Jozef Komorník

Correspondence Between Semi-Measures and Small Systems

Matematický časopis, Vol. 25 (1975), No. 1, 59--62

Persistent URL: `http://dml.cz/dmlcz/127037`

CORRESPONDENCE BETWEEN SEMI-MEASURES AND SMALL SYSTEMS

JOZEF KOMORNÍK

In this paper we present two constructions. They are converse to each other and give the bijective correspondence between equivalence classes of semi-measures and small systems.

The notion of semi-measures was introduced probably in [1], for references see [2]. Small systems were first studied in [3]; for references see [4].

Throughout the whole paper the following symbols are used: S is a ring of subsets of a non-empty set X; R^+ is the set of non-negative real numbers; Z^+ is the set of non-negative integers; $Z' = Z^+ \cup \{\infty\}$. For every $k \in Z^+$ we define: $\infty + k = \infty$, $k^{-\infty} = 0$.

We also define two functions:

$$r\colon \quad R^+ \to Z' \quad r(x) = \inf\{n \in Z^+ : 2^{-n} \leqslant x\}$$

$$t\colon \quad Z' \to Z' \quad t(n) = \begin{cases} 0 & n = 0 \\ n-1 & 0 < n < \infty \\ \infty & n = \infty \end{cases}$$

Definition 1. (i) *A function $P : S \to R^+$ is called a semi-measure on the ring S if it is monotone, subadditive and upper continuous in $\emptyset$;*

(ii) *Two semi-measures P_1, P_2 on S are called equivalent if $P_1(E) = 0 \Leftrightarrow$ $\Leftrightarrow P_2(E) = 0$ for every $E \in S$.*

Construction 1. *Let P be a given semi-measure on S. For every $n \in Z^+$ we define*

$$S_n = \{E \in S : P(E) \leqslant 3^{-n}\}.$$

Lemma 1. *The system $\{S_n\}$ obtained by the construction 1 has the following properties. For every $n \in Z^+$ there holds:*

(1) $S \supset S_0 \supset \ldots \supset \{\emptyset\}$

(2) $A \subset B \in S, \quad B \in S_n \Rightarrow A \in S_n$

(3) $A_1, A_2, A_3 \in S_n \Rightarrow A_1 \cup A_2 \cup A_3 \in S_{t(n)}$

(4) $\{A_m\}_{m=1}^{\infty} \searrow \emptyset, \quad A_m \in S \quad \forall n \in Z^+ \quad \exists M \in Z^+ \quad \forall m \geqslant M : A_m \in S_n$.

Proof. These properties result simply from the definition of $\{S_n\}$ (1), the monotonicity (2), the subadditivity (3) and the upper continuity (4) of P.

Definition 2. (i) *By a small system we mean a sequence $\{S_n\}$ of subsets of S having the properties* (1)—(4). *We put* $S_\infty = \bigcap_{n=0}^{\infty} S_n$.

(ii) *Two small systems $\{S_n\}$, $\{T_n\}$ are said to be equivalent if $S_\infty = T_\infty$.*

Remark. If we have a small system $\{S_n\}$ we can simply observe that the properties (1)—(3) hold for $n = \infty$.

Construction 2. *Let $\{S_n\}$ be a given small system. For every $E \in S$ we define*

$h(E) = \sup \{n \in Z^+, E \in S_n\}$

$f(E) = 2^{-h}(E)$

$p(E) = \inf \{ \sum_{i=1}^{n} f(E_i) : E = \bigcup_{i=1}^{n} E_i,\ E_i \in S,\ n \in Z^+\}.$

Lemma 2. (i) *f is a monotone function.*

(ii) *For every $a \in R^+$ and $E \in S$ there holds $f(E) \leqslant a \Rightarrow E \in S_{r(a)}$.*

Proof: (i) Let $A \subset B$. By the property (2) $\{n : A \in S_n\} \supset \{n : B \in S_n\}$, hence $h(A) \geqslant h(B)$ and therefore $f(A) \leqslant f(B)$.

(ii) Every value of f is by the definition of the type 2^{-n}, $n \in Z'$.

Corollary. *For every $E \in S$ $f(E) = 0 \Leftrightarrow E \in S_\infty$.*

Theorem. (i) *The function p obtained by the construction 2 is a semi-measure.*

(ii) *If we have a semi-measure P and p is a semi-measure obtained by applying the constructions* 1 *and* 2, *then P and p are equivalent.*

Proof. First we prove (i). Let $A \subset B \in S$, $0 < \varepsilon \in R$. There exists $\{B_i\}_{i=1}^{n}$ such that $B = \bigcup_{i=1}^{n} B_i$ and $\sum_{i=1}^{n} f(B_i) \leqslant p(B) + \varepsilon$. We put $A_i = A \cap B_i$, $i = 1, \ldots, n$. By the monotonicity of f we get

$$p(A) \leqslant \sum_{i=1}^{n} f(A_i) \leqslant \sum_{i=1}^{n} f(B_i) \leqslant p(B) + \varepsilon .$$

The subadditivity and upper continuity of p we get also simply from the definition of p.

(ii) To prove the second assertion of the theorem we use the inequality $f(E) \leqslant 2p(E)$ for every $E \in S$.

We show that for every $E \in S$ and any finite decompozition $E = \bigcup_{i=1}^{t} E_i$, $E_i \in S$ we have

$$a = \sum_{i=1}^{n} f(E_i) \geqslant 1/2 \cdot f(E) \quad (\text{i.e. } f(E) \leqslant 2a)$$

It is evident for $a = \infty$ or $n = 1$. In case $a < \infty$ we use the induction with respect to n.

Let $n \geqslant 2$. We consider two cases.

(α) $f(E_i) < a/2$ for $i = 1, \ldots, n$. We put

$$k = \max \{j : \sum_{i=1}^{j-1} f(E_i) < a/2\}.$$

Then we have $1 < k < n$ and

$$\sum_{i=1}^{k-1} f(E_i) < a/2, \quad \sum_{i=1}^{k} f(E_i) \geqslant a/2 \quad \text{i.e.} \sum_{i=k+1}^{n} f(E_i) \leqslant a/2.$$

Because of the inductive assumption we have

$$f(\bigcup_{i=1}^{k-1} E_i) \leqslant 2 \sum_{i=1}^{k-1} f(E_i) \leqslant a, \quad f(\bigcup_{i=k+1}^{n} E_i) \leqslant a$$

and finally

$$f(E_k) \leqslant \sum_{i=1}^{n} f(E_i) = a.$$

Using Lemma 2 (ii) and putting $\alpha = r(a)$, we get

$$\bigcup_{i=1}^{k-1} E_i, \quad E_k, \quad \bigcup_{i=k+1}^{n} E_i \in S_\alpha \overset{\text{prop. (3)}}{\Longrightarrow} E \in S_{t(\alpha)}.$$

According to the definition 2 we obtain $h(E) \geqslant t(\alpha) \geqslant \alpha - 1, f(E) \leqslant 2^{\alpha+1} \leqslant \leqslant 2a$.

(β) $f(E_i) \geqslant a/2$ for some $i = 1, \ldots, n$. We can suppose $i = n$. We obtain $\sum_{i=1}^{n-1} f(E_i) \leqslant a/2$, $f(\bigcup_{i=1}^{n-1} E_i) \leqslant a$. Now we can follow as in the case (α).

To finish the proof of the theorem we use the relations $p(E) \leqslant f(E)$ and $f(E) = 0 \Leftrightarrow P(E) = 0$, which were shown above.

REFERENCES

[1] MAHARAM, D.: An algebraic characterization of measure algebras. Ann. Math., *18*, 1947,.

[2] DOBRAKOV, I.: On measures I, Dissertations Mathematicae. CXII, 1974, Warszawa.

[3] RIEČAN, B.: Abstract formulation of some theorems of measure theory. Mat.-fyz. Čas., *16*, 1966, 268—273.
[4] RIEČAN, B.: Subadditive measures and small systems. Čas. pěstov. Mat., (to appear.)

Received November 13, 1973

Katedra numerickej matematiky
a matematickej štatistiky
Prírodovedeckej fakulty UK
Mlynská dolina
816 31 Bratislava