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A NOTE ON CLASSES OF REGULARITY IN SEMIGROUPS

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Let S be a semigroup. Denote by $\mathscr{R}_S(m, n)$ *classes of regularity* in S (see R. Croisot [1]), i. e.

$$\mathscr{R}_S(m, n) = \{a \mid a \in a^m S a^n\},$$

where m, n are non-negative integers and a^0 means the void symbol.

In [2] I. Fabrici studies sufficient conditions for $\mathscr{R}_S(m, n)$, where $m + n \geqq \geqq 2$, to be subsemigroups of S. In this note we shall study necessary and sufficient conditions for $\mathscr{R}_S(m, n)$ to be subsemigroups, semilattices of groups, right groups and groups, respectively.

It is known [3] that

(1) if $0 \leqq m_1 \leqq m_2$ and $0 \leqq n_1 \leqq n_2$, then $\mathscr{R}_S(2, 2) \subset \mathscr{R}_S(m_2, n_2) \subset \subset \mathscr{R}_S(m_1, n_1)$;

(2) $\mathscr{R}_S(1, 2) = \mathscr{R}_S(1, 1) \cap \mathscr{R}_S(0, 2)$;

(3) $\mathscr{R}_S(2, 1) = \mathscr{R}_S(1, 1) \cap \mathscr{R}_S(2, 0)$.

Denote by E the set of all idempotents of a semigroup S. Then (see Theorem 3 in [2]).

(4) if $1 \leqq m$ and $1 \leqq n$, then $\mathscr{R}_S(m, n) \neq \emptyset$ if and only if $E \neq \emptyset$.

Theorem 1. *The class of regularity $\mathscr{R}_S(1, 1)$ is a subsemigroup of a semigroup S if and only if*

(5) $E \neq \emptyset$ *and* $E^2 \subset \mathscr{R}_S(1, 1)$.

Proof. Let $\mathscr{R}_S(1, 1)$ be a subsemigroup of S. It follows from (4) that $E \neq \emptyset$. Since $E \subset \mathscr{R}_S(1, 1)$, hence $E^2 \subset \mathscr{R}_S(1, 1)$.

Let (5) hold. Then (4) implies that $\mathscr{R}_S(1, 1) \neq \emptyset$. Let $a, b \in \mathscr{R}_S(1, 1)$. Then $a = axa$, $b = byb$ for some $x, y \in S$ and $xa, by \in E$. According to (5) we have $(xa)(by) = (xa)(by)z(xa)(by)$ for some $z \in S$. Therefore, $ab = (axa)(byb) =$

$= a(xa)(by)b = a(xa)(by)z(xa)(by)b = (axa)b(yzx)a(byb) = (ab)u(ab)$, where $u = yzx$. Hence $ab \in \mathscr{R}_S(1, 1)$.

Remark. From [3] (p. 108) it is known that if $\mathscr{R}_S(1, 1)$ is a subsemigroup of S, then $\mathscr{R}_S(1, 1)$ is a regular semigroup.

Corollary 1 (cf. [2], Theorem 4(c)). *If E is a subsemigroup of S, then $\mathscr{R}_S(1, 1)$ is a subsemigroup of S.*

Corollary 2 (cf. [2], Theorem 4(d)). *$\mathscr{R}_S(1, 1)$ is an inverse subsemigroup of a semigroup S if and only if*

(6) *$E \neq \emptyset$ and any two idempotents of S commute.*

Proof. It is known [4] that a semigroup S is inverse if and only if S is regular and any two idempotents of S commute. Evidently (6) implies (5). The rest of the proof follows from Theorem 1 and from the Remark.

Let a be an element of a semigroup S. The right (left) principal ideal generated by a is denoted by $\mathbf{R}(a) = a \cup aS$ ($\mathbf{L}(a) = a \cup Sa$).

Lemma 1. *Let $a, b \in S$.*

1. *If $ab \in \mathscr{R}_S(2, 0)$, then $ab \in \mathbf{R}(aba)$.*
2. *If $ab \in \mathbf{R}(aba)$ and $ba \in \mathbf{R}(bab)$, then $ab \in \mathscr{R}_S(2, 0)$.*

Proof. 1. If $ab \in \mathscr{R}_S(2, 0)$, then $ab = (ab)^2x$ for some $x \in S$. This implies that $ab = aba(bx) \in \mathbf{R}(aba)$.

2. If $ab \in \mathbf{R}(aba)$, then $ab = abax$ for some $x \in S$ or $ab = aba = aba^2$ and in both cases we obtain that $ab = abau$ for some $u \in S$. If $ba \in \mathbf{R}(bab)$, then analogously we can prove that $ba = babv$ for some $v \in S$. Hence we have $ab = (ab)^2z$, where $z = vu$.

Theorem 2. *Let S be a semigroup and $\mathscr{R}_S(2, 0) \neq \emptyset$. Then $\mathscr{R}_S(2, 0)$ is a subsemigroup of S if and only if*

(7) *$ab \in \mathbf{R}(aba)$ for any $a, b \in \mathscr{R}_S(2, 0)$.*

Proof. Let $\mathscr{R}_S(2, 0)$ be a subsemigroup of S. If $a, b \in \mathscr{R}_S(2, 0)$, then $ab \in \mathscr{R}_S(2, 0)$. It follows from Lemma 1 that (7) holds.

Let (7) hold. If $a, b \in \mathscr{R}_S(2, 0)$, then from Lemma 1 it follows that $ab \in \mathscr{R}_S(2, 0)$. This means that $\mathscr{R}_S(2, 0)$ is a subsemigroup of S.

Right identities of an element $a \in \mathscr{R}_S(2, 0)$ of the form ax are called *local right identities.*

Corollary 1 (cf. [2], Theorem 5(b)). *If the product of local right identities of the elements $a, b \in \mathscr{R}_S(2, 0)$ is a right identity of the element ab, then $\mathscr{R}_S(2, 0)$ is a subsemigroup of a semigroup S.*

Proof. If $a, b \in \mathscr{R}_S(2, 0)$, then $a = a^2x$ and $b = b^2y$ for some $x, y \in S$.

The element ax (by) is a local right identity of a (of b). According to the assumption we have $ab = ab(ax)(by) \in \mathbf{R}(aba)$. Hence Theorem 2 implies that $\mathscr{R}_S(2, 0)$ is a subsemigroup of S.

Corollary 2 (cf. [2], Theorem 5 (c)). *If every local right identity of any element of $\mathscr{R}_S(2, 0)$ belongs to the centre of a semigroup S, then $\mathscr{R}_S(2, 0)$ is a subsemigroup of S.*

Proof. If $a, b \in \mathscr{R}_S(2, 0)$, then $a = a^2x$ for some $x \in S$. Therefore $ab = (a^2x)b = a(ax)b = ab(ax) \in \mathbf{R}(aba)$. It follows from Theorem 2 that $\mathscr{R}_S(2, 0)$ is a subsemigroup of S.

Theorem 3. *The class of regularity $\mathscr{R}_S(2, 1)$ is a subsemigroup of a semigroup S if and only if* (5) *and*

(8) $ab \in \mathbf{R}(aba)$ *for any* $a, b \in \mathscr{R}_S(2, 1)$

hold.

Proof. Let $\mathscr{R}_S(2, 1)$ be a subsemigroup of S. It follows from (4) that $E \neq \emptyset$. Since $E \subset \mathscr{R}_S(2, 1)$, hence, by (1), we have $E^2 \subset \mathscr{R}_S(2, 1) \subset \mathscr{R}_S(1, 1)$. This means that (5) holds. If $a, b \in \mathscr{R}_S(2, 1)$, then $ab \in \mathscr{R}_S(2, 1)$. According to (1) we have $ab \in \mathscr{R}_S(2, 0)$. It follows from Lemma 1 that $ab \in \mathbf{R}(aba)$ and thus (8) holds.

Let (5) and (8) hold. Then (4) implies that $\mathscr{R}_S(2, 1) \neq \emptyset$. Let $a, b \in \mathscr{R}_S(2, 1)$. Then by (1) we have $a, b \in \mathscr{R}_S(1, 1)$. Theorem 1 and (5) imply that $\mathscr{R}_S(1, 1)$ is a subsemigroup of S and thus $ab \in \mathscr{R}_S(1, 1)$. According to (8) we have $ab \in \mathbf{R}(aba)$ and $ba \in \mathbf{R}(bab)$. Lemma 1 implies that $ab \in \mathscr{R}_S(2, 0)$. It follows from (3) that $ab \in \mathscr{R}_S(2, 1) = \mathscr{R}_S(1, 1) \cap \mathscr{R}_S(2, 0)$. The class of regularity $\mathscr{R}_S(2, 1)$ is a subsemigroup of S.

Corollary. *$\mathscr{R}_S(2, 1)$ is a subsemigroup of a semigroup S if and only if $\mathscr{R}_S(1, 1)$ is a subsemigroup of S and $\mathscr{R}_S^2(2, 1) \subset \mathscr{R}_S(2, 0)$.*

Lemma 2. *The class of regularity $\mathscr{R}_S(2, 2)$ is a union of all subgroups of a semigroup S.*

Proof. From [3] (pp. 139, 424) it is known that an element $a \in S$ belongs to some subgroup of S if and only if a is totally regular, i. e. $a = axa$ for some $x \in S$ and $xa = ax$. We shall prove that $\mathscr{R}_S(2, 2)$ is the set of all totally regular elements of S.

Let a be a totally regular element of S. Then $a = axa$ for some $x \in S$ and $ax = xa$. This implies that $a = (axa)x(axa) = a^2x^3a^2 \in \mathscr{R}_S(2, 2)$.

Let now $a \in \mathscr{R}_S(2, 2)$. Then $a = a^2ya^2$ for some $y \in S$. Put $x = aya$. Then we have $a = axa$ and $xa = aya^2 = a^2ya^2ya^2 = a^2ya = ax$.

Lemma 3. $\mathscr{R}_S(2, 2) = \mathscr{R}_S(2, 0) \cap \mathscr{R}_S(0,2)$.

(See Lemma 1 in [2].)

Proof. It follows from (1) that $\mathscr{R}_S(2, 2) \subset \mathscr{R}_S(2, 0) \cap \mathscr{R}_S(0, 2)$. Let $x \in \mathscr{R}_S(2, 0) \cap \mathscr{R}_S(0, 2)$. Then $x \in x^2S \subset \mathbf{R}(x^2)$ and $x^2 \in xS \subset \mathbf{R}(x)$. It follows that $\mathbf{R}(x) = \mathbf{R}(x^2)$. Analogously we can prove that $\mathbf{L}(x) = \mathbf{L}(x^2)$. From [5] it is known that x belongs to some subgroup of S. Lemma 2 implies that $x \in \mathscr{R}_S(2, 2)$. Therefore $\mathscr{R}_S(2, 2) = \mathscr{R}_S(2, 0) \cap \mathscr{R}_S(0, 2)$.

Theorem 4. *The class of regularity $\mathscr{R}_S(2, 2)$ is a subsemigroup of a semigroup S if and only if $E \neq \emptyset$ and*

(9) $ab \in \mathbf{R}(aba) \cap \mathbf{L}(bab)$ *for any* $a, b \in \mathscr{R}_S(2, 2)$

holds.

Proof. Let $\mathscr{R}_S(2, 2)$ be a subsemigroup of S. It follows from (4) that $E \neq \emptyset$. If $a, b \in \mathscr{R}_S(2, 2)$, then $ab \in \mathscr{R}_S(2, 2)$. By Lemma 3 we have $ab \in \mathscr{R}_S(2, 0) \cap \cap \mathscr{R}_S(0, 2)$. Lemma 1 and its dual imply that $ab \in \mathbf{R}(aba) \cap \mathbf{L}(bab)$. Hence (9) holds.

Let $E \neq \emptyset$ and let (9) hold. Then (4) implies that $\mathscr{R}_S(2, 2) \neq \emptyset$. Let $a, b \in \in \mathscr{R}_S(2, 2)$, then by (9) we have $ab \in \mathbf{R}(aba) \cap \mathbf{L}(bab)$ and $ba \in \mathbf{R}(bab) \cap \mathbf{L}(aba)$. Lemma 1 and its dual imply that $ab \in \mathscr{R}_S(2, 0) \cap \mathscr{R}_S(0, 2)$. It follows from Lemma 3 that $ab \in \mathscr{R}_S(2, 2)$. The class of regularity $\mathscr{R}_S(2, 2)$ is a subsemigroup of S.

Corollary 1. *If $\mathscr{R}_S(2, 2)$ is a subsemigroup of a semigroup S, then $\mathscr{R}_S(1, 1)$ is a subsemigroup of S.*

Proof. If $\mathscr{R}_S(2, 2)$ is a subsemigroup of S, then $E \neq \emptyset$. Since $E \subset \mathscr{R}_S(2, 2)$, hence, by (1), we have $E^2 \subset \mathscr{R}_S(2, 2) \subset \mathscr{R}_S(1, 1)$. It follows from Theorem 1 that $\mathscr{R}_S(1, 1)$ is a subsemigroup of S.

Corollary 2. *$\mathscr{R}_S(2, 2)$ is an inverse subsemigroup of a semigroup S if and only if* (6) *and* (9) *hold.*

The proof follows from Theorem 4 and from Lemma 2.

Corollary 3. *$\mathscr{R}_S(2, 2)$ is an inverse subsemigroup of a semigroup S if and only if $\mathscr{R}_S(2, 2)$ is a subsemigroup of S and $\mathscr{R}_S(1, 1)$ is an inverse subsemigroup of S.*

Lemma 4. *A semigroup S is a semilattice of groups if and only if S is regular and $E \subset Z$, where Z is the centre of a semigroup S.*

Proof. Let S be a regular semigroup and $E \subset Z$. If $a \in S$, then $a = axa$ for some $x \in S$. Evidently $ax \in E$ and thus we have $a = (ax)a = a^2x$. From this it follows that $S = \mathscr{R}_S(2, 0)$. Analogously we can prove that $S = \mathscr{R}_S(0, 2)$. It follows from Lemma 3 that $S = \mathscr{R}_S(2, 2)$. Lemma 2 implies that S is a union of groups. Hence, by Corollary 2 of Theorem 2 in [6] we obtain that S is a semilattice of groups.

Let S be a semilattice of groups. If $a \in S$, then according to Lemma 6 in [7] we have $\mathbf{R}(a) = \mathbf{L}(a)$. Since S is a regular semigroup, then $aS = a \cup aS = \mathbf{R}(a) = \mathbf{L}(a) = Sa \cup a = Sa$. This means that S is a normal semigroup. It follows from Lemma 1 in [8] that $E \subset Z$.

Theorem 5 (cf. [2], Theorem 6). *Let S be a semigroup and let $1 \leqq m$, $1 \leqq n$. Then the class of regularity $\mathscr{R}_S(m, n)$ is a semilattice of groups if and only if*

(10) $E \neq \emptyset$ *and* $ae = ea$ *for any* $a \in \mathscr{R}_S(m, n)$ *and any* $e \in E$.

Proof. If $\mathscr{R}_S(m, n)$ is a semilattice of groups, then from Lemma 4 and (4) it follows that (10) holds.

Let (10) hold. If $a \in \mathscr{R}_S(m, n)$, then from (1) it follows that $a \in \mathscr{R}_S(1, 1)$. This means that $a = axa$ for some $x \in S$. Since $ax \in E$, hence, by (10), we have $a = (ax)a = a^2x \in \mathscr{R}_S(2, 0)$. Similarly we obtain that $a \in \mathscr{R}_S(0, 2)$. From Lemma 3 we have $a \in \mathscr{R}_S(2, 2)$ and thus $\mathscr{R}_S(m, n) \subset \mathscr{R}_S(2, 2)$. By (1) $\mathscr{R}_S(m, n) = \mathscr{R}_S(2, 2)$.

We shall prove that (9) holds. If $a, b \in \mathscr{R}_S(2, 2)$, then $a = a^2xa^2$ for some $x \in S$. Since $axa^2 \in E$, hence, by (10), $ab = a(axa^2)b = ab(axa^2) \in \mathbf{R}(aba)$. Similarly we obtain that $ab \in \mathbf{L}(bab)$. Theorem 4 implies that $\mathscr{R}_S(2, 2)$ is a subsemigroup of S. It follows from Lemma 2 that $\mathscr{R}_S(2, 2)$ is a regular semigroup. According to Lemma 4 and (10) we obtain that $\mathscr{R}_S(m, n) = \mathscr{R}_S(2, 2)$ is a semilattice of groups.

Corollary. *Let S be a semigroup and let $1 \leqq m$, $1 \leqq n$. If* (10) *holds, then $\mathscr{R}_S(m, n) = \mathscr{R}_S(m + k, n + l)$ for any non-negative integers k, l.*

A semigroup S is called *right simple* if S is the only right ideal of S. A semigroup S is said to be *left cancellative* if in S the left cancellation law holds, that is $ax = ay$ implies $x = y$ for all a, x, y in S. A semigroup S is called a *right group* if it is right simple and left cancellative.

Lemma 5. *A semigroup S is a right group if and only if S is regular and $fe = e$ for any $e, f \in E$.*

Proof. Let S be a regular semigroup and $fe = e$ for any $e, f \in E$. Let $a, b \in S$. Then $a = aua$, $b = bvb$ for some $u, v \in S$. Put $x = ub$. Since $au, bv \in E$, hence $ax = aub = (au)(bv)b = (bv)b = b$. Therefore, S is right simple. Let $ax = ay$ for $a, x, y \in S$. Since S is regular, hence $a = aza$, $x = xux$, $y = yvy$ for some $z, u, v \in S$. Thus we have $axux = ayvy$. Postmultiplying by z, we have $zaxux = zayvy$. Since $za, xu, yv \in E$, then $x = (xu)x = (za)(xu)x = (za)(yv)y = (yv)y = y$. Therefore, S is left cancellative. Thus S is a right group.

Let S be a right group. From Theorem 1.27 in [4] it follows that S is regular and E is a right zero semigroup.

Theorem 6. *Let S be a semigroup and let $1 \leqq m$, $1 \leqq n$. Then the class of regularity $\mathscr{R}_S(m, n)$ is a right group if and only if*

(11) $E \neq \emptyset$ *and* $fe = e$ *for any* $e, f \in E$.

Proof. If $\mathscr{R}_S(m, n)$ is a right group, then from Lemma 5 and (4) it follows that (11) holds.

Let (11) hold. This and (4) imply that $\mathscr{R}_S(1, 1) \neq \emptyset$. It follows from the Remark and from Lemma 5 that $\mathscr{R}_S(1, 1)$ is a right group. Since $\mathscr{R}_S(1, 1)$ is a union of groups, then, by Lemma 2, we have $\mathscr{R}_S(1, 1) \subset \mathscr{R}_S(2, 2)$. According to (1) we obtain that $\mathscr{R}_S(m, n) \subset \mathscr{R}_S(1, 1) \subset \mathscr{R}_S(2, 2) \subset \mathscr{R}_S(m, n)$. Therefore, $\mathscr{R}_S(m, n) = \mathscr{R}_S(1, 1)$ is a right group.

Corollary. *Let S be a semigroup and let $1 \leqq m$, $1 \leqq n$. If* (11) *holds, then* $\mathscr{R}_S(1, 1) = \mathscr{R}_S(2, 1) = \mathscr{R}_S(1, 2) = \mathscr{R}_S(2, 2)$.

Theorem 7 (cf. [2], Corollary of Theorem 4). *Let S be a semigroup and let $1 \leqq m$, $1 \leqq n$. Then the class of regularity $\mathscr{R}_S(m, n)$ is a group if and only if* $\operatorname{card} E = 1$.

The proof follows from Theorem 6 and its dual.

Corollary. *Let S be a semigroup. If* $\operatorname{card} E = 1$, *then* $\mathscr{R}_S(1, 1) = \mathscr{R}_S(2, 1) = \mathscr{R}_S(1, 2) = \mathscr{R}_S(2, 2)$.

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