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PARTIAL GROUPOIDS WITH SOME ASSOCIATIVITY CONDITIONS

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In the theory of semigroups not only subsemigroups of a semigroup $[S,.]$ are of some importance but also subsets of S which with respect to the operation (.) do not form a subsemigroup of $[S,.]$ (e. g. equivalence classes, relative ideals, a. s. o.). However, by the operation (.) a partial operation on every subset of S is induced, so that these objects may be regarded as partial groupoids. Of course, for all elements a, b, c of these partial groupoids $a(bc) = (ab)c$ holds, provided that all „products" are defined. Partial groupoids with this condition have been called associative partial groupoids and they are studied in section II of the present paper. It turns out that a generalization of the notion of a relative ideal of a semigoup and other notions connected with it makes it possible to obtain generalizations of many results from the theory of semigroups. However, some generalizations are rather complicated. It turned out that such generalizations become simple and natural if we consider such an associative partial groupoid where for all its elements a, b, c the following holds: 1) If bc and $(ab)c$ is defined, so is $a(bc)$, 2) If ab and $a(bc)$ is defined, so is $(ab)c$. For such a partial groupoid the term partial semigroup has been chosen and this notion is studied in section III of this paper. In this section also some results which follow from the study of partial semigroups are given for semigroups.

Section I contains such notions and results which can be formulated most generally, i. e. for partial groupoids without a further condition.

I would like to express my gratitude to Professor Š. Schwarz for useful suggestions and criticism.

I

Definition 1,1. *Let P be a set, K a subset of $P \times P$ and (.) a mapping of K into P. The mapping (.) will be called a partial binary operation on P and the pair $[P,.]$ will be called a partial groupoid.*

The set K will be called *the domain* of $[P,.]$ and will be denoted by $K[P,.]$. The set P is called *the carrier* of $[P,.]$.

If $K[P,.] \neq \emptyset$ and $K[P,.] \neq P \times P$, we shall say that $[P,.]$ is *a proper partial groupoid.*

If $K[P,.] = \emptyset$, we identify $[P,.]$ with the set P.

The case $K[P,.] = \emptyset$ and $P = \emptyset$ will be regarded in accordance with [3] as a case of a semigroup. This semigroup will be called *an empty semigroup.*

A partial groupoid $[P,.]$ is said to be *finite* if its carrier P is finite.

The image of an element $(a, b) \in K[P,.]$ will be denoted by $a \,.\, b$ and if there is no risk of misunderstanding, simply by ab.

Instead of $(a, b) \in K[P,.]$ we shall write $ab \in P$; further, $(ab)c \in P$ means $(a, b) \in K[P,.]$ and $(ab, c) \in K[P,.]$. Analogously $a(bc) \in P$ means $(b, c) \in \in K[P,.]$ and $(a, bc) \in K[P,.]$.

Let $A \subset P$, $B \subset P$. If $A \times B \subset K[P,.]$, we denote the image of $A \times B$ in P by $A \,.\, B$ or simply AB and we shall say that the set product AB of the subsets $A \subset P$, $B \subset P$ is defined. If either A or B is a one point set $\{a\}$, we omit the brackets in the set products $A\{a\}$ and $\{a\}B$.

If $A \subset P$, $B = \emptyset$, we define $AB = A$.

If $A = \emptyset$, $B \subset P$, we define $AB = B$.

If $[P,.]$ is a partial groupoid, $Q \subset P$, then $[Q,.]$ means the partial groupoid with an operation induced by the operation of $[P,.]$.

Definition 1,2. *Let $[P,.]$ be a partial groupoid, $Q \subset P$. If $a \in Q$, $b \in Q$, $ab \in P$ imply $ab \in Q$, we shall say that $[Q,.]$ is a stable partial subgroupoid of $[P,.]$. A stable partial subgroupoid $[Q,.]$ of $[P,.]$ is said to be a subgroupoid of $[P,.]$, if for every $a \in Q$, $b \in Q$, $ab \in P$ holds. In particular, if a subgroupoid of $[P,.]$ is a semigroup (group), it will be called a subsemigroup (subgroup) of $[P,.]$.*

Definition 1,3. *By a partial subgroupoid of a partial groupoid $[P,.]$ we mean any partial groupoid $[A,.]$, $A \subset P$.*

Evidently we have:

Lemma 1,1. *Let $[P,.]$ be a partial groupoid.*

a) *If $[Q,.]$ is a stable subgroupoid of $[P,.]$, then $K[Q,.] = K[P,.] \cap (Q \times Q)$.*

b) *If $[Q,.]$, $[R,.]$ are stable subgroupoids of $[P,.]$, then also $[Q \cap R,.]$ is a stable subgroupoid of $[P,.]$ and $K[Q \cap R,.] = K[Q,.] \cap K[R,.]$.*

c) *If $[A,.]$ is a partial subgroupoid of $[P,.]$, then $K[A,.] = K[P,.] \cap \{(a, b) \in \in A \times A : ab \in A\} = K[P,.] \cap A \times A - \{(a, b) \in A \times A : ab \bar{\in} A\}$.*

d) *If $[A,.]$, $[B,.]$ are partial subgroupoids of $[P,.]$, then not only $[A \cap B,.]$ but also $[A \cup B,.]$ is a partial subgroupoid of $[P,.]$ and $K[A \cap B,.] = K[A,.] \cap \cap K[B,.]$, $K[A \cup B,.] \supset K[A,.] \cup K[B,.]$.*

e) *Let $[Q,.]$ be a partial subgroupoid of $[P,.]$. If $[R,.]$ is a stable partial subgroupoid of $[P,.]$ and $R \subset Q$, then $[R,.]$ is also a stable partial subgroupoid of $[Q,.]$.*

The reverse of the statement e) need not hold. A stable partial subgroupoid $[R,.]$ of $[Q,.]$ need not be a stable subgroupoid of $[P,.]$.

Definition 1,4. *Let $[P,.]$ be a partial groupoid. An element $z \in P$ $(e \in P)$ is called a right zero (identity) element of $[P,.]$, if $pz \in P$ implies $pz = z$ $(pe \in P$ implies $pe = p)$ for every $p \in P$. An element $z \in P$ $(e \in P)$ is called a left zero (identity) element of $[P,.]$, if $zp \in P$ implies $zp = z$ $(ep \in P$ implies $ep = p)$ for every $p \in P$.*

An element $z \in P$ $(e \in P)$ is said to be a two-sided zero (identity) element of $[P,.]$, if it is both a right and a left zero (identity) element of $[P,.]$.

We note that neither a zero element nor an identity element of [P,.] (one-sided or two-sided) need be idempotents. Further, in a partial groupoid there can exist many left, right and even two-sided zero and identity elements.

Definition 1,5. *Let $[P,.]$ be a partial groupoid. A right zero (identity) element z (e) of $[P,.]$ is said to be a universal right zero (identity) element of $[P,.]$, if $pz \in P$ $(pe \in P)$ for every $p \in P$.*

A universal left zero (identity) element z (e) of $[P,.]$ is a left zero (identity) element z (e) of $[P,.]$ with the property $zp \in P$ $(ep \in P)$ for every $p \in P$.

If z (e) is both a universal right and a universal left zero (identity) element of $[P,.]$, it will be called a universal two-sided zero (identity) element of $[P,.]$.

Evidently, if there exists in $[P,.]$ a universal two-sided zero (identity) element, then it is unique and no other right or left zero (identity) elements can exist.

Further, if there exist in $[P,.]$ a left and a right zero (identity) element, at least one of them being universal, then they must coincide.

Definition 1,6. *Let $[A,.]$, $[B_1,.]$, $[B_2,.]$ be partial subgroupoids of a partial groupoid $[P,.]$ with the property $B_1A \subset A$ and $AB_2 \subset A$. Then $[A,.]$ will be called a (B_1, B_2)-ideal of $[P,.]$.*

In accordance with [3] let us denote $I(B_1, B_2) = \{A \subset P : B_1A \subset P, AB_2 \subset P\}$ and $I = \{I(B_1, B_2) : B_1 \subset P,\ B_2 \subset P\}$. Then $[A,.]$, $A \in \cup I$ will be called *a relative ideal of* $[P,.]$.

Notation. In the following $A \in I(B_1, B_2)$ means that $[A,.]$ is a (B_1, B_2)-ideal of a partial groupoid $[P,.]$.

We underline that in this paper all definitions and statements concerning a pair of sets B_1, B_2 include the case $B_1 = \emptyset$ or $B_2 = \emptyset$.

Our definition implies:

1) For every partial subgroupoid $[A,.]$ of $[P,.]$ we have $A \in I(\emptyset, \emptyset)$.

2) $\emptyset \in I(B_1, B_2)$ if and only if $B_1 = B_2 = \emptyset$.

Analogously as in the theory of relative ideals of semigroups the following lemma holds:

Lemma 1,2. *Let $[P,.]$ be a partial groupoid and $B_{11}, B_{21}, B_{12}, B_{22}$ subsets of P with $B_{11} \cap B_{12} = B_1$, $B_{21} \cap B_{22} = B_2$. Let $A_1 \in I(B_{11}, B_{21})$ and $A_2 \in \in I(B_{12}, B_{22})$. Then*

1) $A_1 \cup A_2 \in I(B_1, B_2)$.

2) *If $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cap A_2 \in I(B_1, B_2)$.*

In the following $[P,.]$ is a partial groupoid and $B_1 \subset P$, $B_2 \subset P$.

Definition 1,7. *Let $(B_1a)B_2 = B_1(aB_2) = B_1aB_2$, for $a \in P$. Denote the set $a \cup B_1a \cup aB_2 \cup B_1aB_2$ by ${}_{B_1}(a)_{B_2}$. Then the partial subgroupoid $[{}_{B_1}(a)_{B_2},.]$ of $[P,.]$ will be called a (B_1, B_2)-partial subgroupoid of $[P,.]$ generated by the element a.*

Remark. The foregoing definition is, evidently, a formal generalization of the notion of a principal (B_1, B_2)-ideal of a semigroup as defined in [3] in the case when $[B_1,.]$, $[B_2,.]$ are subsemigroups of a semigroup.

Definition 1,8. *Suppose that under the suppositions of the preceding Definition ${}_{B_1}(a)_{B_2} = {}_{B_1}(b)_{B_2}$ for some $a \in P$, $b \in P$. Then we shall say that the elements a and b are ${}_{B_1}\mathscr{I}_{B_2}$-equivalent and write $(a, b) \in {}_{B_1}\mathscr{I}_{B_2}$.*

Evidently we have:

Lemma 1,3. *The relation ${}_{B_1}\mathscr{I}_{B_2}$ is symmetric and transitive on P, thus it is a partial equivalence on the carrier of a partial groupoid with the domain $O({}_{B_1}\mathscr{I}_{B_2}) = \{p \in P : (B_1p)B_2 = B_1(pB_2)\}$.*

Remark. Evidently it can happen that ${}_{B_1}\mathscr{I}_{B_2} = \emptyset$.

Notation. In the following ${}_{B_1}\mathscr{I}$ means ${}_{B_1}\mathscr{I}_{B_2}$ for $B_2 = \emptyset$; analogously $\mathscr{I}_{B_2}$ means ${}_{B_1}\mathscr{I}_{B_2}$ for $B_1 = \emptyset$.

The relations ${}_{B_1}\mathscr{I}$, $\mathscr{I}_{B_2}$, ${}_{B_1}\mathscr{I}_{B_2}$ are generalizations of Green's relations. They are also generalizations of the notions introduced in [3]. We shall call them partial Green's relations (with respect to B_1, B_2).

We note that in contradistinction to the case when $[P,.]$ is a semigroup ${}_{B_1}\mathscr{I} \subset {}_{B_1}\mathscr{I}_{B_2}$, $\mathscr{I}_{B_2} \subset {}_{B_1}\mathscr{I}_{B_2}$ for $B_1 \neq \emptyset$, $B_2 \neq \emptyset$ need not hold if $[P,.]$ is a partial groupoid which is not a semigroup.

On the other hand we can introduce for every pair of partial subgroupoids $[B_1,.]$, $[B_2,.]$ of $[P,.]$ the following two partial equivalences on P.

Definition 1,9. *Let $a \cup B_1a \cup aB_2 \cup (B_1a)B_2 = b \cup B_1b \cup bB_2 \cup (B_1b)B_2$, for $a \in P$, $b \in P$. Then $(a, b) \in {}_1\mathscr{I}$.*

Let $a \cup B_1a \cup aB_2 \cup B_1(aB_2) = b \cup B_1b \cup bB_2 \cup B_1(bB_2)$, for $a \in P$, $b \in P$. Then $(a, b) \in \mathscr{I}_2$.

Corollary 1. *${}_1\mathscr{I}$ and $\mathscr{I}_2$ are partial equivalences on P with the domains*

$O({}_1\mathscr{I}) = \{p \in P : pB_2 \subset P,\ (B_1p)B_2 \subset P\}$, $O(\mathscr{I}_2) = \{p \in P : B_1p \subset P,$ $B_1(pB_2) \subset P\}$.

Corollary 2. *From Lemma* 1,3 *and Definition* 1,9 *it follows*: $O({}_{B_1}\mathscr{I}_{B_2}) \subset O({}_1\mathscr{I})$, $O({}_{B_1}\mathscr{I}_{B_2}) \subset O(\mathscr{I}_2)$ *and on* $O({}_{B_1}\mathscr{I}_{B_2})$ *we have* ${}_1\mathscr{I} = \mathscr{I}_2 = {}_{B_1}\mathscr{I}_{B_2}$. *Thus* ${}_{B_1}\mathscr{I}_{B_2} \subset {}_1\mathscr{I} \cap \mathscr{I}_2$.

It may be shown on examples that ${}_1\mathscr{I} \cap \mathscr{I}_2 \neq \emptyset$ implies in general ${}_{B_1}\mathscr{I}_{B_2} \subsetneqq {}_1\mathscr{I} \cap \mathscr{I}_2$.

Remark. In addition to the foregoing three partial equivalences on the carrier of a partial groupoid we can consider for every pair of partial subgroupoids $[B_1,.]$, $[B_2,.]$ of $[P,.]$ other three partial equivalences on P, ${}_{B_1}\mathscr{L}_{B_2}$, ${}_1\mathscr{L}$, $\mathscr{L}_2$, defined as follows:

1) $(a, b) \in {}_{B_1}\mathscr{L}_{B_2}$, if and only if $B_1aB_2 = B_1bB_2$ with the domain $O({}_{B_1}\mathscr{L}_{B_2}) = \{p \in P : (B_1p)B_2 = B_1(pB_2)\}$, i. e. $O({}_{B_1}\mathscr{I}_{B_2}) = O({}_{B_1}\mathscr{L}_{B_2})$.

2) $(a, b) \in {}_1\mathscr{L}$ if and only if $(B_1a)B_2 = (B_1b)B_2$ with the domain $O({}_1\mathscr{L}) = \{p \in P : (B_1p)B_2 \subset P\}$, i. e. $O({}_1\mathscr{I}) \subset O({}_1\mathscr{L})$.

3) $(a, b) \in \mathscr{L}_2$ if and only if $B_1(aB_2) = B_1(bB_2)$, with the domain $O(\mathscr{L}_2) = \{p \in P : B_1(pB_2) \subset P\}$, i. e. $O(\mathscr{I}_2) \subset O(\mathscr{L}_2)$.

Analogously as in the case of the partial equivalences given by Definition 1,9, we have ${}_{B_1}\mathscr{L}_{B_2} \subset {}_1\mathscr{L} \cap \mathscr{L}_2$ and in general the equality need not hold.

Definition 1,10. *A partial equivalence* ϱ *on* P *is said to be a partial right congruence on* $[P,.]$ *if for each* $p \in P$, $(a, b) \in \varrho$ *implies either that* $(ap, bp) \in \varrho$ *or that neither* ap *nor* bp *belong to the domain of* ϱ.

Analogously we define *the partial left congruence on* $[P,.]$.

Definition 1,11. *Let* $A \in I(B_1, B_2)$ *and there is no* $A' \subset P$, $A' \subsetneqq A$ *such that* $A' \in I(B_1, B_2)$. *Then* $[A,.]$ *is called a minimal* (B_1, B_2)*-ideal of* $[P,.]$.

Notation. In accordance with [3] $A \in I_m(B_1, B_2)$ will mean that $[A,.]$ is a minimal (B_1, B_2)-ideal of $[P,.]$.

Corollary. *Let* $A_1 \in I_m(B_1, B_2)$, $A_2 \in I_m(B_1, B_2)$. *Then either* $A_1 = A_2$ *or* $A_1 \cap A_2 = \emptyset$.

Definition 1,12. *Let* $I_m(B_1, B_2) = \{A_j : j \in J\}$. *The partial subgroupoid* $[\bigcup_{j\in J} A_j,.]$ *of* $[P,.]$ *will be called the* (B_1, B_2)*-socle of* $[P,.]$. $\bigcup_{j\in J} A_j$ *will be denoted by* $\mathfrak{S}(B_1, B_2)$.

Evidently, it may happen that $\mathfrak{S}(B_1, B_2) = \emptyset$ and $[P,.]$ is a finite partial groupoid.

Definition 1,13. *Denote* $aa \doteq a^2$ *if and only if* $aa \in P$. *Let further* $a^2a \in P$, $aa^2 \in P$ *and* $a^2a = aa^2$. *In this case and only in this case we shall say that there* *exists the 3rd power of* $a \in P$. *In general, suppose that recurrently* $aa^{m-1} = a^2a^{m-2} = \ldots = a^{m-1}a$, $m = 2, 3, \ldots n$.

Again in this case and only in this case the element $t = aa^{n-1} = a^2a^{n-2} = \ldots = a^{n-1}a$ *is said to be the n-th power of a and will be denoted by* a^n.

Corollary. *If* a^n *exists for all* $n \geqq 2$, *then* $[\{a, a^2, a^3, \ldots\},.]$ *is a subsemigroup of* $[P,.]$. *If its order is finite, then analogously as in the case of a semigroup there exists a cyclic subgroup in* $[P,.]$.

We note that in contradistinction to the case of a finite semigroup a finite partial groupoid need not contain idempotents.

Definition 1,14. *Let* $[P,.]$ *be a partial groupoid. Suppose that there exists an element* $a \in P$ *such that* $xa \in P$ *for every* $x \in P$. *Then the transformation* ϱ_a *of* P *defined by* $x\varrho_a = xa$ *for all* $x \in P$ *will be called the inner right translation of* $[P,.]$ *corresponding to the element* $a \in P$.

Analogously we define the notion of the inner left translation λ_a of $[P,.]$, corresponding to the element $a \in P$.

Of course, the set of inner right or inner left translations of $[P,.]$ may be empty.

Definition 1,15. *By an inner right translation of* $[P,.]$, *corresponding to the element* $a \in P$ *and relative to the set* $P' \subset P$, $P' \neq \emptyset$, *we mean a mapping* $\varrho_a^{P'}$ *of* P' *into* P *given by* $p\varrho_a^{P'} = pa$ *for every* $p \in P'$. (This means: P' contains only such elements $p \in P$ for which $pa \in P$.)

Remark. Evidently, to an element $a \in P$ there exists a $P' \neq \emptyset$ such that $\varrho_a^{P'}$ is defined if and only if there exists at least one element $p \in P$ with $pa \in P$.

Analogously we define an inner left translation of $[P,.]$, corresponding to the element $a \in P$ and relative to the set $P' \subset P$, $P' \neq \emptyset$.

Definition 1,16. *A transformation* ϱ *of* P *will be called a right translation of* $[P,.]$ *if for* $x \in P$, $y \in P$, $xy \in P$ *and* $x(y\varrho) \in P$ *we have* $(xy)\varrho = x(y\varrho)$. *By a left translation of* $[P,.]$ *we shall mean such a transformation* λ *of* P *that with* $x \in P$, $y \in P$, $xy \in P$ *and* $(x\lambda)y \in P$ *we have* $(xy)\lambda = (x\lambda)y$.

Remark. There exists at least one right and one left translation for every $[P,.]$, namely, the translation given by the identity mapping of P.

Further, we note that, in general, an inner right (left) translation of $[P,.]$ is not a right (left) translation of $[P,.)$.

Definition 1,17. *Let* $[P,.]$, $[T,*]$ *be partial groupoids and* φ *be a mapping of* P *into* T *with the property that* $a \,.\, b \in P$ *implies* $(a\varphi) * (b\varphi) \in T$ *and* $(a \,.\, b)\varphi = (a\varphi) * (b\varphi)$. *Then* φ *will be called a homomorphism of* $[P,.]$ *into* $[T,*]$. *If* φ *is such a one-to-one homomorphism of* $[P,.]$ *onto* $[T,*]$ *that* $a \,.\, b \in P$ *if and only* *if* $(a\varphi)\ {}_*(b\varphi) \in T$, *then* φ *is called an isomorphism of* $[P,.]$ *onto* $[T,_*]$; *the partial groupoids* $[P,.]$ *and* $[T,_*]$ *are then said to be isomorphic.*

From Definition 1,17 there follows:

1) If $K[P,.] = \emptyset$, then by every (one-to-one) mapping of the set P (onto) into T a (an isomorphism) homomorphism of $[P,.]$ (onto) into an arbitrary $[T,_*]$ is given.

2) If $K[T,_*] = \emptyset$ and $K[P,.] \neq \emptyset$, then no mapping of P into T can give a homomorphism of $[P,.]$ into $[T,_*]$.

3) If $K[P,.] = P \times P$, $K[T,_*] = T \times T$, then Definition 1,17 coincides with the definition of a homomorphism of a groupoid into a groupoid.

In the case 1) φ will be called a *trivial* homomorphism (isomorphism).

Example. Let $[G,.]$ be a group, $[H,.]$ a subgroup of $[G,.]$. It is easy to prove that $K[Ha,.] = \emptyset$ for every $a \notin H$. Therefore

1) The one-to-one mapping φ of Ha onto Hb, $a \notin H$, $b \notin H$ defined by $(ha)\varphi = hb$ is a trivial isomorphism of $[Ha,.]$ onto $[Hb,.]$.

2) By the one-to-one mapping φ of the set Ha onto H, $a \notin H$, defined by $(ha)\varphi = h$ a trivial homomorphism is given, but not a trivial isomorphism of $[Ha,.]$ onto $[H,.]$.

It is easy to prove

Lemma 1,4. 1) *Let* φ *be a homomorphism of a partial groupoid* $[P,.]$ *into a partial groupoid* $[T,_*]$. *Then*

a) $[P\varphi,_*]$ *is a partial subgroupoid of* $[T,_*]$ *and for its domain we have* $K[P\varphi,_*] \supset \{(c, d)\}$ *with* $(c\varphi^{-1}) . (d\varphi^{-1}) \in P$.

b) *If* $[R,_*]$ *is a stable partial subgroupoid of* $[P\varphi,_*]$, *then* $[R\varphi^{-1},.]$ *is a stable partial subgroupoid of* $[P,.]$.

c) *If* $[A,.]$ *is a* (B_1, B_2)*-ideal of* $[P,.]$, *then* $[A\varphi,_*]$ *is a* $(B_1\varphi, B_2\varphi)$*-ideal of* $[T,_*]$ *(but also of* $[P\varphi,_*]$*).*

2) *Let* $[P,.]$ *be a groupoid,* $[T,_*]$ *a partial groupoid.*

If φ *is a homomorphism of* $[P,.]$ *into* $[T,_*]$, *then* $[P\varphi,_*]$ *is a subgroupoid of* $[T,_*]$.

As we can see from the following examples, the reverse of b) and c) need not hold.

Let us define on the set $P = \{a, b\}$ the following partial groupoids:

$[P,.]$ with the multiplication table

	a	b
a	–	–
b	–	b

,

$[P,_*]$ with the multiplication table

	a	b
a	b	$-$
b	$-$	b

and

$[P,_+]$ with the multiplication table

	a	b
a	a	$-$
b	$-$	b

.

By the identity mapping of P a homomorphism of $[P,.]$ onto $[P,_*]$ as well as of $[P,.]$ onto $[P,_+]$ are given. We see that

1) $[\{a\},.]$ is a stable partial subgroupoid of $[P,.]$ but its homomorphic image is not a stable partial subgroupoid of $[P,_*]$.

2) $[\{a\},_+]$ is a relative ideal of $[P,_+]$, but $[\{a\},.]$ is not a relative ideal of $[P,.]$.

II

Definition 2,1. *A partial groupoid $[P,.]$ will be called an associative partial groupoid, if for $a \in P$, $b \in P$, $c \in P$, $(ab)c = a(bc)$ if and only if $(ab)c \in P$ and $a(bc) \in P$.*

Our definition implies: If A, B, C are subsets of the carrier of an associative partial groupoid $[P,.]$ with $(AB)C \subset P$ and $A(BC) \subset P$, then $(AB)C = A(BC)$. In this case and only in this case we shall write ABC without brackets; in particular, if $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, we shall write abc.

Example 2,1. Let $[S,.]$ be a semigroup, $B \subset S$. Then every partial groupoid $[B,_*]$ with $K[B,_*] \subset K[B,.]$ and $a_* b = a . b$ for $a \in B$, $b \in B$ and $(a, b) \subset \subset K[B,_*]$ is an associative partial groupoid.

We evidently have:

Lemma 2,1. a) *A partial subgroupoid of an associative partial groupoid is an associative partial groupoid.*

b) *An associative partial groupoid $[P,.]$ is a semigroup if and only if $K[P,.] = = P \times P$.*

Notation. In what follows, $[P,.]_A$ will mean an associative partial groupoid.

In this section we give some results which follow from Section 1 taking the associativity into account.

Theorem 2,1. *Let ${}_{B_1}\mathscr{I}_{B_2}$, ${}_1\mathscr{I}$, $\mathscr{I}_2$ be partial equivalences on the carrier P of $[P,.]_A$ given by Definitions 1,8 and 1,9. Then:*

1) *For the domain $O({}_{B_1}\mathscr{I}_{B_2})$ of the partial equivalence ${}_{B_1}\mathscr{I}_{B_2}$ we have* $O({}_{B_1}\mathscr{I}_{B_2}) = \{p \in P : (B_1 p)B_2 \subset P,\ B_1(pB_2) \subset P\}$.

2) ${}_{B_1}\mathscr{I}_{B_2} = {}_1\mathscr{I} \cap \mathscr{I}_2$.

Proof. The first statement is evident.

2) With regard to Corollary 2 following from Definition 1,9, it is sufficient to prove ${}_1\mathscr{I} \cap \mathscr{I}_2 \subset {}_{B_1}\mathscr{I}_{B_2}$. Suppose that $(a, b) \in {}_1\mathscr{I} \cap \mathscr{I}_2$, i. e. $a \cup B_1a \cup \cup aB_2 \cup (B_1a)B_2 = b \cup B_1b \cup bB_2 \cup (B_1b)B_2$ and $a \cup B_1a \cup aB_2 \cup B_1(aB_2) = = b \cup B_1b \cup bB_2 \cup B_1(bB_2)$. Now in $[P,.]_A$ we have $(B_1a)B_2 = B_1(aB_2)$, $(B_1b)B_2 = B_1(bB_2)$ and so ${}_{B_1}(a)_{B_2} = {}_{B_1}(b)_{B_2}$.

Remark. For partial equivalences ${}_1\mathscr{L}$, $\mathscr{L}_2$, ${}_{B_1}\mathscr{L}_{B_2}$ introduced in the Remark following Definition 1,9 we have in the case $[P,.]_A$ analogously ${}_{B_1}\mathscr{L}_{B_2} = {}_1\mathscr{L} \cap \mathscr{L}_2$.

We show on the following example that in the case of $[P,.]_A$ from $ab \in P$, $bc \in P$ there does not follow $(ab)c \in P$, $a(bc) \in P$.

Example 2,2. Let $[P,.]_A$ be defined on the set $P = \{a, b, c\}$ by the following multiplication table (the sign (—) means that the corresponding pair of elements of P is not contained in $K[P,.]_A$):

	a	b	c
a	a	b	a
b	b	a	b
c	c	–	c .

Here $ca \in P$, $ab \in P$, but $(ca)b \notin P$, $c(ab) \notin P$. Further we have $c(bb) \in P$, but $(cb) \notin P$ and so $(cb)b \notin P$.

This example shows that an associative partial groupoid $[P,.]_A$ cannot be in general embedded in a semigroup $[S,_*]$ by adjoining a zero 0 and defining: If $(a, b) \in K[P,.]_A$, then $a_* b = a \,.\, b$, if $(a, b) \notin K[P,.]_A$, then $a_* b = 0$. (Compare with Conrad's Lemma 3,7 in [1].)

Definition 2,2. *Suppose that on the carrier of* $[P,.]_A$ *another* $[P,_*]_A$ *can be defined in such a way that* $K[P,.]_A \subsetneq K[P,_*]_A$ *and for* $(a, b) \in K[P,.]_A$ $a_* b = a \,.\, b$. *In this case we shall say that there exists an associative continuation of* $[P,.]_A$. *In the reverse case* $[P,.]_A$ *is said to be a closed associative partial groupoid.*

Example 2,2 is an example of a closed associative partial groupoid. For defining $c_* b = a$, the associativity law for the triple (b, c, b) does not hold. Defining $c_* b = b$, the triple (c, b, b) is not associative. Finally defining $c_* b = c$, the triple (b, c, b) is not associative. On the other hand there exist clearly $[P,.]_A$ which have many associative continuations, closed or not closed.

At the end of Section III we shall construct all partial groupoids defined on the two-point set $P = \{a, b\}$. It will turn out that no proper associative partial groupoid defined on this set is closed. The closed associative partial groupoids defined on $P = \{a, b\}$ coincide with semigroups. Example 2,2 shows that on a three-point set at least one closed associative partial groupoid which is not a semigroup can be defined.

In the following we shall give some results concerning *minimal relative ideals* of associative partial groupoids. The following theorems generalize some theorems from the theory of minimal relative ideals of semigroups (compare with [3]). We note that not all statements from the theory of semigroups can be generalized. There does not, e. g. exist a generalization of the following statemant (See [3], Corollary of Theorem 2,2): If $[B_1,.]$, $[B_2,.]$ are subsemigroups of a semigroup $[S,.]$, then for every minimal (B_1, B_2)-ideal $[M,.]$ of $[S,.]$ we have $M = B_1aB_2$ for every $a \in M$. To see it take Example 2,2 and choose $B_1 = \{b\}$, $B_2 = \emptyset$. Then $[M,.]$, $M = \{a, b\}$ is a minimal (B_1, B_2)-ideal of $[P,.]_A$ but neither $B_1a = M$ nor $B_1b = M$ hold.

It can be shown on examples that the assertion of the Theorem just mentioned need not hold even in the case when $[P,.]_A$ is a semigroup and $[B_1,.]$, $[B_2,.]$ are partial subgroupoids of $[P,.]_A$, which are not its subsemigroups. On the other hand, from the following theorem it follows that the statement just mentioned can be generalized in the case when $[P,.]$ is an associative partial groupoid and $[B_1,.]$, $[B_2,.]$ are subsemigroups of $[P,.]$.

We use the notation introduced after Definitions 1,6 and 1,11.

Theorem 2,2. *Let $[B_1,.]$, $[B_2,.]$ be partial subgroupoids of $[P,.]_A$, $L_1 \in I(B_1, \emptyset)$, $L_1 \subset B_1$, $R_2 \in I(\emptyset, B_2)$, $R_2 \subset B_2$.*

a) *If $M \in I_m(B_1, B_2)$, then $M = L_1aR_2$ for every $a \in M$.*

b) *If $[P,.]_A$ is a semigroup, $M \subset P$, then $M \in I_m(B_1, B_2)$ if and only if $M = L_1aR_2$ for every $a \in M$.*

Proof. a) We shall first show that under the suppositions of our Theorem $L_1aR_2 \subset P$, i. e. $(L_1a)R_2 \subset P$ and $L_1(aR_2) \subset P$ for every $a \in M$. Since $L_1 \subset B_1$, $a \in M$, we have $L_1a \subset M \subset P$ and also $(L_1a)R_2 \subset P$, because $(L_1a)R_2 \subset (L_1a)B_2 \subset MB_2 \subset M \subset P$. Analogously we obtain that $L_1(aR_2) \subset P$. This implies in $[P,.]_A$ $(L_1a)R_2 = L_1(aR_2)$. (Recall our convention in using the symbol L_1aR_2). With respect to the foregoing we have $L_1aR_2 \subset M$, hence it is sufficient to prove that $L_1aR_2 \in I(B_1, B_2)$. Since $L_1(aR_2) \subset M$ and $B_1M \subset M \subset P$, we have $B_1(L_1aR_2) = B_1[L_1(aR_2)] \subset P$, moreover, $(B_1L_1)(aR_2) \subset P$, since $B_1L_1 \subset L_1$, $L_1(aR_2) \subset P$. Hence $B_1(L_1aR_2) = (B_1L_1)(aR_2) \subset L_1(aR_2) = L_1aR_2$. Analogously we get $(L_1aR_2)B_2 = [(L_1a)R_2]B_2 = (L_1a)(R_2B_2) \subset (L_1a)R_2 = L_1aR_2$. Since $M \in I_m(B_1, B_2)$, we have $M = L_1aR_2$ for every $a \in M$.

b) With respect to a) it is sufficient to prove that $M \in I_m(B_1, B_2)$ if $M = L_1aR_2$ for every $a \in M$. Let $C \subset M$, $C \in I(B_1, B_2)$. Then for every $c \in C$ we have $L_1cR_2 = M \subset C$, since $L_1 \subset B_1$, $R_2 \subset B_2$, thus $M \in I_m(B_1, B_2)$.

Corollary. 1) *If $[B_1,.]$, $[B_2,.]$ are subsemigroups of $[P,.]_A$ and $[M,.]$ is a minimal (B_1, B_2)-ideal of $[P,.]_A$, then $M = B_1aB_2$ for every $a \in M$.*

b) *If* $[P,.]_A$ *is a semigroup,* $[B_1,.]$, $[B_2,.]$ *are its partial subgroupoids and there exist* $L_1 \in I(B_1, \emptyset)$, $L_1 \subset B_1$ *and* $R_2 \in I(\emptyset, B_2)$, $R_2 \subset B_2$, *then for any* $L \in I_m(B_1, \emptyset)$, $R \in I_m(\emptyset, B_2)$ *and* $c \in P$ *we have* $LcR \in I_m(B_1, B_2)$.

Proof. a) This statement is evident.

b) By a) of the foregoing Theorem we have $L = L_1 l$ for every $l \in L$ and $R = rR_2$ for every $r \in R$. Hence $M = LcR = L_1 lcrR_2$ for every $a = lcr \in M$, thus $LcR \in I_m(B_1, B_2)$ (by b) of the foregoing Theorem).

Remark. In order to obtain an analogous statement as in Corollary b) in the case of a proper associative partial groupoid, it will be necessary to add some supplementary suppositions. For, 1) in $[P,.]_A$ the set LcR need not be defined for any $c \in P$ (this is the case if one or both of the sets $(Lc)R$ and $L(cR)$ are not defined), 2) even if both $(Lc)R$ and $L(cR)$ are defined for some $c \in P$, i. e. $(Lc)R = L(cR)$, it may happen that $LcR \notin I(B_1, B_2)$.

Notation. Unless otherwise stated, $[B_1,.]$, $[B_2,.]$ will mean partial subgroupoids of $[P,.]_A$. We recall that the cases $B_1 = \emptyset$ or $B_2 = \emptyset$ are included in our considerations.

Theorem 2,3. a) *Let* $L \in I_m(B_1, \emptyset)$. *If for some* $c \in P$ $B_1(Lc) \subset P$, *then* $Lc \in I_m(B_1, \emptyset)$.

b) *Let* $R \in I_m(\emptyset, B_2)$. *If for some* $c \in P$ *we have* $(cR)B_2 \subset P$, *then* $cR \in I_m(\emptyset, B_2)$.

Proof. a) If $B_1 = \emptyset$, then for every $M \in I_m(B_1, \emptyset)$ M is a one-point set and conversely. The supposition $B_1(Lc) \subset P$ means in this case $\{ac\} \subset P$, i. e. $ac \in P$. This implies $\{ac\} \in I_m(B_1, \emptyset)$.

Let $B_1 \neq \emptyset$. We then have $B_1(Lc) = (B_1 L)c$, because all products necessary for this equality are defined. Hence $b(lc) = (bl)c$ for every $b \in B_1$, $l \in L$. Further we have $B_1(Lc) = (B_1 L)c \subset Lc$, hence $Lc \in I(B_1, \emptyset)$. Let $M \in I(B_1, \emptyset)$, $M \subset Lc$. Then $M = Bc$, $B = \{t \in L : tc \in M\}$. Let $l \in B \subset L$. Since $M \in I(B_1, \emptyset)$, we have $b(lc) \in bM \subset M$ for every $b \in B_1$. We further have $bl \in bB \subset bL \subset L$, $(bl)c \in M$, so that $bl \in B$. Hence $B \in I(B_1, \emptyset)$, $B \subset L$ and thus $M = Lc$.

The dual case b) can be treated analogously.

Remark. If $[P,.]_A$ is a semigroup, Theorem 2,3 coincides with Lemma 2,1 [3].

If $[P,.]_A$ is a proper associative partial groupoid, then the condition $B_1(Lc) \subset P$ is an essential supposition in a) and the condition $(cR)B_2 \subset P$ is an essential supposition in b) of the foregoing Theorem. The former condition includes the inclusion $Lc \subset P$, the latter condition includes the inclusion $cR \subset P$. However, it is not sufficient to suppose only $Lc \subset P$ instead of $B_1(Lc) \subset P$, $cR \subset P$ instead of $(cR)B_2 \subset P$. We want to show it (for the former case) on the following example:

Example 2,3. Let $[P,.]_A$ be defined on the set $P = \{a, b, c\}$ by the following multiplication table:

	a	b	c
a	a	b	--
b	a	b	c
c	a	b	c

Then $\{b\} \in I_m(\{a\}, \emptyset)$, $\{bc\} = \{c\} \subset P$, but $\{c\} \notin I_m(\{a\}, \emptyset)$ since $\{c\} \notin I(\{a\}, \emptyset)$.

Corollary. *Suppose that the suppositions of Theorem 2,3 are satisfied.*

a) *If $B_1[(Lc)R] \subset P$, then $(Lc)R = \bigcup_{j \in J} L_j$, $L_j \in I_m(B_1, \emptyset)$ and thus by Lemma 2,1 we have $(Lc)R \in I(B_1, \emptyset)$.*

b) *If $[L(cR)]B_2 \subset P$, then $L(cR) = \bigcup_{j \in J'} R_j$, $R_j \in I_m(\emptyset, B_2)$ and by Lemma 1,2 we have $L(cR) \in I(\emptyset, B_2)$.*

We note that, in general, $(Lc)R \notin I(\emptyset, B_2)$ and $L(cR) \notin I(B_1, \emptyset)$.

Lemma 2,2a. *Let $L_1 \in I(B_1, \emptyset)$, $L_1 \in B_1$, $R_2 \in I(\emptyset, B_2)$, $R_2 \subset B_2$. If for a set $M \subset P$ we have*

1) *$M = (L_1a)R_2$ for every $a \in M$,*
2) *$B_1(L_1a) \subset P$ for every $a \in M$,*
3) a) *$B_1M \subset P$,* b) *$MB_2 \subset P$,*

then $[M,.]$ is a minimal (B_1, B_2)-ideal of $[P,.]_A$.

Proof. If follows from 2) and 3a) (by the foregoing Corollary) that $M \in I(B_1, \emptyset)$. From 3b) it follows that $M \in I(\emptyset, B_2)$, since the supposition $[(L_1a)R_2]B_2 \subset P$ implies $[(L_1a)R_2]B_2 = (L_1a)(R_2B_2) \subset (L_1a)R_2$. Hence $M \in I(B_1, B_2)$. Let $M' \subset M$, $M' \in I(B_1, B_2)$. By 1) for every $a \in M'$ we have $M = (L_1a)R_2 \subset M'$, hence $M \in I_m(B_1, B_2)$.

Theorem 2,4a. *Suppose that there exists in $[P,.]_A$ a $(B_1, \emptyset)$-ideal $[L_1,.]$ with $L_1 \subset B_1$ and a $(\emptyset, B_2)$-ideal $[R_2,.]$ with $R_2 \subset B_2$.*

If $[L,.]$ is any minimal $(B_1, \emptyset)$-ideal and $[R,.]$ any minimal $(\emptyset, B_2)$-ideal and moreover for some $c \in P$

1) *$B_1(Lc) \subset P$,*
2) *$B_1[(Lc)R] \subset P$,*
3) *$[(Lc)R]B_2 \subset P$,*

then $[(Lc)R,.]$ is a minimal (B_1, B_2)-ideal of $[P,.]_A$.

Proof. By Theorem 2,2 a) we have $L = L_1l$ for every $l \in L$, $R = rR_2$ for every $r \in R$. By the assumption $lc \in P$ and $L_1(lc) \in P$, since $L_1 \subset B_1$. Hence

$L_1(lc) = (L_1l)c$. By the supposition $(lc)r \in P$ and $[(Lc)r]R_2 \subset P$. This implies $[(Lc)r]R_2 = (Lc)(rR_2)$. Thus we have $(Lc)R = (Lc)(rR_2) = [(Lc)r]R_2 = \{[(L_1l)c]r\}R_2 = \{[L_1(lc)]r\}R_2 = \{L_1[(lc)r]\}R_2$ (since by 2) $[L_1(lc)]r = L_1[(lc)r]$).

Denote $(Lc)R = M$. Then we have $M = (L_1a)R_2$ for every $a \in M$ (since every element $a \in M$ is of the form $a = (lc)r$ with some $l \in L$ and $r \in R$). By the suppositions 2) and 3) we have $B_1M \subset P$, $MB_2 \subset P$ and further $B_1(L_1a) = B_1[(Lc)r] \subset P$ (by the supposition 2)) for every $a \in M$. It follows from Lemma 2,2a that $M = (Lc)R \in I_m(B_1, B_2)$.

Analogously the dual of Lemma 2,2a and of Theorem 2,2a can be proved:

Lemma 2,2b. *Let* $L_1 \in I(B_1, \emptyset)$, $L_1 \subset B_1$, $R_2 \in I(\emptyset, B_2)$, $R_2 \subset B_2$. *If for some set* $M \subset P$ *we have*

1) $M = L_1(aR_2)$ *for every* $a \in M$,
2) $(aR_2)B_2 \subset P$ *for every* $a \in M$,
3) a) $B_1M \subset P$, b) $MB_2 \subset P$,

then $[M,.]$ *is a minimal* (B_1, B_2)*-ideal of* $[P,.]_A$.

Theorem 2,4b. *Replace in suppositions of Theorem* 2,4a *the conditions* 1), 2), 3),

by 1′) $(cR)B_2 \subset P$,
2′) $[L(cR)]B_2 \subset P$,
3′) $B_1[L(cR)] \subset P$.

Then $[L(cR),.]$ *is a minimal* (B_1, B_2)*-ideal of* $[P,.]_A$.

Corollary. *Let* $[S,.]$ *be a semigroup,* $[B_1,.]$, $[B_2,.]$ *partial subgroupoids of* $[S,.]$. *Suppose that there exist in* $[S,.]$ *a minimal* $(B_1, \emptyset)$*-ideal* $[L_0,.]$ *with* $L_0 \subset B_1$ *and a minimal* $(\emptyset, B_2)$*-ideal* $[R_0,.]$ *with* $R_0 \subset B_2$.

Then 1) *for every minimal* (B_1, B_2)*-ideal* $[A,.]$ *of* $[S,.]$ *we have* $A = L_0aR_0$ *with some* $a \in S$,

2) *for every* $a \in S$ $[L_0aR_0,.]$ *is a minimal* (B_1, B_2)*-ideal of* $[S,.]$. (Compare with Theorem 2,4 of [3] differing from our Corollary only by the supposition that $[B_1,.]$, $[B_2,.]$ are subsemigroups of $[S,.]$.)

Remark. As we have shown the generalization of the mentioned Theorem in the case of a proper associative partial groupoid does not have such a symmetric form. A straightforward analogy exists only for the first part of this Theorem, since to obtain an analogous assertion of the second part of this Theorem we are obliged to add further suppositions and none of them can be omitted.

Under the suppositions of the foregoing Corollary the (B_1, B_2)-socle $\mathfrak{S}(B_1, B_2)$ of a semigroup $[S,.]$ is not empty and we have $\mathfrak{S}(B_1, B_2) = L_0SR_0$.

In the case of an associative partial groupoid $[P,.]_A$, which is not a semigroup such an assertion, in general, does not hold since $L_0PR_0 \subset P$ need not hold. Our results imply in this case only the following statement: If at least one minimal (B_1, B_2)-ideal of $[P,.]_A$ exists, then there exists a set $P' \subset P$ with $\mathfrak{S}(B_1, B_2) = L_0P'R_0$ (L_0 and R_0 have the same meaning as in the foregoing Corollary). The description of the set P' is rather complicated and we shall not approach it here.

In the first section we have introduced the notion of the n-th power of an element of a partial groupoid. In the case of an associative partial groupoid $[P,.]_A$ we have:

Theorem 2,5. *In $[P,.]_A$ the n-th power of an element $a \in P$ exists if and only if $a^k a^{m-k} \in P$ for $k = 1, 2, \ldots m - 1$, $m = 2, 3, \ldots n$.*

Proof. 1) With regard to Definition 1,13 it is sufficient to prove the equality of all $a^k a^{m-k}$. For $n = 2$ it is trivial. If $aa^2 \in P$, $a^2a \in P$, then we have $aa^2 = a^2a$, thus our statement is true for $n = 3$. Let a^h be defined for $2 \leqq h < n$, i. e. $aa^{h-1} = a^2a^{h-2} = \ldots = a^{h-1}a$. We shall prove that $aa^{n-1} = a^2a^{n-2} = \ldots = a^{n-1}a$. Let $a^{k_i}a^{n-k_i}$, $a^{k_j}a^{n-k_j}$ be two arbitrary elements of the considered set. Without loss of generality suppose that $k_i < k_j$, i. e. $1 \leqslant k_i < k_j$, $n - k_j < n - k_i \leqslant n - 1$. It follows from the induction supposition that $a^{n-k_i} = a^{k_j-k_i}a^{n-k_j}$, $a^{k_j} = a^{k_i}a^{k_j-k_i}$. Hence we have $a^{k_i}a^{n-k_i} = a^{k_i}(a^{k_j-k_i}a^{n-k_j})$, $a^{k_j}a^{n-k_j} = (a^{k_i}a^{k_j-k_i})a^{n-k_j}$. However, in $[P,.]_A$ this implies the equality of the considered elements.

2) The reverse is trivial.

For translations of an associative partial groupoid introduced by Definitions 1,14—1,16 the following holds:

Theorem 2,6. 1) *If for $a \in P$, $b \in P$, $ab \in P$ the inner right translations ϱ_a, ϱ_b, ϱ_{ab} of $[P,.]_A$ exist, then $\varrho_{ab} = \varrho_a * \varrho_b$. Dually, if the inner left translations λ_a, λ_b, λ_{ab} of $[P,.]_A$ exist, then $\lambda_{ab} = \lambda_b * \lambda_a$. Hereby $(*)$ means the symbol of the operation in the semigroup of transformations on the set P.*

2) *Every inner right (left) translation of $[P,.]_A$ is a right (left) translation of $[P,.]_A$.*

Proof. 1) By assumption we have for every $x \in P$ $x(ab) \in P$, $x(ab) = x\varrho_{ab}$, further $xa = x\varrho_a$, $xb = x\varrho_b$. Hence $(xa)b \in P$ and in $[P,.]_A$, thus $(xa)b = x(ab)$. Analogously for left translations.

2) This statement is evident.

Remark. While the set of right and left translations of every $[P,.]_A$ is not empty, the set of inner right (left) translations of $[P,.]_A$ may be empty. Moreover, the set of right (left) translations of $[P,.]_A$ may contain right (left) translations different from those given by the identity mapping of the set P. This is the case, for example, if $A \in I(\emptyset, B_2)$, $[B_2,.]$ being a partial subgroupoid of $[P,.]_A$. The transformation ϱ^b of A given by $a\varrho^b = ab \in A$ for some $b \in B_2$ is a right translation of the associative partial groupoid $[A,.]$. Evidently, inner right translations of a $(\emptyset, B_2)$-ideal of $[P,.]_A$ need not exist even in the case if $[P,.]_A$ is a semigroup, if $B_2 \neq P$. (If $[P,.]_A$ is a semigroup, $B_2 = P$, then the right translations ϱ^b, for $b \in A$, coincide with the inner right translations of $(\emptyset, B_2)$-ideal $[A,.]$ of $[P,.]_A$.

Let $[P,.]_A$ be a semigroup, $B_2 \neq P$, $A \in I(\emptyset, B_2)$ and there is no $B \subset P$, $B_2 \subsetneq B$ with $A \in I(\emptyset, B)$. Then $[A,.]$ is called a saturated $(\emptyset, B_2)$-ideal of $[P,.]_A$ (compare with [3]) and $[B_2,.]$ is a subsemigroup of $[P,.]_A$ maximal with respect to the property that the inner right translations of $[P,.]_A$ given by $b \in B_2$ induce right translations of $[A,.]$. Inner right translations of $[A,.]$ are then induced by inner right translations ϱ_a of $[P,.]_A$ for which $a \in A \cap B_2$. Hence the set of inner right translations of a saturated $(\emptyset, B_2)$-ideal $[A,.]$ of a semigroup is not empty if and only if $A \cap B_2 \neq \emptyset$.

If $[P,.]_A$ is an associative partial groupoid, then, (using Definition 1,15) we have analogously: Let $[B_2,.]$ be a partial subgroupoid of $[P,.]_A$, $A \in I(\emptyset, B_2)$, $b \in B_2$, and suppose that there exists $A' \subset P$, $A \subsetneq A'$ such that $\varrho_b^{A'}$ is defined. Then $\varrho_b^{A'}$ induces a right translation of $[A,.]$. Evidently, ϱ_b^{A} exists for every $b \in B_2$ and coincides with the mapping ϱ^b mentioned in the foregoing Remark.

Analogous considerations can be made for left translations.

We end this section with an example which shows that a homomorphic image of an associative partial groupoid need again not be an associative partial groupoid, even in the case when the mapping of the carriers is one-to-one.

Example 2,3. By the identity mapping of the set $P = \{a, b\}$ onto itself a homomorphism of the associative partial groupoid $[P,.]_A$ onto the partial groupoid $[P,_*]$ is given, if $[P,.]_A$ is defined by the multiplication table

	a	b
a	–	–
b	–	b

and $[P,_*]$ by the multiplication table

	a	b
a	b	a
b	b	b

The partial groupoid $[P,_*]$ is not associative, $(aa)a \neq a(aa)$.

III

Definition 3,1. *An associative partial groupoid* $[P,.]_A$ *will be called a partial semigroup, if* 1) $(ab)c \in P$, $bc \in P$ *implies* $a(bc) \in P$ *and* 2) $a(bc) \in P$, $ab \in P$ *implies* $(ab)c \in P$ *for* $a \in P$, $b \in P$, $c \in P$.

Example 3,1. Let $[S,.]$ be a semigroup, $B \subset S$. Then the partial subgroupoid $[B,.]$ of $[S,.]$ is a partial semigroup. Namely, if $(ab)c \in B$, $bc \in B$, then we necessarily have $a(bc) \in B$, since $a(bc) \in S$ and $a(bc) = (ab)c$. Analogously we get that $[B,.]$ satisfies the condition 2) of the foregoing Definition.

We recall Example 2,1 and note that an associative partial groupoid $[B,_*]$ with the domain $K[B,_*] \subsetneqq K[B,.]$ need not be a partial semigroup.

Notation. In what follows $[P,.]_{AC}$ means a partial semigroup.

It follows from Definition 3,1:

1) In $[P,.]_{AC}$ $(ab)c \in P$, $bc \in P$ holds if and only if $a(bc) \in P$, $ab \in P$ holds.

2) If $[P',.]$ is a partial subgroupoid of $[P,.]_{AC}$, then $[P',.]$ is a partial semigroup. We shall call it a *partial subsemigroup of* $[P,.]_{AC}$. In particular, if $[P',.]$ is a semigroup, we shall say that $[P',.]$ is a *subsemigroup of* $[P,.]_{AC}$.

Clearly, $[S,.]$ is a subsemigroup of $[P,.]_{AC}$ if and only if S is such a subset of P that for every $s \in S$, $t \in S$ we have $st \in S$.

Theorem 3,1. *Let the carrier of* $[P,.]_{AC}$ *contain an element* x_0 *with* $px_0 \in P$ *or* $x_0p \in P$ *for every* $p \in P$. *Then there exists a non empty subsemigroup of* $[P,.]_{AC}$.

Proof. Suppose the former case and denote $S_1 = \{x \in P : px \in P$ for every $p \in P\}$. Then $[S_1,.]$ is a subsemigroup of $[P,.]_{AC}$. For, if $s \in S_1$, $t \in S_1$, then $ps \in P$ and $pt \in P$. Thus we have $st \in P$ and $(ps)t \in P$ for every $p \in P$. This implies in $[P,.]_{AC}$ $p(st) \in P$ for every $p \in P$, hence $st \in S_1$.

Analogously in the second case $S_2 = \{x \in P : xp \in P$ for every $p \in P\}$ is a non-empty semigroup.

The statement of Theorem 3,1 may be visualized as follows: If there exists a non-empty set $S_1 \subset P$ ($S_2 \subset P$) such that the corresponding columns (rows) in the multiplication table do not contain the sign (—), then $[S_1,.]$ ($[S_2,.]$) is a subsemigroup of $[P,.]_{AC}$.

It is possible that $S_1 = \emptyset$ and $S_2 = \emptyset$. However, $S_1 = \emptyset$ and $S_2 = \emptyset$ do not imply that there are no subsemigroups of $[P,.]_{AC}$. This is to be seen on the Example 2,3. On the other hand, there may be $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, $S_1 \neq S_2$. This is the case of Example 2,2, where $S_1 = \{a, c\}$, $S_2 = \{a, b\}$. Clearly, if there exists a universal zero (at least one-sided) element in $[P,.]_{AC}$, then at least one of the sets S_1 and S_2 is non-empty.

We emphasize that the assertion of Theorem 3,1 need not hold for an associative partial groupoid which is not a partial semigroup. This is shown on the following

Example 3,2. Let $[P,.]_A$ be defined on the set $P = \{a, b\}$ by the multiplication table

	a	b
a	b	b
b	b	–

In this case $S_1 = S_2 = \{a\}$, but $[\{a\},.]$ is not a subsemigroup of $[P,.]_A$. Notice that $[P,.]_A$ is not a partial semigroup (his is due the triple (a, a, b)).

Further one can show on examples that a partial semigroup is not necessarily a closed associative partial groupoid.

We shall next deal with relative ideals in partial semigroups.

Lemma 3,1. *Let $[P,.]_{AC}$ be a partial semigroup, $B_{11} \subset P$, $B_{21} \subset P$, $B_{12} \subset P$, $B_{22} \subset P$, $A_1 \in I(B_{11}, B_{21})$, $A_2 \in I(B_{12}, B_{22})$. If $A_1A_2 \subset P$, then $A_1A_2 \in I(B_{11}, B_{22})$.*

The proof is evident.

Notation. In the following $[B_1,.]$, $[B_2,.]$ mean partial subsemigroups of a partial semigroup $[P,.]_{AC}$ (including the case $B_1 = \emptyset$ or $B_2 = \emptyset$).

Lemma 3,2. *Let $[P,.]_{AC}$ be a partial semigroup, $L \in I(B_1, \emptyset)$, $R \in I(\emptyset, B_2)$.*

a) *If $(Lc)R \subset P$ for some $c \in P$, then* 1) $B_1(Lc) \subset P$, 2) $B_1[(Lc)R] \subset P$, 3) $[(Lc)R]B_2 \subset P$.

b) *If $L(cR) \subset P$ for some $c \in P$, then* 1′) $(cR)B_2 \subset P$, 2′) $B_1[L(cR)] \subset P$, 3′) $[L(cR)]B_2 \subset P$.

Proof. a)

1) $[(Lc)R] \subset P$ includes $Lc \subset P$. It follows that $B_1(Lc) \subset P$, since $(B_1L)c \subset Lc \subset P$.

2) Since $(Lc)R \subset P$ and $B_1(Lc) = (B_1L)c \subset Lc$, we have $[B_1(Lc)]R \subset P$. In $[P,.]_{AC}$ this imlies $B_1[(Lc)R] \subset P$.

3) From $(Lc)R \subset P$ it follows that $(Lc)(RB_2) \subset P$, since $(Lc)(RB_2) \subset (Lc)R$. Thus $[(Lc)R]B_2 = (Lc)(RB_2)$ and therefore $[(Lc)R]B_2 \subset P$.

b) This case can be treated analogously.

Theorem 3,2. 1) *If $L \in I_m(B_1, \emptyset)$ and $Lc \subset P$ for some $c \in P$, then $Lc \in I_m(B_1, \emptyset)$.*

2) *If $R \in I_m(\emptyset, B_2)$ and $cR \subset P$ for some $c \in P$, then $cR \in I_m(\emptyset, B_2)$.*

The proof follows directly from Theorem 2,3 and from the foregoing Lemma.

Corollary. *If under suppositions of Theorem* 3,2

1) $(Lc)R \subset P$, *then* $(Lc)R \in I(B_1, B_2)$, $(Lc)R = \bigcup_{j \in J} L_j$, $L_j \in I_m(B_1, \emptyset)$,

2) $L(cR) \subset P$, *then* $L(cR) \in I(B_1, B_2)$, $L(cR) = \bigcup_{j \in J'} R_j$, $R_j \in I_m(\emptyset, B_2)$.

Lemma 3,3. *Let* $L_1 \in I(B_1, \emptyset)$, $L_1 \subset B_1$, $R_2 \in I(\emptyset, B_2)$, $R_2 \subset B_2$. *Then* $[M,.]$, $M \subset P$ *is a minimal* (B_1, B_2)*-ideal of* $[P,.]_{AC}$ *if and only if* $M = (L_1a)R_2$ *for every* $a \in M$ *or* $M = L_1(aR_2)$ *for every* $a \in M$.

Proof. 1) If $M \in I_m(B_1, B_2)$, then by Theorem 2,2a we have $M = L_1aR_2$ for every $a \in M$. It means that $M = (L_1a)R_2$ and simultaneously $M = L_1(aR_2)$ for every $a \in M$.

2) Suppose the former case, i. e. $M = (L_1a)R_2$ for every $a \in M$. This means that $(L_1a)R_2 \subset P$ for every $a \in M$. By Lemma 3,2a) $B_1(L_1a) \subset P$, $B_1M \subset P$, $MB_2 \subset P$, hence by Lemma 2,2a we have $M \in I_m(B_1, B_2)$. The second case can be treated analogously.

From Lemma 3,3 there follows

Theorem 3,3. *Suppose that in* $[P,.]_{AC}$ *there exists a* $(B_1, \emptyset)$*-ideal* $[L_1,.]$ *with* $L_1 \subset B_1$ *and a* $(\emptyset, B_2)$*-ideal* $[R_2,.]$ *with* $R_2 \subset B_2$. *Then* $[M,.]$ *is a minimal* (B_1, B_2)*-ideal of* $[P,.]_{AC}$ *if and only if* $M = L_1aR_2$ *for every* $a \in M$. (Compare with Theorem 2,2b of the present paper.)

Remark. It follows from Lemma 3,3 that in order to have $M = L_1aR_2$ for every $a \in M$ it is sufficient to suppose only $M = (L_1a)R_2$ for every $a \in M$ or $M = L_1(aR_2)$ for every $a \in M$.

Theorem 3,4. *Suppose that in* $[P,.]_{AC}$ *there exists a* $(B_1, \emptyset)$*-ideal* $[L_1,.]$ *with* $L_1 \subset B_1$ *and a* $(\emptyset, B_2)$*-ideal* $[R_2,.]$ *with* $R_2 \subset B_2$.

If $[L,.]$ *is any minimal* $(B_1, \emptyset)$*-ideal,* $[R,.]$ *any minimal* $(\emptyset, B_2)$*-ideal of* $[P,.]_{AC}$ *and for some* $c \in P$ *we have*

1) $(Lc)R \subset P$, *then* $[(Lc)R,.]$ *is a minimal* (B_1, B_2)*-ideal of* $[P,.]_{AC}$,

2) $L(cR) \subset P$, *then* $[L(cR),.]$ *is a minimal* (B_1, B_2)*-ideal of* $[P,.]_{AC}$

Proof. 1) The statement follows from Theorem 2,4a and Lemma 3,2a).

2) The statement follows from Theorem 2,4b and Lemma 3,2b).

Remark. Theorem 3,4 is an analogy of Theorem 2,4a and 2,4b. It means that in the case of a partial semigroup the conditions 1), 2), 3) from Theorem 2,4a can be replaced by a simple and natural condition $(Lc)R \subset P$, the conditions 1′, 2′, 3′ from Theorem 2,4b by the condition $L(cR) \subset P$.

Corollary 1. *Suppose that in* $[P,.]_{AC}$ *there exists at least one minimal* $(B_1, \emptyset)$*-ideal* $[L_0,.]$ *with* $L_0 \subset B_1$ *and at least one minimal* $(\emptyset, B_2)$*-ideal* $[R_0,.]$ *with* $R_0 \subset B_2$. *Then*

a) *for every minimal* (B_1, B_2)*-ideal* $[M,.]$ *of* $[P,.]_{AC}$ *we have* $M = L_0cR_0$ *for some* $c \in P$;

b) *for every* $c \in P$ *either* $L_0cR_0 \not\subset P$ *or* $[L_0cR_0,.]$ *is a minimal* (B_1, B_2)*-ideal of* $[P,.]_{AC}$;

c) *if for some* $c \in P$ *we have* $L_0cR_0 \not\subset P$ *and either* $(L_0c)R_0 \subset P$ (or $L_0(cR_0) \subset P$), *then either* $[(L_0c)R_0,.]$ — (*or* $[L_0(cR_0),.]$) — *is a minimal* (B_1, B_2)*-ideal of* $[P,.]_{AC}$.

We note that in order to have in $[P,.]_{AC}$ $L_0cR_0 \subset P$, it is sufficient to suppose $(L_0c)R_0 \subset P$ and $cR_0 \subset P$ or $L_0(cR_0) \subset P$ and $L_0c \subset P$.

Corollary 2. *Suppose that in* $[P,.]_{AC}$ *there exists a minimal* $(B_1, \emptyset)$*-ideal* $[L_0,.]$ *with* $L_0 \subset B_1$ *and a minimal* $(\emptyset, B_2)$*-ideal* $[R_0,.]$ *with* $R_0 \subset B_2$. *Suppose further that* $[P,.]_{AC}$ *contains at least one minimal* (B_1, B_2)*-ideal Then for the* (B_1, B_2)*-socle of* $[P,.]_{AC}$ *we have*:

$$\mathfrak{S}(B_1, B_2) = L_0P'R_0,$$

where $P' \subset P$ *has the following property*: $L_0P'R_0 \subset P$, *but there does not exist a set* $P'' \subset P$, $P' \subsetneqq P''$ *such that* $L_0P''R_0 \subset P$.

Of course, there may exist a set $S \subset P$, $P' \subsetneqq S$ with either $L_0P'R_0 = (L_0S)R_0$ or $L_0P'R_0 = L_0(SR_0)$.

In the following we shall deal with the partial equivalences ${}_{B_1}\mathscr{I}_{B_2}$, ${}_1\mathscr{I}$, $\mathscr{I}_2$, introduced by Definitions 1,8 and 1,9 on the carrier of a partial semigroup.

Theorem 3,5. *In* $[P,.]_{AC}$ *we have* ${}_{B_1}\mathscr{I}_{B_2} = {}_1\mathscr{I} = \mathscr{I}_2$.

Proof. With respect to Corollary 2 according to Definition 1,9 it is sufficient to prove that the domains of these three partial equivalences coincide. But this is true, since in $[P,.]_{AC}$ the equality $(B_1p)B_2 = B_1(pB_2)$ (which defines the domain of ${}_{B_1}\mathscr{I}_{B_2}$) is equivalent with the pair of inclusions $(B_1p)B_2 \subset P$, $pB_2 \subset P$ and simultaneously $B_1(pB_2) \subset P$, $B_1p \subset P$. These pairs of inclusions, however, define the domains of ${}_1\mathscr{I}$ and $\mathscr{I}_2$.

Remark. As already mentioned, we include the case $B_1 = \emptyset$ or $B_2 = \emptyset$. For $B_1 = B_2 = \emptyset$ the relation ${}_{B_1}\mathscr{I}_{B_2}$ is the equality relation on P.

In the following we shall give some results concerning special cases of the partial equivalence ${}_{B_1}\mathscr{I}_{B_2}$ on the carrier of a partial semigroup. We also complete some results proved in [3] concerning the theory of semigroups.

Theorem 3,6. *Let* $[B_1,.]$, $[B_2,.]$ *be partial subsemigroups of* $[P,.]_{AC}$.

1) *If for some* $a \in P$ *we have* $B_1a \subset P$, *i. e.* $a \in O({}_{B_1}\mathscr{I})$, *then the partial equivalence* ${}_{B_1}\mathscr{I}$ *on* P *is a partial right congruence on* $[P,.]_{AC}$.

2) *If for some* $a \in P$ *we have* $aB_2 \subset P$, *i. e.* $a \in O(\mathscr{I}_{B_2})$, *then the partial equivalence* $\mathscr{I}_{B_2}$ *on* P *is a partial left congruence on* $[P,.]_{AC}$.

Proof. Let $(a, b) \in {}_{B_1}\mathscr{I}$, i. e. $a \cup B_1a = b \cup B_1b$ and $ap \in O({}_{B_1}\mathscr{I})$, i. e. $B_1(ap) \subset P$ for some $p \in P$. Since $B_1(ap) \subset P$, $B_1a \subset P$, we have $B_1(ap) = (B_1a)p$. This implies $(b \cup B_1b)p = ap \cup B_1(ap) = bp \cup (B_1b)p$, hence $bp \in P$ and further $(B_1b)p = B_1(bp)$. Thus we get $ap \cup B_1(ap) = bp \cup B_1(bp)$, i. e.

$(ap, bp) \in {}_{B_1}\mathscr{I}$. If $(a, b) \in {}_{B_1}\mathscr{I}$ and for some $p \in P$ we have $ap \notin O({}_{B_1}\mathscr{I})$, then by the foregoing $bp \in O({}_{B_1}\mathscr{I})$ cannot hold.

2) The second statement can be proved analogously.

Remark. In the case of associative partial groupoids which are not partial semigroups, the partial equivalences ${}_{B_1}\mathscr{I}$, $\mathscr{I}_{B_2}$ are not in general partial right (left) congruences. We show on the following example that in $[P,.]_A$, $(a, b) \in \mathscr{I}_{B_2}$ and $pa \in O(\mathscr{I}_{B_2})$, $p \in P$ does not imply $(pa, pb) \in \mathscr{I}_{B_2}$ even in the case if $p \in O(\mathscr{I}_{B_2})$.

Example 3,2. Let $[P,.]_A$ be defined on the set $P = \{b, c, d, e\}$ by the multiplication table:

	b	c	d	e
b	–	b	–	–
c	d	c	d	–
d	c	d	–	–
e	b	b	–	e

If we take $B_2 = \{b, c\}$, then $O(\mathscr{I}_{B_2}) = \{c, d, e\}$, $(c, d) \in \mathscr{I}_{B_2}$, $dc \in O(\mathscr{I}_{B_2})$, but $dd \notin O(\mathscr{I}_{B_2})$. Note that $[P,.]_A$ is not a partial semigroup (this is due to the triple (d, c, b)).

It is known (see [3]) that in the case when $[P,.]_{AC}$ is a semigroup, $[B_1,.]$, $[B_2,.]$ are subsemigroups of $[P,.]_{AC}$; then the (B_1, B_2)-partial subgroupoid ${}_{B_1}(a)_{B_2}$ (introduced by Definition 1,7) is a (B_1, B_2)-ideal of $[P,.]_{AC}$, further ${}_{B_1}\mathscr{I}$ is a right congruence on $[P,.]_{AC}$, $\mathscr{I}_{B_2}$ is a left congruence on $[P,.]_{AC}$ and ${}_{B_1}\mathscr{I} \subset {}_{B_1}\mathscr{I}_{B_2}$, $\mathscr{I}_{B_2} \subset {}_{B_1}\mathscr{I}_{B_2}$.

Suppose in the following that $[P,.]_{AC}$ is a semigroup, $[B_1,.]$, $[B_2,.]$ are its partial subsemigroups and denote in this case $[P,.]_{AC}$ by $[S,.]$. Then $B_1aB_2 \in S$ for every $a \in S$, so that $O({}_{B_1}\mathscr{I}_{B_2}) = S$. But, in general, ${}_{B_1}(a)_{B_2} \notin I(B_1, B_2)$.

Note also that it may happen that ${}_{B_1}(a)_{B_2} \in I(B_1, B_2)$, even if $[B_1,.]$, $[B_2,.]$ are not subsemigroups of the semigroups $[S,.]$.

On the following example we show that (by choosing suitable $[B_1,.]$, $[B_2,.]$ which are not subsemigroups of $[S,.]$) ${}_{B_1}(x)_{B_2} \in I(B_1, B_2)$ holds for every $x \in S$.

Example 3,3. Let $[S,.]$ be a semigroup defined on $S = \{a, b, c, d, e\}$ by the following multiplication table:

	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a

a) Let $B_1 = B_2 = \{c\}$. Then $[\{c\},.]$ is not a subsemigroup of $[S,.]$ but ${}_{B_1}(a)_{B_2} = \{a\} \in I(B_1, B_2)$, ${}_{B_1}(b)_{B_2} = \{b, c\} = {}_{B_1}(c)_{B_2} \in I(B_1, B_2)$, ${}_{B_1}(d)_{B_2} = \{d\} \in I(B_1, B_2)$, ${}_{B_1}(e)_{B_2} = \{e, d\} \in I(B_1, B_2)$.

b) If we choose in the same example $B_1 = \{c, d\}$, $B_2 = \emptyset$, then ${}_{B_1}(a)_{B_2} = {}_{B_1}(d)_{B_2} = \{a, d\} \in I(B_1, B_2)$, ${}_{B_1}(e)_{B_2} = \{e, a, d\} \in I(B_1, B_2)$, but ${}_{B_1}(b)_2 = {}_{B_1}(c)_{B_2} = \{b, c, d\} \notin I(B_1, B_2)$. In the sense of Definition 1,7 $[{}_{B_1}(b)_{B_2},.]$ and $[{}_{B_1}(c)_{B_2},.]$ are merely (B_1, B_2)-partial subgroupoids of $[S,.]$ generated by b and c respectively, hence partial subsemigroups of $[S,.]$ which are not (B_1, B_2)-ideals of $[S,.]$.

Clearly, in the case when $[B_1,.]$, $[B_2,.]$ are partial subsemigroups of a semigroup $[S;.]$, we have (analogously as in the case when $[B_1,.]$, $[B_2,.]$ are subsemigroups of $[S,.]$) the following: ${}_{B_1}\mathscr{I} \subset {}_{B_1}\mathscr{I}_{B_2}$, $\mathscr{I}_{B_2} \subset {}_{B_1}\mathscr{I}_{B_2}$, ${}_{B_1}\mathscr{I}$ is a right congruence, $\mathscr{I}_{B_2}$ is a left congruence on $[S,.]$.

It follows from the foregoing that in this case we can consider besides the classes corresponding to the equivalences ${}_{B_1}\mathscr{I}_{B_2}$, ${}_{B_1}\mathscr{I}$, $\mathscr{I}_{B_2}$ two further classes of elements of S given by the following

Lemma 3,4. *Let $[S,.]$ be a semigroup, $[B_1,.]$, $[B_2,.]$ partial subsemigroups of $[S,.]$, ${}_{B_1}T^+_{B_2} = \{a \in S : {}_{B_1}(a)_{B_2} \in I(B_1, B_2)\}$, ${}_{B_1}T^-_{B_2} = \{a \in S : {}_{B_1}(a)_{B_2} \notin I(B_1, B_2)\}$. Then*

1) *$(a, b) \in {}_{B_1}\mathscr{I}_{B_2}$ implies either $a \in {}_{B_1}T^+_{B_2}$, $b \in {}_{B_1}T^+_{B_2}$ or $a \in {}_{B_1}T^-_{B_2}$, $b \in {}_{B_1}T^-_{B_2}$,*

2) *${}_{B_1}T^+ \cap T^+_{B_2} \subset {}_{B_1}T^+_{B_2}$, ${}_{B_1}T^- \cup T^-_{B_2} \supset {}_{B_1}T^-_{B_2}$ (${}_{B_1}T$, T_{B_2}* mean *${}_{B_1}T_{B_2}$ with $B_2 = \emptyset$ and $B_1 = \emptyset$, respectively).*

Proof. 1) This statement is evident.

2) Let $a \in {}_{B_1}T^+$, $a \in T^+_{B_2}$, i. e. $B_1(a \cup B_1a) = B_1a \cup B_1B_1a \subset a \cup B_1a$, $(a \cup aB_2)B_2 = aB_2 \cup aB_2B_2 \subset a \cup aB_2$. Then $B_1(a \cup B_1a \cup aB_2 \cup B_1aB_2) \subset a \cup B_1a \cup aB_2 \cup B_1aB_2$ and $(a \cup B_1a \cup aB_2 \cup B_1aB_2)B_2 \subset a \cup B_1a \cup aB_2 \cup B_1aB_2$, hence $a \in {}_{B_1}T^+_{B_2}$. If $a \in {}_{B_1}T^-_{B_2}$, then from the foregoing we have $a \notin {}_{B_1}T^+ \cap T^+_{B_2}$, hence $a \in {}_{B_2}T^- \cup T^-_{B_2}$.

It follows from the foregoing Lemma:

1) The classes corresponding to the equivalences ${}_{B_1}\mathscr{I}_{B_2}$ are refinements of classes ${}_{B_1}T^+_{B_2}$, ${}_{B_1}T^-_{B_2}$.

2) Taking $B_1 = \emptyset$, $B_2 = \emptyset$, we get ${}_{B_1}T^+_{B_2} = S$, ${}_{B_1}T^-_{B_2} = \emptyset$.

Note that the same as in 2) holds if $[B_1,.]$, $[B_2,.]$ are subsemigroups of $[S,.]$. However, we have seen in Example 3,3 that ${}_{B_1}T^+_{B_2} = S$, ${}_{B_1}T^-_{B_2} = \emptyset$ may hold also in other cases.

Further we note that the inclusions in 2) of Lemma 3,4 cannot be replaced by equalities.

Theorem 3,7. *If under suppositions of Lemma* 3,4

1) *${}_{B_1}T^+ \neq \emptyset$, then ${}_{B_1}T^+ \in I(\emptyset, S)$,*

2) *$T^+_{B_2} \neq \emptyset$, then $T^+_{B_2} \in I(S, \emptyset)$.*

Proof. 1) Let $a \in {}_{B_1}T^+$, i. e. $B_1(a \cup B_1a) = B_1a \cup B_1B_1a \subset a \cup B_1a$. Then $B_1as \cup B_1B_1as = B_1(as \cup B_1as) \subset as \cup B_1as$ for every $s \in S$, hence $as \in {}_{B_1}T^+$ for every $s \in S$.

2) The second statement can be proved analogously.

We illustrate the foregoing results on Example 3,3:

Taking $B_1 = B_2 = \{c, d\}$, we get ${}_{B_1}T^+ = \{a, d, e\}$, $T^+_{B_2} = \{a, d\}$, ${}_{B_1}T^+_{B_2} = S$, ${}_{B_1}T^- = \{b, c\}$, $T^-_{B_2} = \{b, c, e\}$, ${}_{B_1}T^-_{B_2} = \emptyset$. Here $[\{a, d, e\},.]$ is a right and $[\{a, d\},.]$ is a left ideal of $[S,.]$. For the corresponding Green's relations with respect to B_1, B_2 we have ${}_{B_1}\mathscr{I} = \mathscr{I}_{B_2} = {}_{B_1}\mathscr{I}_{B_2}$, thus they give a congruence on $[S,.]$; the classes corresponding to this congruence are $M_1 = \{a, d\}$, $M_2 = \{b, c\}$, $M_3 = \{e\}$.

In the following theorem we consider again a partial semigroup $[P,.]_{AC}$ while $[B_1,.]$, $[B_2,.]$ are subsemigroups of $[P,.]_{AC}$.

Theorem 3,8. *Let $[B_1,.]$, $[B_2,.]$ be subsemigroups of a partial semigroup $[P,.]_{AC}$. Let $B_1aB_2 \subset P$ for some $a \in P$ (i. e. at least one of the two pairs of inclusions: $(B_1a)B_2 \subset P$, $aB_2 \subset P$ or $B_1(aB_2) \subset P$, $B_1a \subset P$ - holds). Then ${}_{B_1}(a)_{B_2} \in I(B_1, B_2)$.*

Proof. Since the case $B_1 = \emptyset$ or $B_2 = \emptyset$ is included, it is sufficient to prove $B_1(B_1aB_2) \subset B_1aB_2$, $(B_1aB_2)B_2 \subset B_1aB_2$. By suppositions $B_1^2 \subset B_1 \subset P$, hence $B_1(B_1aB_2) = B_1B_1(aB_2) = B_1^2(aB_2) \subset B_1(aB_2) = B_1aB_2$. The second case can be treated analogously.

Corollary. *Under the suppostiions of Theorem 3,8 we have ${}_{B_1}T^+_{B_2} = O({}_{B_1}\mathscr{I}_{B_2})$, ${}_{B_1}T^-_{B_2} = \{a \in P : {}_{B_1}(a)_{B_2} \not\subset P\}$.*

Altogether we can conclude that the cases which are close to the known results concerning relative ideals in semigroups are the following:

1) $[S,.]$ is a semigroup, $[B_1,.]$, $[B_2,.]$ are partial subsemigroups of $[S,.]$,

2) $[P,.]_{AC}$ is a partial semigroup, $[B_1,.]$, $[B_2,.]$ are subsemigroups of $[P,.]_{AC}$.

In section II we have proved an assertion concerning the existence of the n-th power of an element in an associative partial groupoid. In the case of a partial semigroup we have

Theorem 3,9. *Let $[P,.]_{AC}$ be a partial semigroup. The n-th power of an element $a \in P$ exists if and only if $a^ka^{n-k} \in P$ for some $k \in \{1, 2, \dots n - 1\}$.*

Proof. 1) Let $a^h \in P$ for $2 \leqslant h < n$, i. e. $a^h = aa^{h-1} = a^2a^{h-2} = \dots = a^{h-1}a$ for $h = 2, 3, \dots n - 1$. Let $a^{k_i}a^{n-k_i} \in P$ for some $k_i \in \{1, 2, \dots n - 1\}$. We shall prove that then $a^{k_j}a^{n-k_j} \in P$, for every $k_j \neq k_i$, $k_j \in \{1, \dots n - 1\}$. If $k_i < k_j$, then $n - k_i > n - k_j$ and $n - k_i = n - k_j + k_j - k_i$ imply $a^{k_i}a^{n-k_i} = a^{k_i}(a^{k_j-k_i}a^{n-k_j})$. Since $a^{k_i}a^{k_j-k_i} \in P$, we have $a^{k_j}a^{n-k_j} \in P$. If $k_i > k_j$, then $n - k_i < n - k_j \leqslant n - 1$, $a^{k_i}a^{n-k_i} = (a^{k_j}a^{k_i-k_j})a^{n-k_i} \in P$, $a^{k_i-k_j}a^{n-k_i} \in P$ (by the induction supposition), hence $a^{k_j}a^{n-k_j} \in P$.

2) The reverse is evident.

The following theorem shows that the set of idempotents E of a partial semigroup (under supposition that $E \neq \emptyset$) can be partially ordered in the same way as in the case of a semigroup.

Theorem 3,10. *Let the set E of idempotents of $[P,.]_{AC}$ be nonempty. Then the relation $\leqq$ defined by : $e \leqq f$ if and only if $ef = fe = e$, $e \in E, f \in E$, is a partial ordering on E.*

Proof. The reflexivity and antisymmetricity are evident. Let $e \leqq f, f \leqq g$, i. e. $ef = fe = e$, $fg = gf = f$. Hence $e(fg) = e$, thus $e(fg) \in E \subset P$. Since $ef \in E \subset P$, we have $(ef)g \in P$ and $(ef)g = eg = e(fg) = e$. Analogously $e = (gf)e = g(fe) = ge$, hence $e \leqq g$.

In the following we shall give some results concerning right and left translations of a partial semigroup.

Notation. $[Z_{P},_{*}]$ means the semigroup of all transformations of a set P.

Theorem 3,11. 1) *Let Π_0 be the set of all inner right translations of a partial semigroup $[P,.]_{AC}$. Then $[\Pi_0,_{*}]$ is a subsemigroup of $[Z_{P},_{*}]$.*

2) *Let Λ_0 be the set of all inner left translations of $[P,.]_{AC}$. Then $[\Lambda_0,_{*}]$ is a subsemigroup of $[Z_{P},_{*}]$.*

Proof. 1) Let $\Pi_0 \neq \emptyset$. By Theorem 3,1 $\Pi_0 = \{\varrho_x : x \in S_1\}$, $[S_1,.]$ is a subsemigroup of $[P,.]_{AC}$. By Theorem 2,6 $\varrho_x * \varrho_y = \varrho_{xy}$ for every $\varrho_x \in \Pi_0$, $\varrho_y \in \Pi_0$, with $\varrho_{xy} \in \Pi_0$.

2) This statement can be proved analogously.

Corollary. *Let $[P,.]_{AC}$ be a partial semigroup.*

1) *If the set of inner right translations of $[P,.]_{AC}$ is given by the set $S_1 \neq \emptyset$, $S_1 \subset P$, then the mapping $a \to \varrho_a$, $a \in S_1$ is a representation of the semigroup $[S_1,.]$ by transformations of the set P.*

2) *If the set of inner left translations of $[P,.]_{AC}$ is given by the set $S_2 \neq \emptyset$, $S_2 \subset P$, then the mapping $a \to \lambda_a$, $a \in S_2$ is an antirepresentation of the semigroup $[S_2,.]$ by transformations of the set P.*

Remark. We see that by a natural generalization of the notion of a regular representation (antirepresentation) of a semigroup for the case of a partial semigroup we have obtained again a representation (antirepresentation) of a semigroup. However, in this case the representation (antirepresentation) is given by transformations of a set on which is defined a partial semigroup and which contains the carrier of the represented semigroup.

We note that by the set of inner translations (left or right) of an associative partial groupoid which is not a partial semigroup need not be a subsemigroup of given a $[Z_{P},_{*}]$ (this is shown on Example 3,2).

In Definition 1,16 a generalization of a right (left) translation of a semigroup has been given. It is known that in the case of a semigroup $[S,.]$ the following holds: $[\Pi,_*]$ and $[\Lambda,_*]$ where Π is the set of right and Λ the set of left translations of $[S,.]$, are subsemigroups of $[Z_S,_*]$. Moreover, $[\Pi_0,_*]$ is a subsemigroup of $[\Pi,_*]$ and $[\Lambda_0,_*]$ is a subsemigroup of $[\Lambda,_*]$. On the following example we show that in the case of a partial semigroup which is not a semigroup this assertion need not hold.

Example 3,4. Let $[P,.]_{AC}$ be a partial semigroup defined on the set $P = \{a, b\}$ by the following multiplication table:

	a	b
a	a	–
b	b	–

The mappings $\varrho_1 : a\varrho_1 = b,\ b\varrho_1 = b$ and $\varrho_2 : a\varrho_2 = b,\ b\varrho_2 = a$ are right translations of $[P,.]_{AC}$, but the mapping $\varrho_3 = \varrho_1 * \varrho_2$ is not a right translation of $[P,.]_{AC}$, since $(ba)\varrho_3 \neq b(a\varrho_3)$.

Remark. If $[P,.]_{AC}$ is a partial semigroup, Π the set of its right translations and Λ the set of its left translations, then $[\Pi,_*]$ and $[\Lambda,_*]$ are, in general, merely partial subsemigroups of $[Z_P,_*]$. As we have seen these partial semigroups contain subsemigroups of inner translations of $[P,.]_{AC}$ (it may happen that these semigroups are empty), but they can contain also other subsemigroups. We can obtain an example of such a semigroup by using the following result of [3]: If $[S,.]$ is a semigroup, $[P,.]_{AC}$ a partial subsemigroup of $[S,.]$ such that there exists a subset $B \neq \emptyset$, $B \subset S$ with $P \in I(\emptyset, B)$, then $P \in I(\emptyset, H)$ for some subsemigroup $[H,.]$ of $[S,.]$. Evidently, the transformations ϱ^h given by $a\varrho^h = ah$, $h \in H$, $a \in P$, are right translations of $[P,.]_{AC}$ and $[\{\varrho^h\},_*]$, $h \in H$ is a semigroup.

We end our considerations by giving a complete list of all partial groupoids, associative partial groupoids, partial semigroups and semigroups defined on the set $P = \{a, b\}$. Those of them which are isomorphic are included in classes denoted by T_n.

1. If the domain is a one-point set, we obtain four classes. The multiplication tables are as follows:

T_1:

	a	b
a	a	–
b	–	–

,

	a	b
a	–	–
b	–	b

;

T_2:

	a	b
a	–	–
b	a	–

,

	a	b
a	–	b
b	–	–

;

T_3:

	a	b
a	–	–
b	–	a

,

	a	b
a	b	–
b	–	–

;

T_4:

$$\begin{array}{c|cc} & a & b \\ \hline a & - & a \\ b & - & - \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & - & - \\ b & b & - \end{array}.$$

All these partial groupoids are associative, as a matter of fact they are partial semigroups.

2. If the domain contains two elements, we get the partial groupoids with the following multiplication tables:

T_1: $$\begin{array}{c|cc} & a & b \\ \hline a & b & - \\ b & - & a \end{array};$$

T_2: $$\begin{array}{c|cc} & a & b \\ \hline a & a & - \\ b & - & b \end{array};$$

T_3: $$\begin{array}{c|cc} & a & b \\ \hline a & a & - \\ b & a & - \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & - & b \\ b & - & b \end{array};$$

T_4: $$\begin{array}{c|cc} & a & b \\ \hline a & a & - \\ b & - & a \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & b & - \\ b & - & b \end{array};$$

T_5: $$\begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & - & - \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & - & - \\ b & b & b \end{array};$$

T_6: $$\begin{array}{c|cc} & a & b \\ \hline a & - & a \\ b & a & - \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & - & b \\ b & b & - \end{array};$$

T_7: $$\begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & - & - \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & - & - \\ b & a & b \end{array};$$

T_8: $$\begin{array}{c|cc} & a & b \\ \hline a & a & - \\ b & b & - \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & - & a \\ b & - & b \end{array};$$

T_9: $$\begin{array}{c|cc} & a & b \\ \hline a & b & - \\ b & b & - \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & - & a \\ b & - & a \end{array};$$

T_{10}: $$\begin{array}{c|cc} & a & b \\ \hline a & - & - \\ b & a & a \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & b & b \\ b & - & - \end{array};$$

T_{11}: $$\begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & - & - \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & - & - \\ b & b & a \end{array};$$

T_{12}: $$\begin{array}{c|cc} & a & b \\ \hline a & - & b \\ b & - & a \end{array}, \quad \begin{array}{c|cc} & a & b \\ \hline a & b & - \\ b & a & - \end{array};$$

T_{13}: $$\begin{array}{c|cc} & a & b \\ \hline a & - & b \\ b & a & - \end{array};$$

T_{14}: $$\begin{array}{c|cc} & a & b \\ \hline a & - & a \\ b & b & - \end{array}.$$

All these partial groupoids are associative. Those of them which are defined by $T_1 - T_8$ are partial semigroups. The groupoids $T_9 - T_{14}$ are associative partial groupoids but not partial semigroups.

3. If the domain contains three elements, then the multiplication tables are as follows:

T_1:

	a	b
a	a	a
b	a	$-$,

	a	b
a	$-$	b
b	b	b ;

T_2:

	a	b
a	a	$-$
b	a	a ,

	a	b
a	b	b
b	$-$	b ;

T_3:

	a	b
a	a	$-$
b	a	b ,

	a	b
a	a	b
b	$-$	b ;

T_4:

	a	b
a	a	$-$
b	b	a ,

	a	b
a	b	a
b	$-$	b ;

T_5:

	a	b
a	a	a
b	$-$	b ,

	a	b
a	a	$-$
b	b	b ;

T_6:

	a	b
a	$-$	a
b	a	b ,

	a	b
a	a	b
b	b	$-$;

T_7:

	a	b
a	a	b
b	a	$-$,

	a	b
a	$-$	b
b	a	b ;

T_8:

	a	b
a	a	a
b	$-$	a ,

	a	b
a	b	$-$
b	b	b ;

T_9:

	a	b
a	a	b
b	$-$	a ,

	a	b
a	b	$-$
b	a	b ;

T_{10}:

	a	b
a	a	a
b	b	$-$,

	a	b
a	$-$	a
b	b	b ;

T_{11}:

	a	b
a	b	a
b	a	$-$,

	a	b
a	$-$	b
b	b	a ;

T_{12}:

	a	b
a	$-$	a
b	a	a ,

	a	b
a	b	b
b	b	$-$;

T_{13}:

	a	b
a	b	a
b	$-$	a ,

	a	b
a	b	$-$
b	b	a :

T_{14}:

	a	b
a	$-$	a
b	b	a ,

	a	b
a	b	a
b	b	$-$;

T_{15}:

	a	b
a	b	b
b	$-$	a ,

	a	b
a	b	$-$
b	a	a ;

T_{16}:

	a	b
a	b	b
b	a	$-$,

	a	b
a	$-$	b
b	a	a .

In this case the partial groupoids $T_1 - T_6$ are partial semigroups; those of $T_7 - T_{14}$ are associative partial groupoids but not partial semigroups. In T_{15} and T_{16} we have partial groupoids, which are not associative.

4. Finally, if the domain is $P \times P$, we get the following classes:

T_1:

	a	b
a	a	a
b	a	a

	a	b
a	b	b
b	b	b

T_2:

	a	b
a	b	a
b	a	b

	a	b
a	a	b
b	b	a

T_3:

	a	b
a	a	a
b	a	b

	a	b
a	a	b
b	b	b

T_4:

	a	b
a	a	a
b	b	b

T_5:

	a	b
a	a	b
b	a	b

T_6:

	a	b
a	b	a
b	b	a

T_7:

	a	b
a	b	b
b	a	a

T_8:

	a	b
a	b	a
b	b	b

	a	b
a	a	a
b	b	a

T_9:

	a	b
a	a	b
b	a	a

	a	b
a	b	b
b	a	b

T_{10}:

	a	b
a	b	a
b	a	a

	a	b
a	b	b
b	a	a

The groupoids defined in $T_1 - T_6$ are associative, thus they are semigroups; those of $T_7 - T_{10}$ are not semigroups.

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