the other cases are dual).

If $p \in L$ and $\langle st \rangle \in Q_r$, e. g. if $[xy] \subseteq r$ is projective with $[st]$, we denote $v = (s \vee p) \wedge t = s \vee (p \wedge t)$. Again the intervals $[(v \vee x) \wedge y, y]$, $[vt]$ are projective. On the other hand, $[vt] \sim [v \vee p, t \vee p] = [s \vee p, t \vee p]$ holds. Hence we have $\langle s \vee p, t \vee p \rangle \in Q_r$. Dually $\langle s \wedge p, t \wedge p \rangle \in Q_r$ is proved.

1.9. *If* $[uv] \subseteq \mathrm{D}(\bar{r})$, *then there is* $[st] \subseteq [uv]$ *such that* $\langle us \rangle$, $\langle vt \rangle \in R_r$, $\langle st \rangle \in Q_r$.

Proof. Let us denote $x = \bar{r}(u)$, $y = \bar{r}(v)$, $s = (x \vee u) \wedge v$, $t = (y \wedge v) \vee u$. Then we have $s \vee x = ((x \vee u) \wedge v) \vee x = (u \vee x) \wedge (v \vee x) = (u \wedge v) \vee \vee x = u \vee x$, $t \vee y = (y \wedge v) \vee u \vee y = u \vee ((y \wedge v) \vee y) = u \vee y$ and therefore $y \wedge (s \vee x) = y \wedge (u \vee x) = x$, $y \vee (s \vee x) = y \vee u \vee x = y \vee \vee u = t \vee y$, i. e. $[xy] \sim [s \vee x, t \vee y]$. Dually we see that $[s \wedge x, t \wedge y] \sim \sim [xy]$, thus by 2.1, 2.2 [1] the intervals $[xy]$, $[st]$ are projective and the proof is complete according to 1.4, 1.8.

1.10. *R_r is the maximal congruence orthogonal to Q_r, $R_r = Q_r^{\perp}$.*

Proof. Let us suppose that R is a congruence orthogonal to Q_r and $\langle uv \rangle \in R$, $\langle uv \rangle \notin R_r$. Hence also $\langle u \wedge v, u \vee v \rangle \in R$, $\langle u \wedge v, u \vee v \rangle \notin R_r$, so that we may assume $u \leqslant v$. We cannot have $[uv] \subseteq \mathrm{D}(\bar{r})$, because then 1.9 would give $[st] \subseteq [uv]$, $\langle st \rangle \in Q_r$, $s \neq t$ and $\langle st \rangle \in R$. Thus let, e. g., $u \notin \mathrm{D}(\bar{r})$, therefore $u \in R_r^0$ and r has no least element. Then for any $x \in r$, $x \leqslant \bar{r}(v)$ (or for any $x \in r$, if $v \notin \mathrm{D}(\bar{r})$, $v \in R_r^1$) $\langle (u \vee x) \wedge v, x \rangle \in R_r$ and $(u \vee x) \wedge v \in [uv]$ hold. Hence we have $[yz] \subseteq [uv] \cap \mathrm{D}(\bar{r})$, $\langle yz \rangle \notin R_r$, $\langle yz \rangle \in R$ and again 1.9 gives a contradiction with the orthogonality of R, Q_r.

1.11. *If r is projective with a convex chain p, then $Q_r = Q_p$, $R_r = R_p$ and $\bar{r}(t) = \bar{r}(\bar{p}(t))$ for any $t \in L$.*

Proof. The first equality follows from the definition of Q_r, the second one from 1.10. Let $t \in L$, then by 1.7(iii) $\langle \bar{p}(t)t \rangle \in R_p$ and therefore $\langle \bar{p}(t)t \rangle \in R_r$ hold. By 1.7(iii) we have also $\langle \bar{r}(\bar{p}(t))\bar{p}(t) \rangle$, $\langle \bar{r}(t)t \rangle \in R_r$, hence $\langle \bar{r}(\bar{p}(t))\bar{r}(t) \rangle \in R_r$ and $\bar{r}(\bar{p}(t)) = \bar{r}(t)$.

2. Linear congruences

2.0. Let L be a distributive lattice in this section. We shall study the extending of a linear congruence from a convex sublattice onto the whole L. We describe also the maximal congruence which is orthogonal to a given linear one.

2.1. *Let Q be a congruence on L, let B be a finite independent system over X in L/Q. Then there is an independent system b over x with the same cardinality as B such that $x \in X$, $y \in Y \in B$ (and therefore $\langle xy \rangle \notin Q$) for any $y \in b$.*

Proof. For every $Y \in B$ let us choose $v(Y) \in Y$, then for $Y \neq Z$, $v(Y) \wedge \wedge v(Z) \in X$ holds. We denote $x = \bigvee \{v(Y) \wedge v(Z);\ Y, Z \in B \mathbin{\&} Y \neq Z\}$, $b = \{x \vee v(Y);\ Y \in B\}$. Evidently $x \in X$, $x \vee v(Y) \in Y$ for any $Y \in B$ and $(x \vee v(Y)) \wedge (x \vee v(Z)) = x \vee (v(Y) \wedge v(Z)) = x$. So $x \vee v(Y) \neq x \vee v(Z)$ for $Y \neq Z$ and b is an independent system over x with the same cardinality as B.

2.2. *Let M be a set of congruences on L, let a linear congruence Q_M be the least congruence on L containing all elements of M, i. e. $Q_M = \sum M$. Then $R_M = \cap \{Q^\perp; Q \in M\} = \cap \{R_r;\ r \in L/Q \mathbin{\&} Q \in M\} = \cap \{R_r;\ r = [xy] \mathbin{\&} \langle xy \rangle \in Q \in \in M\} = Q_M^\perp$.*

Proof. Evidently $Q_M = \sum \{Q_r; r \in L/Q \mathbin{\&} Q \in M\} = \sum \{Q_r; r = [xy] \mathbin{\&} \langle xy \rangle \in \in Q \in M\}$. According to 1.10, the assertion follows from the properties of the ordering of congruences by the set inclusion.

2.3. *Let Q be a linear congruence on L, let us denote $Q^\perp = R$. Then L/R is a chain.*

Proof. By 2.2, $R = \cap \{R_r;\ r = [xy] \mathbin{\&} \langle xy \rangle \in Q\}$. Let $x, y \in L$ such that $[x]_R \parallel [y]_R$. Then $\langle x \wedge y, x \rangle, \langle x \wedge y, y \rangle \notin R$, i. e. $\langle x \wedge y, x \rangle \notin R_r$, $\langle x \wedge y, y \rangle \notin \notin R_p$ for some $r = [uv]$, $p = [st]$, $\langle uv \rangle \in Q$, $\langle st \rangle \in Q$. We have $\mathrm{D}(\bar{r}) = \mathrm{D}(\bar{p}) = L$ by 1.5(v), therefore using 1.9 and the properties of transposed intervals we can find x', y' such that $x' \parallel y'$, $\langle x' \wedge y', x' \rangle \in Q_r$, $\langle x' \wedge y', y' \rangle \in Q_p$, i. e. $\langle x'y' \rangle \in Q$, which is a contradiction with the linearity of Q.

2.4. *Let Q_1 be a linear congruence on a convex sublattice L_1 of L, then $Q = \cup \{Q_r; r \in L_1/Q_1\} = \cup \{Q_r; r = [xy] \mathbin{\&} \langle xy \rangle \in Q_1\}$ is a linear congruence on L. Q is the minimal congruence on L containing Q_1 and the restriction $Q \cap (L_1 \times L_1)$ of Q to L_1 is equal to Q_1. If we denote $Q^\perp = R$, $Q_1^\perp = R_1$, then $R \cap (L_1 \times L_1) = = R_1$.*

Proof. At first let $u < v < w$, let the interval $[uv]$ be projective with $p = [x_1y_1]$, let $r = [vw]$ be projective with $q = [x_2y_2]$ and let $\langle x_1y_1 \rangle, \langle x_2y_2 \rangle \in Q_1$. By 1.11 we have $R_r = R_q$ and $\bar{r}(y_1 \wedge v) = v$, $\bar{r}(y_1) = \bar{r}(x_1 \vee (y_1 \wedge v)) = = \bar{r}(x_1) \vee v = \bar{r}(x_1)$. Let us denote $t = \bar{q}(\bar{r}(x_1))$, by 1.11 we get $t = \bar{q}(x_1) = = \bar{q}(y_1)$ and set $z = \bar{p}(t)$.

If $z > x_1$, $t > x_2$ holds, then $[x_1 \wedge t, z \wedge t] \sim [x_1z]$, $[x_2 \wedge z, z \wedge t] \sim [x_2t]$ give us $x_1 \wedge t, x_2 \wedge z < z \wedge t$, $\langle x_1 \wedge t, z \wedge t \rangle, \langle x_2 \wedge z, z \wedge t \rangle \in Q_1$ and $(x_1 \wedge t) \vee \vee (x_2 \wedge z) = ((x_1 \wedge t) \vee x_2) \wedge ((x_1 \wedge t) \vee z) = t \wedge z$, which is a contradiction with the linearity of Q_1.

Analogously, if $z < y_1$, $t < y_2$, then $[zy_1] \sim [z \vee t,\ y_1 \vee t]$, $[ty_2] \sim \sim [z \vee t, z \vee y_2]$, which gives $z \vee t < y_1 \vee t$, $z \vee y_2$, $\langle z \vee t, y_1 \vee t \rangle$, $\langle z \vee t, z \vee y_2 \rangle \in Q_1$ and $(y_1 \vee t) \wedge (z \vee y_2) = ((y_1 \vee t) \wedge z) \vee ((y_1 \vee t) \wedge$

$\wedge\, y_2) = z \vee t$, which is a contradiction with the linearity of Q_1.

Hence we have $z = x_1$, $t = y_2$ or $z = y_1$, $t = x_2$.

In the first case according to 1.11 we may assume $y_2 \leqslant x_1$. Then the elements $a = (x_1 \wedge v) \vee y_2 = x_1 \wedge (v \vee y_2)$, $b = (y_1 \wedge v) \vee x_2 = y_1 \wedge (v \vee x_2)$ fulfil $a \wedge b = x_1 \wedge (v \vee y_2) \wedge y_1 \wedge (v \vee x_2) = (x_1 \wedge y_1) \wedge ((v \vee y_2) \wedge \wedge (v \vee x_2)) = x_1 \wedge (v \vee x_2) = (x_1 \wedge v) \vee x_2$ and $a, b \in [x_2 y_1]$, i. e. $a, b \in L_1$. Hence we have $b \wedge x_1 = y_1 \wedge (v \vee x_2) \wedge x_1 = a \wedge b$, $b \vee x_1 = ((y_1 \wedge v) \vee \vee x_2) \vee x_1 = (y_1 \wedge v) \vee x_1 = y_1$, thus $[a \wedge b, b] \sim [x_1 y_1]$. Analogously $y_2 \wedge \wedge (a \wedge b) = y_2 \wedge x_1 \wedge (v \vee x_2) = y_2 \wedge (v \vee x_2) = x_2$, $y_2 \vee (a \wedge b) = y_2 \vee \vee (x_1 \wedge v) \vee x_2 = y_2 \vee (x_1 \wedge v) = a$, thus $[x_2 y_2] \sim [a \wedge b, a]$ holds. We have obtained a contradiction with the linearity of Q_1.

So $z = y_1$, $t = x_2$ is true. According to 1.11 we can assume that $y_1 \leqslant x_2$. Denoting $a = (x_1 \vee u) \wedge y_2 = x_1 \vee (u \wedge y_2)$, $b = (x_1 \vee v) \wedge y_2 = x_1 \vee \vee (v \wedge y_2)$, $c = (x_1 \vee w) \wedge y_2 = x_1 \vee (w \wedge y_2)$ we have $x_1 \leqslant a \leqslant b \leqslant c \leqslant \leqslant y_2$, i. e. $a, b, c \in L_1$. Then we have $y_1 \wedge a = y_1 \wedge (x_1 \vee u) \wedge y_2 = y_1 \wedge \wedge (x_1 \vee u) = x_1$, $y_1 \vee a = y_1 \vee ((x_1 \vee u) \wedge y_2) = (y_1 \vee x_1 \vee u) \wedge (y_1 \vee y_2) = = (y_1 \vee u) \wedge y_2 = (x_1 \vee v) \wedge y_2 = b$, thus $[x_1 y_1] \sim [ab]$ and dually $[bc] \sim \sim [x_2 y_2]$. That means $\langle ab \rangle, \langle bc \rangle \in Q_1$, therefore also $\langle ac \rangle \in Q_1$. Then $s = [ac]$ is a chain and we have $\langle uv \rangle, \langle vw \rangle \in Q_s$, thus $\langle uw \rangle \in Q_s$. From $\bar{s}(u) = a$, $\bar{s}(v) = b$, $\bar{s}(w) = c$ it follows that $[ac]$, $[uw]$ are projective intervals.

We have proved now that if the elements $u, v \in L$ can be connected by a finite Q_1-sequence, i. e. if there are finitely many elements $t_0 = u$, $t_1, t_2, \ldots, t_n = v$ such that every interval $[t_{i-1} t_i]$ $(i = 1, \ldots, n)$ is projective with some $[x_i y_i] \subseteq L_1$, $\langle x_i y_i \rangle \in Q_1$, then there is $r = [xy] \subseteq L_1$, $\langle xy \rangle \in Q_1$ such that $\langle uv \rangle \in Q_r$, $\langle uv \rangle \in Q$.

It is a known fact that the congruence Q_1 can be extended to a congruence Q_0 on L (without the assumption of the convexity of L_1 and the linearity of Q_1) defined as follows: $\langle uv \rangle \in Q_0$, if the elements $u \wedge v$, u and the elements u, $u \vee v$ can be connected by a finite Q_1-sequence. In our case $\langle uv \rangle \in Q_0$ gives $\langle u \wedge v, u \vee v \rangle \in Q$, therefore also $\langle uv \rangle \in Q$. That shows that $Q_0 \subseteq Q$. The converse inclusion is trivial. So Q is a congruence on L. The remaining assertions follow easily from the definition of Q.

2.5. *Let Q be a linear congruence on L, $R = Q^\perp$. If L/R consists just of two elements R^0, R^1, then $\langle xy \rangle \in Q$ if and only if $x \in R^0$, $y \in R^1$, $x \prec y$.*

Proof. Q is not the identity on L, otherwise L/R would have only one element L. Hence there are $x < y$, $\langle xy \rangle \in Q$. The orthogonality of Q, R gives $x \in R^0$, $y \in R^1$, $x \prec y$. Now let $u \in R^0$, $v \in R^1$, $u \prec v$, then $y \wedge u$, $x \vee (y \wedge u) \in \in R^0$, thus $x \leqslant x \vee (y \wedge u) < y$ gives $y \wedge u \leqslant x$ and $x \wedge u = y \wedge u$. Analogously we get $x \wedge u = x \wedge v$ and dually $x \vee v = y \vee v = y \vee u$. By 2.1, 2.2 [1], the intervals $[xy]$, $[uv]$ are projective and $\langle uv \rangle \in Q$.

3. k-chains

3.0. From now on we suppose that L is a distributive lattice of finite local dimension $k > 0$. We show that if r is a k-chain, then Q_r, R_r have some special properties.

3.1. *If b is a k-system over x, then for any $y \in b$, the interval $[xy]$ is a chain.*

Proof. Let us suppose that there are incomparable elements $u, v \in [xy]$. We denote $x' = u \wedge v$, $b' = \{z \vee x';\ z \in b \ \&\ z \neq y\}$, $z_0 = \bigvee (b - \{y\})$ and get $[xz_0] \sim [x', z_0 \vee x']$. So b' is a $(k - 1)$-system over x' and for $z \in b$, $z \neq y$ we have $(z \vee x') \wedge u = (z \wedge u) \vee (x' \wedge u) = x \vee x' = x' = (z \vee x') \wedge v$. We see that $b' \cup \{uv\}$ is a $(k + 1)$-system over x', which is a contradiction.

3.2. *If b is a k-system over x and $x < t$, then there is $y \in b$ such that $x < t \wedge y$. If, moreover $[xt]$ is a chain, then $x = t \wedge z$ for any $z \in b$, $z \neq y$ and the elements y, t are comparable.*

Proof. For any $y \in b$ we have $x \leqslant t \wedge y$. The assumption $x = t \wedge y$ for any $y \in b$ leads to a contradiction, hence $x < t \wedge y$ for some $y \in b$. If $x < t \wedge z$, $z \in b$, $z \neq y$, then $(t \wedge z) \wedge (t \wedge y) = x$, therefore $t \wedge z$, $t \wedge y$ are incomparable and $[xt]$ is not a chain. If $x = t \wedge z$ for any $z \in b$, $z \neq y$, then $\{t \vee y\} \cup (b - \{y\})$ is a k-system over x, therefore by 3.1 $[x, t \vee y]$ is a chain and y, t are comparable.

3.3. *Let r be a convex chain, x a lower k-element in L. If $x, u, v \in r$, $x < u < v$, then* lodim $[u]_{R_r} < k$.

Proof. Let us suppose that lodim $[u]_{R_r} = k$, i. e. let there exist a k-system b over y, $\langle yu\rangle$, $\langle zu\rangle \in \bar{r}$ for every $z \in b$. Hence $u \vee y = (x \vee (u \wedge y)) \vee y = x \vee ((u \wedge y) \vee y) = x \vee y$ holds. Let a be a k-system over x, by 3.2 we can assume $u \in a$. In the same way as in the proof of 2.4 [1] we denote $a_2' = \{z \vee y; z \in a \ \&\ z \wedge (x \vee y) = x\}$, $b_2' = \{z \vee x;\ z \in b \ \&\ z \wedge (x \vee y) = y\}$. From the mentioned proof it follows that $d = \{z \wedge t;\ z, t \in a_2' \cup b_2' \ \&\ z \wedge t > x \vee y\}$ is a k-system over $x \vee y$. If $z \in a$, $z \wedge (x \vee y) = x$, i. e. $z \neq u$, then by 3.2 $z \wedge v = x$ holds, hence $(z \vee y) \wedge (v \vee y) = (z \wedge v) \vee y = x \vee y$. For $z \in b$ we have $v \wedge (u \vee z) = u = x \vee (u \wedge z)$ and $u \wedge z = (v \wedge (u \vee z)) \wedge z = v \wedge ((u \vee z) \wedge z) = v \wedge z$, therefore $(z \vee x) \wedge (v \vee y) = (z \wedge v) \vee (x \wedge v) \vee (z \wedge y) \vee (x \wedge y) = (z \wedge u) \vee x \vee y \vee (x \wedge y) = ((z \wedge u) \vee x) \vee y = u \wedge y = x \vee y$. We have shown $z \wedge (v \vee y) = x \vee y$ for $z \in a_2' \cup b_2'$, hence for $z \wedge t \in d$ we have $(z \wedge t) \wedge (v \vee y) = x \vee y$. Now $[uv] \sim [x \vee y, v \vee y]$ gives $v \vee y > x \vee y$. Thus $d \cup \{v \vee y\}$ is a $(k + 1)$-system over $x \vee y$, which is a contradiction.

3.4. *Let r be a lower k-chain, let us denote $L_r = L - (R_r^0 \cup R_r^1)$, then* lodim $L_r/Q_r < k$.

Proof. Let us suppose lodim $L_r/Q_r \geqslant k$, then by 2.1 there is a k-system b over x in L_r such that $\langle xy\rangle \notin Q_r$ for any $y \in b$. As L/R_r is a chain, according to 3.3 there is $y \in b$ such that $\langle xy \rangle \notin R_r$, $\langle xz \rangle \in R_r$ for any $z \in b$, z y. Evidently $L_r \subseteq \mathrm{D}(\bar{r})$ holds, hence by 1.9 there is an $[st] \subseteq [xy]$ such that $\langle xs \rangle$, $\langle yt \rangle \in R_r$, $\langle st \rangle \in Q_r$. The assumption $s > x$ gives a contradiction according to 3.3, thus $s = x$. Then $t < y$ must hold and $\{y\} \cup \{t \vee z;\, z \in b \;\&\; z$ $y\}$ is a k-system over t, which is a contradiction with 3.3.

3.5. *If r is a lower k-chain, then there is a set M of upper k-chains such that $Q_r = \cup \{Q_s;\, s \in M\}$.*

Proof. By 3.2, for any $p = [xy] \subseteq r$, $x < y$ we can choose a k-system b over $x, y \in b$. Then we denote $x' = \bigvee b$, $z' = \bigvee (b - \{z\})$ for any z $b' = \{z';\, z \in b\}$ and from the properties of transposed intervals we get b' is a k-system under x, i. e. $p' = [y'x']$ is an upper k-chain, projective wi By 1.11 we have $Q_p = Q_{p'}$, hence the fact that $Q_r = \cup \{Q_p;\, p = [xy]$ completes the proof.

3.6. Let Q_0, Q be congruences on L. We shall say that Q is a k-extens of Q_0, if there is a set M of lower k-chains such that $Q = Q_0 + \sum \{Q_r;\, r \in M\}$. If Q_0 is the identity on L, we say that Q is a k-generated congruence on L

Note. By 3.5 the definition will not be changed if we replace „lower by „upper“ in it.

4. k-generated congruences

4.0. If we want to show that L is a subdirect product of k chains, we can do it by induction and it is necessary and sufficient to find a linear congruence Q in L such that $\operatorname{lodim} L/Q < k$. Namely, if we denote $R = Q^{\perp}$, then L/R is a chain and L is a subdirect product of $L/R \times L/Q$. In the previous section we have seen that, for any k-chain r in L, Q_r fulfils a weaker condition $\operatorname{lodim} L_r/Q_r < k$. This fact will be generalized now.

4.1. Let Q be a linear congruence on L, $R = Q^{\perp}$. We denote by R^0 the least element in L/R, or set $R^0 = 0$ if such an element does not exist. R^1 is defined dually. Finally we set $L(R) = L - (R^0 \cup R^1)$.

Note. If r is a convex chain in L, then $L_r = L(R_r)$.

4.2. By $\mathscr{A}(L)$ we shall denote the set of all non-identical linear congruences Q on L satisfying the condition $\operatorname{lodim} L(Q^{\perp})/Q < k$. $\mathscr{A}_k(L)$ will be the set of all k-generated $Q \in \mathscr{A}(L)$.

If $Q_0 \in \mathscr{A}(L)$, then $\mathscr{A}(Q_0, L)$ is the set of all k-extensions Q of Q_0, $Q \in \mathscr{A}(L)$.

4.3. *$\mathscr{A}(L)$, $\mathscr{A}_k(L)$, $\mathscr{A}(Q_0, L)$ satisfy the condition of maximality, if ordered by inclusion.*

Proof. The proof suffices to be done for $\mathscr{A}(L)$, $\mathscr{A}(Q_0, L)$. Hence let M be a chain in $\mathscr{A}(L)$. Let us denote $Q_M = \cup M$, evidently Q_M is a linear congruence on L. If M is a chain in $\mathscr{A}(Q_0, L)$, then Q_M is a k-extension of Q_0. If we denote $M^{\perp} = \{Q^{\perp};\, Q \in M\}$, then by 2.2 we have $Q_M^{\perp} = R_M = \cap M^{\perp}$ and clearly also $R_M^0 = \cap \{R^0;\, R \in M^{\perp}\}$, $R_M^1 = \cap \{R^1;\, R \in M^{\perp}\}$. So $L(R_M) = L -$

$- (R^0_M \cup R^1_M) = L - (\cap \{R^0;\ R \in M^\perp\} \cup \cap \{R^1;\ R \in M^\perp\}) = L \cap - \cap \cap \{R^0 \cup R^1;\ R \in M^\perp\} = L \cap \cup \{- (R^0 \cup R^1);\ R \in M^\perp\} = \cup \{L \cap - (R^0 \cup R^1);\ R \in M^\perp\} = \cup \{L(R);\ R \in M^\perp\}$ holds, if we use the fact that R^0, R^1 for $R \in M^\perp$ are linearly ordered by inclusion.

Now let b be a k-system over x in $L(R_M)$, then there is $Q \in M$, $Q^\perp = R$ such that $b \subseteq L(R)$, $x \in L(R)$, therefore $\langle xy \rangle \in Q$ and $\langle xy \rangle \in Q_M$ for some $y \in b$. So $\operatorname{lodim} L(R_M)/Q_M < k$ and $Q_M \in \mathscr{A}(L)$, $\mathscr{A}(Q_0, L)$ respectively.

4.4. *Let Q be a maximal element in $\mathscr{A}(L)$ or in $\mathscr{A}(Q_0, L)$. If $[xy]$ is a k-chain and $\langle xy \rangle \notin R = Q^\perp$, then $\langle xy \rangle \in Q$.*

Proof. Let us suppose $\langle xy \rangle \notin Q$. The assumption $x \in' R^1$ gives $y \in R^1$ and $\langle xy \rangle \in R$, hence $x \notin R^1$ must hold.

First we shall assume that for any $t \in [xy]$, $t \notin R^1$, there is $\langle xt \rangle \in Q$. Then evidently $y \in R^1$ must be true. We denote $r = [xy]$ and show that $\langle uv \rangle \in Q_r$, $u \leqslant v$, $v \notin R$ imply $\langle uv \rangle \in Q$. Namely, $[uv]$ is projective with $[pq] \subseteq [xy]$. If $u \neq v$, by 1.4 we have $\bar{r}(v) = q$, $q = x \vee (q \wedge v)$ and $x \notin R^1$, $v \notin R^1$ give $q \wedge v \notin R^1$, $q \notin R^1$. Thus $\langle xq \rangle$, $\langle pq \rangle \in Q$ and $\langle uv \rangle \in Q$.

We define a relation N in L as follows: $\langle uw \rangle$, $\langle wu \rangle \in N$, if there is $v \in L$, $u \leqslant v \leqslant w$ such that $\langle uv \rangle \in Q$ and $\langle vw \rangle \in Q_r$. The reflexivity, the symmetry and the compatibility of N with the lattice operations are evident. To verify the transitivity of N let be $\langle uw \rangle$, $\langle wq \rangle \in N$.

Let $u \leqslant v \leqslant w \leqslant p \leqslant q$, $\langle uv \rangle$, $\langle wp \rangle \in Q$, $\langle vw \rangle$, $\langle pq \rangle \in Q_r$. If $w = p$, then $\langle vq \rangle \in Q_r$, $\langle uq \rangle \in N$. If $w \neq p$, then $w \notin R^1$, $\langle vw \rangle \in Q$, $\langle up \rangle \in Q$, $\langle uq \rangle \in N$.

Further let $u \leqslant v \leqslant w$, $q \leqslant p \leqslant w$, $\langle uv \rangle$, $\langle qp \rangle \in Q$, $\langle vw \rangle$, $\langle pw \rangle \in Q_r$. Then $\langle vp \rangle \in Q_r$ holds and v, p are comparable, let e. g., $v \leqslant p$. There is $v \leqslant v \vee q \leqslant p$, i. e. $\langle v, v \vee q \rangle$, $\langle v \wedge q, q \rangle \in Q_r$, and $q \leqslant v \vee q \leqslant p$, i. e. $\langle q, v \vee q \rangle \in Q$. If $q < v \vee q$, then $q \notin R^1$, $\langle v \wedge q, q \rangle$, $\langle v, v \vee q \rangle \in Q$ and $\langle vq \rangle \in Q$, i. e. $\langle uq \rangle \in Q, N$.

If $q = v \vee q$, then $\langle uq \rangle \in N$.

Finally, let $w \leqslant v \leqslant u$, $w \leqslant p \leqslant q$, $\langle wv \rangle$, $\langle wp \rangle \in Q$, $\langle vu \rangle$, $\langle pq \rangle \in Q_r$. Hence $\langle vp \rangle \in Q$ and v, p are comparable, let e. g., $p \leqslant v$. Then $w \leqslant v \wedge q \leqslant v$, i. e. $\langle v \wedge q, v \rangle$, $\langle q, q \vee v \rangle \in Q$, and $p \leqslant v \wedge q \leqslant q$, i. e. $\langle v \wedge q, q \rangle \in Q_r$. If $v \wedge q < q < v \vee q$, then $q \notin R^1$, $\langle v \wedge q, q \rangle \in Q$, $\langle vq \rangle \in Q$, $v \parallel q$, which is a contradiction. Hence $q = v \wedge q$ and $\langle uq \rangle \in N$, or $q = v \vee q$, $v = v \wedge q$, $\langle vq \rangle \in Q_r$, $\langle uq \rangle \in Q_r, N$. The proof of the transitivity of N is complete. We see that N is a linear congruence on L and that $N = Q + Q_r$, hence N is a k-extension of Q.

Let $u \in R^0$, then $\langle x, x \vee u \rangle \in R$, $\langle x, (x \vee u) \wedge y \rangle \in R$. Therefore $(x \vee u) \wedge \wedge y \notin R^1$ and $\langle x, (x \vee u) \wedge y \rangle \in Q$, i. e. $x = (x \vee u) \wedge y$. By 1.1, 1.2, $\bar{r}(u) = x$ and therefore $u \in R^0_r$ holds. Thus $R^0 \subseteq R^0_r$.

Now let $u \in R^1_r$, i. e. $\bar{r}(u) = y$. Then $y \in R^1$, $y = x \vee (y \wedge u)$ give $y \wedge u \in R^1$ and $u \in R^1$. Hence $R^1_r \subseteq R$.

We denote $N^{\perp} = P$ and by 2.2 we get $P^0 = R^0 \cap R_r^0 = R^0$, $P^1 = R^1 \cap R_r^1 = R_r^1$. Therefore $L(P) = L - (R^0 \cup R_r^1)$.

Let c be a k-system in $L(P)$ over u, such that $\langle uv \rangle \notin N$ for any $v \in c$. Then $u \in L_r$ would give $c \subseteq L_r$ and $\langle uv \rangle \notin Q_r$ for any $v \in c$, which is a contradiction according to 3.4. Thus we have $u \in R_r^0$, $\bar{r}(u) = x$ and $u \notin R^1$. By the linearity of L/R and by lodim $L(R)/Q < k$, there is $v \in c$ such that $\langle uv \rangle \notin R$ and $\langle uw \rangle \in R$ for $w \in c$, $v \neq w$. As $\langle uv \rangle \notin Q$ holds, there is $v \in R^1$. Then $\bar{r}(v) = y \wedge (\bar{r}(v) \vee v) \in R^1$, hence $x < \bar{r}(v)$ and $\langle uv \rangle \notin R_r$. By the linearity of L/R_r we have $\langle uw \rangle \in R_r$, $\bar{r}(w) = x$ for any $w \in c$, $v \neq w$. By 1.9, there is $[st] \subseteq [uv]$ such that $\langle st \rangle \in Q_r$, $\langle su \rangle$, $\langle tv \rangle \in R_r$. As $v \in L_r$, $\bar{r}(v) < y$, then the assumption $t < v$ leads to a contradiction by 3.3. Further, $\langle su \rangle \in R_r$ gives $\bar{r}(s) = x$ and $s \notin R^1$. Therefore $\langle us \rangle \in Q$ holds, and from $\langle st \rangle \in Q_r$, $t = v$ we get $\langle uv \rangle \in N$. Thus we have proved lodim $L(P)/N < k$ and therefore N belongs to $\mathscr{A}(L)$ or $\mathscr{A}(Q_0, L)$, respectively. Here $\langle xy \rangle \notin Q$, $\langle xy \rangle \in N$ give a contradiction with the maximality of Q; therefore our assumption at the beginning of the proof cannot be fulfilled.

So there is $t \in [xy]$, $t \in R^1$, $\langle xt \rangle \notin Q$. In an analogous way as in the proof of 3.5 we find an upper k-chain $[t'x']$ transposed to $[xt]$. We get $x' \notin R^1$, $\langle t'x' \rangle \notin Q$ and $\langle x's \rangle \in Q$ for any $s \in [t'x']$, $s \notin R^0$. If $L(R) \neq 0$, we can choose $t \in L(R)$. Then also $x' \notin R^0$ holds and the situation is dual to the previous one. If $L(R) = 0$, then by 2.5 there is a prime interval $p = [uv]$ such that $Q = Q_p$. Then $x \in R^0$, $y \in R^1$ gives $u_0 = (x \vee u) \wedge y \in R^0$, $v_0 = (x \vee v) \wedge y \in R^1$ and $\langle u_0 v_0 \rangle \in Q$ implies that $p_0 = [u_0 v_0]$ is a prime interval projective with p. Hence $Q = Q_{p_0}$ and $Q_r \supseteq Q$ is again a contradiction with the maximality of Q. That proves $\langle xy \rangle \in Q$.

5. Dimensions

5.0. We have mentioned already in the previous section that the proof of the equality of the local and lattice dimensions of L will be done by induction through k. Now we formulate the induction hypothesis.

5.1. Induction hypothesis. If L_1 is a distributive lattice and lodim $L_1 < k$, then for any $Q_1 \in \mathscr{A}(L_1)$ there is $N_1 \in \mathscr{A}(Q_1, L_1)$ such that lodim $L_1/N_1 <$ lodim L_1.

5.2. *Let Q be a maximal element in $\mathscr{A}(L)$ or in $\mathscr{A}(Q_0, L)$, $R = Q^{\perp}$. If $x \in R^1$ is a lower k-element, then x is an upper bound of $-R^1$, i. e. $x \in \mathscr{U}(-R^1)$.*

Proof. Let x be a lower k-element, $x \in R^1$, $z \notin R^1$. For any k-system b over x let us denote $b_1 = \{y; y \in b \;\&\; y \wedge (z \vee x) = x\}$, $b_2 = \{y; y \in b \;\&\; y \wedge (z \vee x) > x\}$, $t = \bigvee b_1$, $s = x \wedge z = t \wedge z$; $L_1 = [st]$. Then $b_2' = \{y \wedge z; y \in b_2\}$ is an independent system over s of the same cardinality as b_2, orthogonal to L_1 over s. That implies lodim $L_1 \leqslant k -$ card $b_2 =$ card b_1. On the other hand we have lodim $L_1 \geqslant$ card b_1, so lodim $L_1 =$ card b_1. Analogically lodim $[x, x \vee z] =$ card b_2 is shown, therefore card b_2 depends only on the elements x, z. Let us denote $\mathrm{m}(x, z) =$ card b_2 and set $m = \max \{\mathrm{m}(x, z);$ x being a lower k-element in R^1 & $z \notin R^1\}$. In the following we shall suppose that card $b_2 = m > 0$, i. e. x is not comparable with z. As $x > s$ holds, we have lodim $L_1 =$ card $b_1 > 0$, so $m < k$ and $b_1 \neq 0$.

Further, if $L(R) \neq 0$, the element z can be chosen so that $z \in L(R)$. If $L(R) = 0$, then L/R has only two elements R^0, R^1 and according to 2.5, we can choose z in such a way that there is $z' \in R^1$, $z \prec z' \leqslant z \vee x$ and $\langle zz' \rangle \in Q$. Then $s' = x \wedge z' \in R^1$, $\langle ss' \rangle \in Q$ and by 2.5 we have $s \prec s' \leqslant x$. Hence for any $y \in L_1 \cap R^0$, $y \prec y' = y \vee s' \in L_1 \cap R^1$ holds. Therefore there is lodim $(L_1 \cap R^0) <$ lodim L_1 as well as in the case $z \in L(R)$.

Let us denote $Q_1 = Q \cap (L_1 \times L_1)$, $R_1 = R \cap (L_1 \times L_1)$ the restrictions of Q, R to L_1. Analogously as in 1.10 we can show, using 1.9, that $R_1 = Q_1^{\perp}$. Evidently $L(R_1) \subseteq L_1 \cap L(R)$ and, as $\langle sy \rangle \in R$ holds for any $y \in b_2'$, the orthogonality of L_1 and b_2' over s implies lodim $L(R_1)/Q_1 < k - m =$ lodim L_1. Hence the linear congruence Q_1 belongs to $\mathscr{A}(L_1)$.

By induction hypothesis there is a congruence N_1 on L_1, which is a $(k - m)$-extension of Q_1 and fulfils lodim $L_1/N_1 < k - m$. By 2.4, $N = \cup \{Q_r;$ $r \in L_1/N_1\} = \cup \{Q_r;\ r = [uv]\ \&\ \langle uv \rangle \in N_1\}$ is a linear congruence on L and $N_1 = N \cap (L_1 \times L_1)$.

If $\langle uv \rangle \in N$, $u \leqslant v$, $v \notin R^1$, then $\langle uv \rangle \in Q$. To prove it let us assume that $\langle uv \rangle \notin Q$. Then $[uv]$ is projective with some $[u_1v_1] \subseteq L_1$, $\langle u_1v_1 \rangle \in N_1$, $\langle u_1v_1 \rangle \notin Q$. By the definition of N_1 then $[u_1v_1]$ (or some its non-trivial subinterval) is projective with a lower $(k - m)$-chain $[x_1y_1]$ in L_1, $\langle x_1y_1 \rangle \in N$, $\langle x_1y_1 \rangle \notin Q_1$. By 3.2, there is a lower $(k - m)$-system $c_1 \subseteq L_1$ over x_1 such that $y_1 \in c_1$. We denote $c_2 = \{y \vee x_1;\ y \in b_2'\}$, then $c = c_1 \cup c_2$ is a k-system in L over x_1. For any $y \in c_2$ we have $\langle x_1y \rangle \in R$ and for $y \in c_1$ we have $\langle x_1y \rangle \notin Q$. Thus, by 4.4, $\langle x_1y \rangle \in R$ holds for any $y \in c_1$. As the element z was chosen so that lodim $(L_1 \cap R^0) < k - m$, $c_1 \subseteq R^1$, $x_1 \in R_1$. If we denote $z_1 = z \vee (v \wedge y_1) \notin$ $\notin R^1$, then for $y \in b_2'$ there is $y \leqslant z_1$ and $(y \vee x_1) \wedge (z_1 \vee x_1) = y \vee x_1 > x_1$. There is also $y_1 \wedge (z_1 \vee x_1) = y_1 \wedge (z \vee (v \wedge y_1) \vee x_1 = (y_1 \wedge z) \vee (y_1 \wedge v) \vee$ $\vee (y_1 \wedge x_1) = s \vee (y_1 \wedge (v \vee x_1)) = s \vee y_1 = y_1 > x_1$, using the projectivity of $[uv]$, $[x_1y_1]$. Hence $m(x_1, z_1) > m$ and we have obtained a contradiction with the maximality of m.

Now we define a relation J in L as follows: $\langle uw \rangle, \langle wu \rangle \in J$, if there is $v \in L$, $u \leqslant v \leqslant w$ such that $\langle uv \rangle \in Q$ and $\langle vw \rangle \in N$. Evidently, J is reflexiv, symmetric and compatible with the lattice operations in L. The transitivity of J is proved quite analogously to the transitivity of N in the proof of 4.4. Thus, J is a linear congruence on L and, as the orthogonality of L_1 and b_2'

over s implies that every lower $(k - m)$-chain in L_1 is a lower k-chain in L, J is a k-extension of Q. We denote $N_1^{\perp} = P_1$, $N^{\perp} = P$, $J^{\perp} = K$.

Let $y \notin K^0$, then there is $r = [uv]$, $\langle uv \rangle \in J$, $u \neq v$ such that $\bar{r}(y) = v$. Then $[u \wedge y, v \wedge y] \sim [uv]$ holds, therefore r can be chosen so that $v \leqslant y$. The assumption $y \in R^0$ gives us $u, v \in R^0$, $\langle uv \rangle \in Q$, $u = v$. Thus $y \notin R^0$ and we have proved $R^0 \subseteq K^0$, i. e. $R^0 = K^0$ according to 2.2.

The definition of N_1 implies that there is $y_1 \in b_1$ such that $\langle xy_1 \rangle \in N_1$ and $\langle xy \rangle \in P_1$ for $y \in b_1$, $y_1 \neq y$. Let us denote $p = [xy_1]$.

First let $u \in R_p^1$, $u \notin R^1$, then $y_1 \wedge u \notin R^1$. As $[x \wedge u, y_1 \wedge u] \sim [xy_1]$ holds, there is $\langle x \wedge u, y_1 \wedge u \rangle \in N$ and therefore $\langle x \wedge u, y_1 \wedge u \rangle \in Q$, $\langle xy_1 \rangle \in Q$, $x = y_1$. This is a contradiction with $y_1 \in b_1$ and we have proved $R_p^1 \subseteq R^1$.

Now let $u \in R_p^1$, $u \notin K^1$. Then also for $u_1 = (u \wedge y_1) \vee s \in L_1$, $u_1 \in R_p^1$, $u_1 \notin K^1$ holds. Thus there are $v, w \in L$ such that $\langle vw \rangle \in J$, $v < w$ and $u_1 \wedge v = u_1 \wedge w$. Then $[vw] \sim [u_1 \vee v, u_1 \vee w]$ shows that we may assume $u_1 \leqslant v$. But $u_1 \in R_p^1$ and $u_1 \in R^1$ implies $\langle vw \rangle \in N$. Thus there are $v_1, w_1 \in L$ such that $[vw]$, $[v_1w_1]$ are projective and again we may assume $u_1 \leqslant v_1$. Hence we have $\langle v_1w_1 \rangle \in N_1$, $u_1 \leqslant v_1 < w_1 \leqslant t$. On the other hand, if we denote $s_1 = \bigvee (b_1 - \{y_1\})$, we get $[xy_1] \sim [s_1t] = q$, therefore $u_1 \in R_q^1$, i. e. $u_1 \vee s = u_1 \vee t = t$ and $\langle s_1t \rangle \in N_1$. From the properties of transposed intervals it follows that we have obtained a contradiction with the linearity of N_1. So $R_p^1 \subseteq K^1$ has been proved. At the same time we have $K^1 \subseteq P^1 \subseteq R_p^1$, thus $K^1 = P^1 = R_p^1$. Therefore $L(K) = L - (R^0 \cup R_p^1)$.

Let d be a k-system in $L(K)$ over w, such that $\langle wv \rangle \notin J$, i. e. $\langle wv \rangle \notin Q$ and by 4.4 $\langle wv \rangle \in R$ for any $v \in d$. Then $d \subseteq R^1$, $w \in R^1$ must hold.

We denote $x_1 = \bigvee \{\bar{p}(v); v \in d\}$, $r = [x_1y_1]$ and have $x_1 \in p$, $x_1 < y_1$, $d \subseteq R_r^0$. Then r is a $(k - m)$-chain in L_1 and there are a $(k - m)$-system a_1 in L_1 and a k-system a in L over x_1 such that $y_1 \in a_1$, $\bigvee a_1 = t$, $a_1 \subseteq a$ and for $y \in a$, $(z \vee x_1) \wedge y = x_1$ if and only if $y \in a_1$.

If we denote $u = x_1 \wedge w$, then $u \in R^1$ and denoting $a_0 = \{w \wedge y; y \in a \,\&\, (x_1 \vee w) \wedge y > x_1\}$, $d_0 = \{x_1 \wedge v; v \in d \,\&\, (x_1 \vee w) \wedge v > w\}$, $c_0 = \{y \wedge v; y \in a \,\&\, (x_1 \vee w) \wedge y = x_1 \,\&\, v \in d \,\&\, (x_1 \vee w) \wedge v = w \,\&\, (y \vee w) \wedge (v \vee x_1) > x_1 \vee w\}$ we obtain from the proof of 2.4 [1] that $c = a_0 \cup c_0 \cup d_0$ is a k-system over u.

We have $w \in R_r^0$ and therefore $(x_1 \vee w) \wedge y_1 = x_1$. As $y \in R_r^0$ holds for $y \in a$, $y \neq y_1$, there is $a_0 \subseteq R_r^0$. For any $v \in d, v \in R_r^0$ holds, hence $(y_1 \vee w) \wedge (v \vee x_1) = x_1 \vee w$ and $d_0, c_0 \subseteq R_r^0$. We have shown $c \subseteq R_r^0$.

Let us denote $c_1 = \{v; v \in c \,\&\, v \wedge (z \vee u) = u\}$, $c_2 = \{v; v \in c \,\&\, v \wedge (z \vee u) > u\}$. The maximality of m implies that card $c_2 \leqslant m$. On the other hand, we denote $u_1 = s \vee u$, $v_1 = s \vee v$, $v' = t \wedge v$, $v_1' = s \vee v' = t \wedge v_1$ for any $v \in c_1$, $c_1' = \{v_1'; v \in c\}$ and get $(z \vee x_1) \wedge (x_1 \vee v) = (z \vee u \vee x_1) \wedge \wedge (x_1 \vee v) = ((z \vee u) \wedge v) \vee x_1 = u \vee x_1 = x_1$ for any $v \in c_1$. As $[uv]$ is a chain, then $[x_1, x_1 \vee v]$ is also a chain. If $x_1 < x_1 \vee v$, then, by 3.2, $x_1 \vee v$ is comparable exactly with one element $y \in a_1$ ($y \neq y_1$ as $v \in R_r^0$) and $x_1 < < y \wedge (x_1 \vee v) = t \wedge (x_1 \vee v)$. As $[x_1 \wedge v, v] \sim [x_1 \wedge v_1, v_1] \sim [x_1, x_1 \vee v] = = [x_1, x_1 \vee v_1]$, also $x_1 \wedge v < v'$, $x_1 \wedge v_1 < v_1'$, i. e. $u_1 < v_1'$ must hold. If $x_1 = x_1 \vee v$, then $v \leqslant x_1 \leqslant t$, $v_1 \leqslant t$, $v_1' = v_1$ and again $u_1 < v_1'$. Hence c_1' is an independent system in L_1 over u_1 with the same cardinality as c_1, therefore card $c_1 \leqslant k - m$ and card $c_2 = m$, card $c_1 =$ card $c_1' = k - m$.

By the definition of N_1, there is $v \in c_1$ such that $\langle u_1 v_1' \rangle \in N_1$. The linearity of N_1 implies that $x_1 < x_1 \vee v$ is impossible, so that $v \leqslant x_1$, $v, v_1 \leqslant t$, $v_1' = v_1$, $\langle u_1 v_1 \rangle \in N_1$. Now $u \leqslant v \wedge u_1 = v \wedge (s \vee u) \leqslant v \wedge (z \vee u) = u$, $v \vee u_1 = = v \vee s \vee u = v \vee s = v_1$, $[uv] \sim [u_1 v_1]$ give $\langle uv \rangle \in N$.

From $v \leqslant x_1$ we get $v \in d_0$ (see also the proof of 2.4 [1]), hence there is $q \in d$ such that $q' = (x_1 \vee w) \wedge q > w$ and $v = x_1 \wedge q$. We have $v \wedge w = = x_1 \wedge q \wedge w = x_1 \wedge w = u$ and $v \vee w = (x_1 \wedge q) \vee w = (x_1 \vee w) \wedge \wedge (q \vee w) = q'$, i. e. $[uv] \sim [wq']$ and $\langle wq' \rangle \in N$. By the linearity of N, for any $y \in d$, $y \neq q$, $w < y' \leqslant y$ there is $\langle wy' \rangle \notin N$.

If $q' < q$, then $d' = \{q\} \cup \{q' \vee y;\ y \in d \,\&\, y \neq q\}$ is again a k-system in $L(K)$, this time over q', and again there is $y' \in d'$ such that $\langle q'y'' \rangle \in N$ for some y'', $q' < y'' \leqslant y'$. If $y' \neq q$, i. e. $y' = q' \vee y$, $y \in d$, $y \neq q$, then $[wy] \sim \sim [q'y']$ and $[w, y \wedge y''] \sim [q'y'']$, i. e. $[w, y \wedge y''] \in N$, $w < y \wedge y'' \leqslant y$, which is impossible. Therefore $y' = q$ and $\langle q'q'' \rangle \in N$, $q' < q'' \leqslant q$. Then $[q'q''] \sim \sim [x_1 \vee w, x_1 \vee q'']$, i. e. $x_1 \vee w < x_1 \vee q''$, $\langle x_1 \vee w, x_1 \vee q'' \rangle \in N$. On the other hand $x_1 \vee w, x_1 \vee q'' \in R_r^0$ give $[x_1 y_1] \sim [x_1 \vee w, y_1 \vee w]$ and $(y_1 \vee w) \wedge \wedge (x_1 \vee q'') = x_1 \vee w$. That is a contradiction with the linearity of N.

Thus $q' = q$ holds and $\langle wq \rangle \in N$, $\langle wq \rangle \in J$. We have proved that lodim $L(K)/J < k$. Therefore $J \in \mathscr{A}(L)$, $\mathscr{A}(Q_0, L)$, respectively, and $\langle xy_1 \rangle \notin Q$, $\langle xy_1 \rangle \in J$ give a contradiction with the maximality of Q. Hence the assumption $m > 0$ is false, $m = 0$ holds and the proof is complete by 3.2.

5.3. *Let us define a relation $\mathscr{P}$ in $\mathscr{A}(L)$ as follows: $\langle NQ \rangle \in \mathscr{P}$ if $N = Q$ or if $P = N^\perp$, $R = Q^\perp$, $P^1 \cup R^0 = L$ holds.*

Then $\mathscr{P}$ is a partial order in $\mathscr{A}(L)$.

Proof. The reflexivity of $\mathscr{P}$ follows immediately. Let $\langle NQ \rangle, \langle QN \rangle \in \mathscr{P}$, then have $P^0 \subseteq R^0$, $P^0 \supseteq R^0$, i. e. $P^0 = R^0$ and analogously $P^1 = R^1$. Hence $R^1 \cup R^0 = L$, L/R is the two-element lattice and $P = R$. From 2.5 also $N = Q$ follows. Therefore $\mathscr{P}$ is antisymmetric. To prove the transitivity of $\mathscr{P}$, let us set $\langle JN \rangle, \langle NQ \rangle \in \mathscr{P}$, $J \neq N$, $N \neq Q$, $K = J^\perp$. Then $K^1 \cup P^0 = L$ gives $K^1 \supseteq P^1$ and $P^1 \cup R^0 = L$ implies $K^1 \cup R^0 = L$, i. e. $\langle JQ \rangle \in \mathscr{P}$.

5.4. *Let M be a chain in $\langle \mathscr{A}(L), \mathscr{P} \rangle$. Then the congruence $Q_M = \sum M$ is linear.*

Proof. First let us define the relation Q_M as follows: $\langle uv\rangle$, $\langle vu\rangle \in Q_M$ if there are $u_0 = u \leqslant u_1 \leqslant \ldots \leqslant u_n = v$, $n \geqslant 0$, such that $\langle u_{i-1}u_i\rangle \in \cup M$, $i = 1, \ldots, n$. If we show that Q_M defined in this way is transitive, then evidently Q_M is a congruence fulfilling $Q_M \supseteq Q \in M$, $Q_M \subseteq \sum M$, i. e. $Q_M = \sum M$.

Thus let $x < y$, $x < z$, $\langle xy\rangle \in N \in M$, $\langle xz\rangle \in Q \in M$, $P = N^\perp$, $R = Q^\perp$, $N \neq Q$. The congruences N, Q are $\mathscr{P}$-comparable, let e. g., $\langle NQ\rangle \in \mathscr{P}$. There is $x \notin P^1$, otherwise $y \in P^1$, $x = y$. Therefore $x \in R^0$ and $z \notin R^0$, $z \in P^1$. If $y \in R^0$, then $y \wedge z \in R^0$ and $x \leqslant y \wedge z \leqslant z$ gives $\langle x, y \wedge z\rangle \in Q$, $x = y \wedge z$. Hence $[xy] \sim [z, y \vee z]$ and $z < y \vee z$, $\langle z, y \vee z\rangle \in N$, which is a contradiction with $z \in P^1$. Therefore $y \in P^1$ and $y \wedge z \in P^1$, $x \leqslant y \wedge z \leqslant y$ imply $\langle y \wedge z, y\rangle \in N$, $y \wedge z = y, y \leqslant z$, $\langle yz\rangle \in Q$. The transitivity of $\mathscr{P}$ is proved by iteration of the previous and dual reasoning. The linearity of Q_M follows immediately from the definition.

5.5. *Let M be a maximal chain in $\langle \mathscr{A}(Q_0, L) \cup \mathscr{A}_k(L), \mathscr{P}\rangle$, fulfilling*

(i) *if b is a k-system over x and $\langle xy\rangle \notin Q_M$ for any $y \in b$, then for any $Q \in M$, $R = Q^\perp$ there is $x \in \mathscr{U}(-R^1)$ or $\bigvee b \in \mathscr{L}(-R^0)$.*

Then lodim $L/Q_M < k$.

Proof. Let us suppose lodim $L/Q_M \geqslant k$. By 2.1, there is a k-system b in L over x such that for any $y \in b$, $\langle xy\rangle \notin Q_M$ holds. We denote $a_0 = \{Q; R = Q^\perp \;\&\; x \in \mathscr{U}(-R^1)\}$, $a_1 = \{Q; R = Q^\perp \;\&\; \bigvee b \in \mathscr{L}(-R^0)\}$, then $a = \langle a_0a_1\rangle$ is a cut in M. Further we denote $L^0 = \cap \{\mathscr{L}(-R^0);\ R = Q^\perp \;\&\; Q \in a_1\}$, $L^1 = \cap \{\mathscr{U}(-R^1);\ R = Q^\perp \;\&\; Q \in a_0\}$, $L_a = L^0 \cap L^1$ and see that L_a is a convex sublattice of L and $[x, \bigvee b] \subseteq L_a$, hence $\mathscr{A}_k(L_a) \neq 0$. Let N_a be a maximal element in $\mathscr{A}_k(L_a)$, let N be the congruence induced in L by N_a, let $P_a = N_a^\perp$, $P = N^\perp$. N is linear by 2.4.

As N_a is k-generated in L_a, there is a set M_N of lower k-chains in L_a such that $N_a = \sum \{Q_r \cap (L_a \times L_a); r \in M_N\}$. By 2.2, 2.4, then $N = \sum \{Q_r;\ r \in M_N\}$, i. e. $N \in \mathscr{A}_k(L)$ and $P^0 = \cap \{R_r^0; r \in M_N\}$. If $Q \in a_0$, $R = Q^\perp$, then for any $r \in M_N$ we have $r \subseteq L_a$, therefore $r \subseteq \mathscr{U}(-R^1)$, $R_r^0 \supseteq -R^1$, i. e. $P^0 \supseteq -R^1$, $P^0 \cup R^1 = L$, $\langle QN\rangle \in \mathscr{P}$. Dually we can prove $\langle NQ\rangle \in \mathscr{P}$ for $Q \in a_1$, hence $K = M \cup \{N\}$ is a proper extension of M to a chain in $\langle \mathscr{A}(Q_0, L) \cup \mathscr{A}_k(L), \mathscr{P}\rangle$. We shall show that K also fulfils (i), which will be a contradiction with the maximality of M.

Let d be a k-system over u, for any $v \in d$ let $\langle uv\rangle \notin Q_K$, hence $\langle uv\rangle \notin Q_M$. Let a cut $c = \langle c_0c_1\rangle$ in M be determined by d in the same way as the cut a by b.

At first let $a \neq c$, let there exist $Q \in a_1 \cap c_0$ (the case $Q \in a_0 \cap c_1$ is dual). Then $\langle NQ\rangle \in \mathscr{P}$, i. e. $P^1 \cup R^0 = L$ holds. Then $P^1 \supseteq R^1$, $-P^1 \subseteq -R^1$ and $\mathscr{U}(-P^1) \supseteq \mathscr{U}(-R^1)$. Therefore $u \in \mathscr{U}(-R^1)$ gives us $u \in \mathscr{U}(-P^1)$.

Now let $a = c$, i. e. $[u, \bigvee d] \subseteq L_a$. As $\langle uv\rangle \notin N$, $\langle uv\rangle \notin N_a$ holds for any

$v \in d$, the maximality of N_a implies $u \in P^1_a$ or $\bigvee d \in P^0_a$. Let $u \in P^1_a$, then by 2.4 $u \in P^1$. Let us suppose $u \notin \mathscr{U}(-P^1)$, i. e. let there be $z \notin P^1$, $z \leqslant u$. As $u \in L^1 = \bigcap \{\mathscr{U}(-R^1); R = Q^\perp \& Q \in a_0\}$, $z \in R^1$ must hold for any $R = Q^\perp$, $Q \in a_0$. But $v \in R^1$ for any $v \in d$ holds too and therefore $u \in R^1$ and $s = u \wedge z \in \in \bigcap \{R^1;\ R = Q^\perp \& Q \in a_0\}$. Hence by 5.2, any lower k-chain r in $[s, \bigvee d]$ fulfils $r \subseteq L^1$. As $\bigvee d \in L^0$, evidently also $r \subseteq L^0$ holds, i. e. $r \subseteq L_a$. The proof of 5.2 gives us an extension N' of N, generated by a set of Q_r, where r are lower k-chains in $[s, \bigvee d]$, hence $N'_a = N' \cap (L_a \times L_a)$ is a k-extension of N_a in L_a. Moreover, we have, denoting $P' = N'^\perp$, $P'_a = N'^\perp_a$, $u \notin P'^1$ and $u \notin P'^1_a$, therefore N'_a is a proper k-extension of N_a, which is a contradiction with the maximality of N_a. We have proved $u \in \mathscr{U}(-P^1)$. In the case $\bigvee d \in P^0_a$, $\bigvee d \in \mathscr{L}(-P_0)$ is shown dually. The proof is complete.

5.6. *For any* $Q_0 \in \mathscr{A}(L)$ *there is* $N \in \mathscr{A}(Q_0, L)$ *such that* lodim $L/N < k$.

Proof. By 4.3, there is a maximal element $Q \in \mathscr{A}(Q_0, L)$. By 4.4, 5.2 the one-element chain $\{Q\}$ in $\langle \mathscr{A}(Q_0, L) \cup \mathscr{A}_k(L), \mathscr{P}\rangle$ fulfils 5.5(i). As the set of all chains in $\langle \mathscr{A}(Q_0, L) \cup \mathscr{A}_k(L), \mathscr{P}\rangle$ fulfilling 5.5(i) satisfies the condition of maximality, there is a maximal chain M in $\langle \mathscr{A}(Q_0, L) \cup \mathscr{A}_k(L), \mathscr{P}\rangle$ fulfiling 5.5(i) such that $Q \in M$, i. e. $Q_M \supseteq Q$. Then evidently $Q_M \in \mathscr{A}(Q_0, L)$ and lodim $L/Q_M < k$ by 5.5.

5.7. Theorem. *If L is a distributive lattice and* lodim L *is finite, then* lodim $L =$ $=$ dim $L =$ ldim L.

Proof. By induction is given by 4.0, 5.0, 5.1 and 5.6.

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