Eugen Futáš

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Matematický časopis, Vol. 22 (1972), No. 4, 271--281

Persistent URL: http://dml.cz/dmlcz/127032

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EXTENSION OF LINEAR OPERATORS

EUGEN FUTÁŠ,
Košice

§ 1. Basic notions.

Consider a lattice X with operations $\cup$ and $\cap$ interpreted as usual. Let $x_n \in X$, $x_n \subseteq x_{n+1}$. Then we shall write $x_n \nearrow x$, if x is the l. u. b. of the sequence $\{x_n\}_{n=1}^{\infty}$. Analogously, $x_n \searrow x$ for $x_n \geqslant x_{n+1}$. Let us define in the lattice X two more operations: $+$ and $-$. Let X, together with operations $+$, $\cup$, $\Rightarrow$ and $\cap$ be a lattice-ordered Abelian group.

Furthemore, let the following hold in X:

S_1: For $x_n \in X$, $x_n \leqslant x_{n+1} \leqslant x_0 \in X$ there exists $x \in X$ such that $x_n \nearrow x$.

Let A be a sublattice of X which is closed under operations $+$ and $-$ (and is therefore a group). On A let us define a function φ_0 whose values lie in some Banach space Y. For φ_0, let the following relations hold.

M_1: For $x, y \in A$ we have $\varphi_0(x + y) = \varphi_0(x) + \varphi_0(y)$.

M_2: For $x_n \in A$, $x_n \searrow 0$ we have $\lim\limits_{n\to\infty} \varphi_0(x_n) = 0$.

M_3: For $x \in A$ we have $\sup \{|\varphi_0(y)| : y \geqslant 0;\ y \in A,\ y \leqslant |x|\} < \infty$.

For any $x \in X$ we define $|x| = x \cup (-x)$, $x^+ = x \cup 0$, $x^- = (-x) \cup 0$. Evidently $|x| = x^+ + x^-$, $x = x^+ - x^-$.

The following well-known lemmas which will be necessary in the sequel are stated without proof for completeness' sake.

Lemma 1.1. *For* $x_n \geqslant x_{n+1} \geqslant x_0 \in X$ *there exists* $x \in X$ *such that* $x_n \searrow x$.

Lemma 1.2. $x, y \geqslant 0 \Rightarrow (x - y)^- \leqslant y$.

Lemma 1.3. $|x + y| \leqslant |x| + |y|$.

Lemma 1.4. $|x - y| \leqslant |x - z| + |z - y|$.

Lemma 1.5. $|(a \cup b) - (c \cup d)| \leqslant |a - c| + |b - d|$.

Lemma 1.6. $|(a \cap b) - (c \cap d)| \leqslant |a - c| + |b - d|$.

Lemma 1.7. $|(a - b) - (c - d)| \leqslant |(a - c)| + |(d - b)|$.

Lemma 1.8. $\bigcup_{n=1}^{\infty} (x_n + y) = \bigcup_{n=1}^{\infty} x_n + y$.

Lemma 1.9. $\bigcap_{n=1}^{\infty} (x_n + y) = \bigcap_{n=1}^{\infty} x_n + y$.

Definition 1.1. $\sum_{n=1}^{\infty} |x_n| = \bigcup_{k=1}^{\infty} \sum_{n=1}^{k} |x_n|$.

Lemma 1.10. $y = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} y_i \Rightarrow |y - y_n| \leqslant \sum_{i=n}^{\infty} |y_{i+1} - y_i|$.

Proof. $|y - y_n| = |\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} y_i - y_n| =$

$$= |\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} (y_i - y_n)| \leqslant \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} |\sum_{j=n}^{i-1} (y_{i+1} - y_j)| \leqslant$$

$$\leqslant \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \sum_{j=n}^{i-1} |y_{j+1} - y_j| =$$

$$= \bigcap_{k=1}^{\infty} \bigcup_{p=k-1}^{\infty} \sum_{j=n}^{k} |y_{j+1} - y_j| \leqslant \bigcup_{p=k}^{\infty} \sum_{j=n}^{k} |y_{i+1} - y_j| =$$

$$= \sum_{j=n}^{\infty} |y_{i+1} - y_j|.$$

§ 2. Construction

Definition 2.1. *For* $x \in A$ *put* $\|\varphi\|(x) = \sup\{|\varphi_0(y)| : y \geqslant 0,\ y \in A,\ y \leqslant |x|\}$.

Evidently:

1. $\|\varphi\|(0) = 0$.
2. $\|\varphi\|(x) \geqslant 0$.
3. $\|\varphi\|(|x|) = \|\varphi\|(x)$.
4. $x \geqslant 0 \Rightarrow |\varphi_0(x)| \leqslant \|\varphi\|(x)$.
5. $x \geqslant y \geqslant 0 \Rightarrow \|\varphi\|(x) \geqslant \|\varphi\|(y)$.

Lemma 2.1. $|x| \leqslant \sum_{n=1}^{\infty} |x_n| \Rightarrow \|\varphi\|(x) \leqslant \sum_{n=1}^{\infty} \|\varphi\|(x_n)$.

Proof. Clearly we can suppose that $x \geqslant 0$, $x_n \geqslant 0$. First let $\|\varphi\|(x) < \infty$.

Let us choose an arbitrary $\varepsilon > 0$. From Definition 2.1 we see that there exists $y \in A$ such that $0 \leqslant y \leqslant x$ and $|\varphi_0(y)| + \varepsilon > \|\varphi\|(x)$.

Let $y_1 = y \cap x_1$,

$$y_{n+1} = (y - \sum_{i=1}^{n} y_i) \cap x_{n+1}.$$

Evidently $0 \leqslant y_n \leqslant x_n$, $y_n \in A$.

We prove by induction that

$$\sum_{i=1}^{n} y_i = y \cap \sum_{i=1}^{n} x_i .$$

Evidently $y_1 = y \cap x_1$. If $\sum_{i=1}^{n} y_i = y \cap \sum_{i=1}^{n} x_i$ for some n, then

$$\sum_{i=1}^{n+1} y_i = \sum_{i=1}^{n} y_i + y_{n+1} = \sum_{i=1}^{n} y_i + (y - \sum_{i=1}^{n} y) \cap x_{n+1} =$$

$$= [\sum_{i=1}^{n} y_i + (y - \sum_{i=1}^{n} y_i)] \cap [\sum_{i=1}^{n} y_i + x_{n+1}] =$$

$$= y \cap [(y \cap \sum_{i=1}^{n} x_i) + x_{n+1}] =$$

$$= y \cap (y + x_{n+1}) \cap (\sum_{i=1}^{n} x_i + x_{n+1}) = y \cap \sum_{i=1}^{n+1} x_i .$$

Therefore $y = \sum_{n=1}^{\infty} y_n$, so that

$$\varphi_0(y) = \sum_{n=1}^{\infty} \varphi_0(y_n)$$

and $|\varphi_0(y)| \leqslant \sum_{n=1}^{\infty} |\varphi_0(y_n)|$, $\|\varphi\|(x) - \varepsilon < |\varphi_0(y)| \leqslant \sum_{n=1}^{\infty} |\varphi_0|(y_n) \leqslant \sum_{n=1}^{\infty} \|\varphi\|(y_n) \leqslant$

$\leqslant \sum_{n-1}^{\infty} \|\varphi\|(x_n)$, giving $\|\varphi\|(x) \leqslant \sum_{n=1}^{\infty} \|\varphi\|(x_n)$.

Now let $\|\varphi\|(x) = \infty$. We choose a natural number N. By Definition 2.1 there exists $y \in A$ such that $0 \leqslant y \leqslant x$ and $|\varphi_0(y)| < N$.

In a manner similar to that of the previous case we could construct the sequence $\{y_n\}_{n=1}^{\infty}$ with the following properties:

$$0 \leqslant y_n \leqslant x_n,\ |\varphi_0(y)| \leqslant \sum_{n=1}^{\infty} |\varphi_0(y_n)|,\ y_n \in A,$$

$$N < |\varphi_0(y)| \leqslant \sum_{n=1}^{\infty} |\varphi_0(y_n)| \leqslant \sum_{n=1}^{\infty} \|\varphi\|(y_n) \leqslant \sum_{n=1}^{\infty} \|\varphi\|(x_n) \Rightarrow \sum_{n=1}^{\infty} \|\varphi\|(x_n) = \infty.$$

Definition 2.2. $\omega(x) = \inf \{\sum_{n=1}^{\infty} \|\varphi\|(x_n) : x_n \in A, \; |x| \leqslant \sum_{n=1}^{\infty} |x_n|\}$.

Evidently:

1. $\omega(0) = 0$,
2. $\omega(x) \geqslant 0$,
3. $\omega(|x|) = \omega(x)$,
4. $|x| \geqslant |y| \Rightarrow \omega(x) \geqslant \omega(y)$,
5. $x \in A \Rightarrow \omega(x) = \|\varphi\|(x)$.

Lemma 2.2. $|x| \leqslant \sum_{n=1}^{\infty} |x_n| \Rightarrow \omega(x) \leqslant \sum_{n=1}^{\infty} \omega(x_n)$.

Proof. We take all $x_n^m \in A$ such that $|x_n| \leqslant \sum_{m=1}^{\infty} |x_n^m|$. Then

$$\omega(x) \leqslant \inf \{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \|\varphi\|(x_n^m) : x_n^m \in A,$$

$$|x_n| \leqslant \sum_{m=1}^{\infty} |x_n^m|\} \leqslant \sum_{n=1}^{\infty} \{\inf \sum_{m=1}^{\infty} \|\varphi\|(x_n^m) : x_n^m \in A,$$

$$|x_n| \leqslant \sum_{m=1}^{\infty} |x_n^m|\} = \sum_{n=1}^{\infty} \omega(x_n).$$

According to M_3, for every $x \in A$, $\omega(x) < \infty$. This supposition is not always satisfied, as the following example shows.

Example 2.1. Let $M = \{0, 1, 2, \ldots\}$. Let X be the system of all real functions defined on M we permit their values to be infinite. Let N be the system of all finite subsets of M and their complements. On N we define the function ν in the following way: $\nu(E)$ — is the number of elements in E, if E is finite and the number of elements in the complement of E — multiplied by -1, if E is infinite.

Let $A = \{f \in X : f = \sum_{i=1}^{\infty} \alpha_i \; \varkappa_{E_i}, \; E_i \in N, \; \alpha_i; \; \alpha_i$ are integers, $E_i \cap E_k = \emptyset$ for $i \neq k\}$ and $\varphi_0(f) = \sum_{i=1}^{n} \alpha_i \nu(E_i)$.

Let $f = 1$. Then $\|\varphi\|(f) > |\varphi_0(\varkappa_L)| = n$ for any natural number n, where $L = \{1, 2, 3, \ldots, n\}$. Therefore clearly $\omega(f) = \|\varphi\|(f) = \infty$.

Definition 2.3. $F = \{x \in X : \omega(x) < \infty\}$.
Note. The above supposition is equivalent to saying that $A \subset F$.

Definition 2.4. $\varrho(x, y) = \omega(x - y)$ *for* $x, y \in F$.

Lemma 2.3. *The function* ϱ *is a pseudometric on* F.
Proof. 1. $\varrho(x, x) = \omega(x - x) = \omega(0) = 0$.
2. $\varrho(x, y) = \omega(x - y) = \omega(|x - y|) = \omega(|y - x|) = \omega(y - x) = = \varrho(y, x)$.
3. $\varrho(x, y) = \omega(x - y) = \omega(|x - y|) \leqslant \omega(|x - z| + |z - y|) \leqslant \leqslant \omega(x - z) + \omega(z - y) = \varrho(x, z) + \varrho(z, y)$.

Theorem 2.1. *The pseudometric space* (F, ϱ) *is complete.*
Proof. Consider a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$. From among its members se select a subsequence $\{y_n\}_{n=1}^{\infty}$ such that $\varrho(y_n, y_{n+1}) < 2^{-n}$. To prove the convergence of $\{x_n\}_{n=1}^{\infty}$, it is enough to prove that $\{y_n\}_{n=1}^{\infty}$ is convergent.
From among the members of the sequence $\{y_n\}_{n=1}^{\infty}$ we choose the element y mentioned in Lemma 1.10. Then $\varrho(y, y_n) = \omega(y - y_n) \leqslant \sum_{k=n}^{\infty} \omega(y_{k+1} - y_k) \leqslant \leqslant \sum_{k=n}^{\infty} 2^{-k} = 2^{1-n}$, so that $\lim_{n\to\infty} \varrho(y, y_n) = 0$.
Evidently $|y| \leqslant |y_1| + \sum_{n=1}^{\infty} |y_{n+1} - y_n|$.

Therefore $\omega(y) \leqslant \omega(y_1) + \sum_{n=1}^{\infty} \omega(y_{n+1} - y_n) = \omega(y_1) + \sum_{n-1}^{\infty} \varrho(y_{n+1} - y_n) \leqslant \leqslant \omega(y_1) + \sum_{n=1}^{\infty} 2^{-n} = \omega(y_1) + 1 < \infty$, so that $y \in F$.

Lemma 2.4. $y, x \in A \Rightarrow |\varphi_0(x) - \varphi_0(y)| \leqslant 2\varrho(x, y)$.

Proof. $|\varphi_0(x) - \varphi_0(y)| = |[\varphi_0(x - (x \cap y)) + \varphi_0(x \cap y)] - [\varphi_0(y - (x \cap y)) + + \varphi_0(x \cap y)]| = |\varphi_0(x - (x \cap y)) - \varphi_0(y - (x \cap y))| \leqslant \|\varphi\|(x - (x \cap y)) + \|\varphi\|(y - - (x \cap y)) = \omega(x - (x \cap y)) + \omega(y - (x \cap y)) \leqslant \omega(x - y) + \omega(x - y) = 2\varrho(x, y)$.

The last estimate is true, since

$$|x - (x \cap y)| = |(x - x) \cup (x - y)| = |(x - y)^+| = (x - y)^+ \leqslant |x - y|,$$

$$|y - (x \cap y)| \leqslant |x - y|.$$

Lemma 2.5. $x, y \in A$; $\varrho(x, y) = 0 \Rightarrow \varphi_0(x) = \varphi_0(y)$.
Proof. $|\varphi_0(x) - \varphi_0(y)| \leqslant 2\varrho(x, y) = 0$.

Theorem 2.2. *The function* φ_0 *is uniformly continuous on* A.

Proof. This follows from Lemma 2.4.

Definition 2.5. $R = \bar{A}$.

According to Theorem 2.1, R is complete. By Lemmas 1.5, 1.6, 1.7 and Definition 2.4, R is a lattice-ordered group. By Theorem 2.2 and the extension theorem for uniformyl continuous functions there exists exyctly one uniformly continuous function ď defined on R such that for every $x \in A \Rightarrow \varphi_0(x) = \varphi(x)$.

Lemma 2.6. $x, y \in R \Rightarrow |\varphi(x) - \varphi(y)) \leqslant 2\varrho(x, y)$.

Proof. Let $\varepsilon > 0$. As φ is a continuous functions and A is dense in R, there exist x_n, $y_n \in A$ such that $\varrho(x_n, x) < 1/n$, $\varrho(y_n, y) < 1/n$ and $|\varphi(x) - \varphi(x_n)| < \varepsilon$; $|\varphi(y) - \varphi(y_n)| < \varepsilon$, $|\varphi(x) - \varphi(y)| \leqslant |\varphi(x) - \varphi(x_n)| + |\varphi(y) - \varphi(y_n)| + |\varphi(x_n) - \varphi(y_n)| < 2\varepsilon + 2\varrho(x_n, y_n)$.

The continuity of metric gives us

$$|\varphi(x) - \varphi(y)| \leqslant 2\varepsilon + 2\varrho(x, y); \quad |\varphi(x) - \varphi(y)| \leqslant 2\varrho(x, y).$$

Lemma 2.7. *Let* $x \in R$. *Suppose that there exists* $x_n \in A$ *such that* $|x| \leqslant \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} |x_i|$ *and* $\sum_{n=1}^{\infty} \omega(x_n) < \infty$. *Then* $\omega(x) = 0$.

Proof. Let $\varepsilon > 0$. Take k such that $\sum_{i=k}^{\infty} \omega(x_i) < \varepsilon$. Then $|x| \leqslant \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} |x_i| \leqslant \bigcup_{i=k}^{\infty} |x_i|$, hence $\omega(x) \leqslant \sum_{i=k}^{\infty} \omega(|x_i|) < \varepsilon$.

Lemma 2.8. *Let* $x_n \in A$; $x_n \geqslant x_{n+1} \geqslant 0$ *and* $\omega(\bigcap_{n=1}^{\infty} x_n) = 0$. *Then* $\lim_{n\to\infty} |\varphi_0(x_n)| = 0$.

Proof. Let $\bigcap_{n=1}^{\infty} x_n = x$; $\varepsilon > 0$. There exists $y_n \in A (y_n \geqslant 0)$ such that $|x| \leqslant \sum_{n=1}^{\infty} y_n$ and $\sum_{n=1}^{\infty} \omega(y_n) = \sum_{n=1}^{\infty} \|\varphi\|(y_n) < \varepsilon$.

Evidently $x_n = (x_n - \sum_{i=1}^{n} y_i)^+ - (x_n - \sum_{i=1}^{n} y_i)^- + \sum_{i=1}^{n} y_i$,

$$\varphi_0(x_n) = \varphi_0((x_n - \sum_{i=1}^{n} y_i)^+) - \varphi_0((x_n - \sum_{i=1}^{n} y_i)^-)^+ + \varphi_0(\sum_{i-1}^{n} y_i).$$

Clearly

$$(x_n - \sum_{i=1}^{n} y_i)^+ \searrow 0, \text{ so that } \lim_{n\to\infty} |\varphi_0((x_n - \sum_{i=1}^{n} y_i)^+)| = 0.$$

From Lemma 1.2 we have

$$(x_n - \sum_{i=1}^{n} y_i)^- \leqslant \sum_{i=1}^{n} y_i,$$

$$|\varphi_0((x_n - \sum_{i=1}^{n} y_i)^-)| \leqslant \|\varphi\|((x_n - \sum_{i=1}^{n} y_i)^-) =$$

$$= \omega((x_n - \sum_{i=1}^{n} y\ )^-) \leqslant \omega(\sum_{i=1}^{n} y_i) \leqslant \sum_{i=1}^{n} \omega(y_i) \leqslant \sum_{i=1}^{\infty} \omega(y_i) < \varepsilon,$$

$$|\varphi_0(\sum_{i=1}^{n} y_i)| \leqslant \|\varphi\|(\sum_{i=1}^{n} y_i) = \omega(\sum_{i=1}^{n} y_i) \leqslant \sum_{i=1}^{n} \omega(y_i) \leqslant \sum_{i=1}^{\infty} \omega(y_i) < \varepsilon,$$

therefore

$$|\varphi_0(x_n)| \leqslant |\varphi_0((x_n - \sum_{i=1}^{n} y_i)^+)| + 2\varepsilon, \quad \limsup_{n\to\infty} |\varphi_0(x_n)| \leqslant 2\varepsilon,$$

and therefore $\lim_{n\to\infty} |\varphi_0(x_n)| = 0$.

Theorem 2.3. *Let $x_n \in R$, $x_n \searrow 0$. Then* $\lim_{n\to\infty} \varphi(x_n) = 0$.

Proof. Let $\varepsilon > 0$. Consider $y_n \in A$, $y_n \geqslant 0$ such that $\varrho(x_n, y_n) < \varepsilon \,.\, 2^{-n}$. We put $z_n = \bigcap_{i=1}^{n} y_i$. Clearly $z_n \in A$ and

$$\omega(z_n - x_n) = \omega(\bigcap_{i=1}^{n} y_i - \bigcap_{i=1}^{n} x_i) \leqslant \sum_{i=1}^{n} \omega(y_i - x_i) =$$

$$= \sum_{i=1}^{n} \varrho(x_i, y_i) < \varepsilon \,.\, 2^{1-n} \leqslant \varepsilon.$$

Evidently

$$|\bigcap_{n-1}^{\infty} z_n| = \bigcap_{n=1}^{\infty} z_n = \bigcap_{n=1}^{\infty} y_n = \bigcap_{n=1}^{\infty} \{[y_n - (y_n \cap x_n)] + (x_n \cap y_n)\} =$$

$$= \bigcap_{n=1}^{\infty} [y_n - (y_n \cap x_n)] + \bigcap_{n=1}^{\infty} (x_n \cap y_n) = \bigcap_{n=1}^{\infty} [y_n - (y_n \cap x_n)] =$$

$$= \bigcap_{n=1}^{\infty} |y_n - (y_n \cap x_n)|.$$

In Lemma 1.6. put $a = b = c = y_n$; $d = x_n$.
This yields

$$|y_n - (y_n \cap x_n)| \leqslant |y_n - x_n|,$$

hence

$$|\bigcap_{n=1}^{\infty} z_n| = \bigcap_{n=1}^{\infty} |y_n - (y_n \cap x_n)| \leqslant \bigcap_{n=1}^{\infty} |y_n - x_n| \leqslant \bigcap_{n-1}^{\infty} \bigcup_{i-1}^{\infty} |x_i - y_i|,$$

$$|\bigcap_{n-1}^{\infty} z_n| \leqslant \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} |x_i - y_i|$$

and

$$\sum_{n=1}^{\infty} \omega(x_n - y_n) = \sum_{n=1}^{\infty} \varrho(x_n, y_n) < \varepsilon < \infty.$$

By Lemma 2.7 $\omega(\bigcap_{n=1}^{\infty} z_n) = 0$.

From the definition of z_n, we have $z_n \geqslant z_{n+1}$; therefore, according to Lemma 2.8 $\lim_{n\to\infty} |\varphi_0(z_n)| = 0$.

By Lemma 2.6

$$|\varphi(x_n) - \varphi(z_n)| \leqslant 2\varrho(x_n, z_n) < \varepsilon \,.\, 2^{2-n} \leqslant 2\varepsilon,$$

so that $\lim_{n\to\infty} |\varphi(x_n) - \varphi(z_n)| = 0$; $\lim_{n\to\infty} |\varphi(z_n)| = 0$.

This gives us $\lim_{n\to\infty} |\varphi(x_n)| = 0 \Rightarrow \lim_{n\to\infty} \varphi(x_n) = 0$.

Theorem 2.4. *Let $x, y \in R$. Then $\varphi(x + y) = \varphi(x) + \varphi(y)$.*

Proof. We take $u, v \in A$ such that $\varrho(x, u) < \varepsilon$; $\varrho(y, v) < \varepsilon$.
Then $\varrho(x + y, u + v) = \omega(x + y - u - v) \leqslant \omega(x - u) + \omega(y - v) < 2\varepsilon$,

$$|\varphi(x) + \varphi(y) - \varphi(x + y)| \leqslant |\varphi(x) - \varphi(u)| + |\varphi(y) - \varphi(v)| + |\varphi(u) + \varphi(v) - \varphi(u + v)| + |\varphi(u + v) - \varphi(x + y)| \leqslant 2\varrho(x, u) + 2\varrho(x + y, u + v) < 2\varepsilon + 2\varepsilon + 4\varepsilon = 8\varepsilon.$$

Theorem 2.5. *Let $x, x_n \in R$. If $x_n \searrow x$ or $x_n \nearrow x$, then $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$.*

Proof. Let $x_n \searrow x$. Then $x_n - x \searrow x - x = 0 \Rightarrow \lim_{n\to\infty} \varphi(x_n - x) = 0 \Rightarrow \lim_{n\to\infty} \varphi \,.\, (x_n) = \varphi(x)$.

Let $x_n \nearrow x$. Then $x - x_n \searrow x - x = 0 \Rightarrow \lim_{n\to\infty} \varphi(x - x_n) = 0 \Rightarrow \lim_{n\to\infty} \varphi(x_n) = \varphi(x)$.

Definition 2.6. *For $x \in R$ put $\|\varphi_1\|(x) = \sup \{|\varphi(y)| : y \in R;\ y \geqslant 0;\ y \leqslant |x|\}$. Evidently $\|\varphi_1\|(x) \geqslant \|\varphi\|(x)$ for $x \in A$.*

Theorem 2.6. $x \in A \Rightarrow \|\varphi\|(x) = \|\varphi_1\|(x)$.

Proof. It suffices to prove that for $x \in A$, $\Rightarrow \|\varphi_1\|(x) \leqslant \|\varphi\|(x)$. It is therefore enough to prove the existence of an element $z \in A$; $z \geqslant 0$; $z \leqslant |x|$, for which $\|\varphi_1\|(x) - |\varphi(z)|$ is small enough.

Let $\|\varphi_1\|(x) < \infty$ and $\varepsilon > 0$. There exists $y \in R$ such that $y \geqslant 0$; $y \leqslant |x|$ and $|\varphi(y)| + \varepsilon > \|\varphi_1\|(x)$. We choose $u \in A$ such that $\varrho(y, u) < \varepsilon$ and put $z = u^+ \cap |x|$.

Now $\varrho(y, z) = \varrho(u^+ \cap |x|, y \cap |x|) \leqslant \varrho(u^+, y) = \varrho(u \cup 0, y \cup 0) \leqslant \varrho(u, y) < < \varepsilon$, $|\varphi(y) - \varphi(z)| \leqslant 2\varrho(y, z) < 2\varepsilon \Rightarrow |\varphi(y)| < 2\varepsilon + |\varphi(z)|$, $\|\varphi_1\|(x) < |\varphi(y)| + + \varepsilon < |\varphi(z)| + 3\varepsilon$.

Now let $\|\varphi_1\|(x) = \infty$. We choose a natural number N. There exists $y \in R$ such that $y \geqslant 0$, $y \leqslant |x|$ and $|\varphi(y)| > N$. In a manner analogous to the above construction, we construct an element $z \in A$ such that $z \geqslant 0$, $z \leqslant |x|$ and $\varrho(y, z) < \varepsilon$.

Now $|\varphi(y) - \varphi_0(z)| \leqslant 2\varrho(y, z) < 2\varepsilon \Rightarrow N < |\varphi(y)| \leqslant |\varphi_0(z)| + 2\varepsilon \Rightarrow |\varphi_0(z)| \geqslant \geqslant N - 2\varepsilon$ therefore $\|\varphi\|(x) = \infty$; $x \in A$, which is a contradiction.

From Theorem 2.6 it can be seen that repeating the original construction would not yield any further extension of the functional φ. We should get the same pseudometric ϱ and $\bar{R} = R$, since R is closed with respect to φ.

§ 3. Saturability

Let $A \subset X$ be a lattice ordered group. On A let us define a functional φ_0 for which relations $M_1 - M_3$ hold.

Let φ be an extension of φ_0 on R. We shall use the notation of § 2.

Definition 2.1. *We say that functional φ_0 is saturable, if for every $x_n \in A$ and $x_0 \in R$ such that $\sum_{n=1}^{\infty} |x_n| \leqslant x_0$, we have $\sum_{n=1}^{\infty} |\varphi_0(x_n)| < \infty$.*

Lemma 3.1. $x_n \in R$, $x_n \geqslant 0$; $x_0 \in R$, $\sum_{n=1}^{\infty} x_n \leqslant x_0 \Rightarrow x = \sum_{n=1}^{\infty} x_n \in R$.

Proof. $0 \leqslant x \leqslant x_0 \Rightarrow \omega(x) \leqslant \omega(x_0) < \infty \Rightarrow x \in F$.

Let $u_n = \sum_{i=1}^{\infty} x_i$, $\varrho(x, u_n) = \omega(\sum_{i=n+1}^{\infty} x_i) \leqslant \sum_{i=n+1}^{\infty} \|\varphi\|(x_i)$.

We must prove $\lim_{n \to \infty} \varrho(x, u_n) = 0$. For that it is enough to prove $\sum_{n=1}^{\infty} \|\varphi\|(x_n) < < \infty$. Let $\varepsilon > 0$. There exists $v_n \in A$, $|v_n| \leqslant x_n$ such that $\|\varphi\|(x_n) < |\varphi_0(v_n)| + + \varepsilon \,.\, 2^{-n}$. Then $\sum_{n=1}^{\infty} \|\varphi\|(x_n) \leqslant \sum_{n=1}^{\infty} |\varphi_0(v_n)| + \varepsilon < \infty$ because $\sum_{n=1}^{\infty} |v_n| \leqslant \sum_{n=1}^{\infty} x_n \leqslant \leqslant x_0 \in R$. Thus $x \in R$.

Theorem 3.1. $x_n \in R,\ x_n \leqslant x_{n+1} \leqslant x_0 \in R,\ x = \bigcup_{n=1}^{\infty} x_n \Rightarrow x \in R$ and $\varphi(x) = \lim \varphi(x_n)$.

Proof. $x_n \nearrow x$. Let $y_1 = x_1,\ y_{n+1} = x_{n+1} - x_n \geqslant 0,\ y_n \in R,\ x_1 + \sum_{n=2}^{\infty} y_n = x,\ \sum_{n=2}^{\infty} y_n = x - x_1 \leqslant x_0 - x_1 \in R.$

By the abovementioned lemma we have $\sum_{n=1}^{\infty} y_n \in R$, therefore also $x \in R$. The second assertion of Theorem 3.1 follows from Theorem 2.4.

Theorem 3.2. $x_n \in R,\ x_n \geqslant x_{n+1} \geqslant x_0 \in R \Rightarrow x = \bigcap_{n=1}^{\infty} x_n \in R$ and $\varphi(x) = \lim_{n \to \infty} \varphi(x_n)$.

Proof. Let $y_n = -x_n;\ y_0 = -x_0$. Evidently $y_0, y_n \in R$ and $y_n \leqslant y_{n+1} \leqslant y_0,\ y = \bigcup_{n=1}^{\infty} y_n = \bigcup_{n=1}^{\infty} (-x_n) = -(\bigcap_{n=1}^{\infty} x_n) = -x.$

By the Theorem 3.1 $y \in R$, therefore also $x = -y \in R$. Using Theorem 3.1, we get

$$\varphi(x) = \varphi(-y) = -\varphi(y) = -\lim_{n \to \infty} \varphi(y_n) = \lim_{n \to 1} [-\varphi(y_n)] = \lim_{n \to \infty} \varphi(-y_n) = \lim_{n \to \infty} \varphi(x_n).$$

Theorem 3.3. $x_n \in R;\ x_n \leqslant x_0 \in R \Rightarrow x = \bigcup_{n=1}^{\infty} x_n \in R.$

Proof. Let $y_n = \bigcup_{i=1}^{n} x_i \in R$, as R is a lattice ordered group, $y_n \leqslant y_{n+1} \leqslant x_0$.

By Theorem 3.1. $x = \bigcup_{n=1}^{\infty} y_n = \bigcup_{n=1}^{\infty} x_n \in R.$

Theorem 3.4. $x_n \in R;\ x_n \geqslant x_0 \in R \Rightarrow x = \bigcap_{n=1}^{\infty} x_n \in R.$

Proof. Analogous to that of Theorem 3.3.

§ 4. Measure and integral

We shall now show that the abovementioned construction may be used to extend a vector measure and integral.

1. Let X be the system of all real — valued functions defined on the set M.

The operations $+$, $-$, $\cup$ and $\cap$ together with the relation $\leqslant$, are interpreted in the usual way.

Let A be the system of all simple integrable functions. Let $\varphi_0(f) = \int f \mathrm{d}\,\mu$. Let φ_0 satisfy the axiom M . Then the construction of § 2 gives us an ex tension theorem for the vector integral, where R is the system of all integrable functions and $g \in R \to \int g \mathrm{d}\,\mu = \varphi(g)$.

2. Let μ be a vector measure. Let A be the system of all real valued functions which can be written in form $f = \sum\limits_{i-1}^{n} \alpha_i \; \chi_{E_i}$ where α_i are real numbers, E_i measurable pairwise disjoint subsets and χ_{E_i} the characteristic function of the set E_i. Then $\varphi_0(f) = \sum\limits_{i=1}^{n} \alpha_i \; \mu(E_i)$. Only vector measures μ such that φ_0 satisfies the axiom M_3 will be considered. X will again be the system of all real — valued functions defined on M. Using the construction of § 2, we can extend the functional φ_0 to φ. Then $\nu(E) = \varphi(\chi_E)$ is the required extension of the vector measure μ.

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Received May 4, 1969

Katedra matematiky
Prírodovedeckej fakulty Univerzity P. J. Šafárika
Košice