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ON CERTAIN CLASSES OF SETS OF NATURAL NUMBERS

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In number theory there were studied the properties of such infinite sets $A \subset N = \{1, 2, 3, \dots n, \dots\}$, for which the following holds: if $a, b \in A$, $a \neq b$, then $a \nmid b$, $b \nmid a$ (see [1], [2], [3] II, p. 18—20). Denote by $\mathscr{S}_2$ the class of all such sets $A \subset N$.

Denote by $\mathscr{S}_1$ the class of all such infinite sets $A \subset N$, which have the following property: if $a \in A$, then A contains every natural divisor of the number a. Š. Znám called attention to these sets.

Obviously $\mathscr{S}_1 \cap \mathscr{S}_2 = \emptyset$ and the sets belonging to $\mathscr{S}_1$ have in a certain sense the properties which are opposite to those of sets from $\mathscr{S}_2$.

We shall define other two classes of sets $A \subset N$ more. Let F be a function from N to $\mathscr{U}$, $\mathscr{U}$ being the class of all infinite sets $A \subset N$. We shall write F_d instead of $F(d)$. Denote by $\mathscr{T}_1(F)$ ($\mathscr{T}_2(F)$) the class of all such $A \in \mathscr{U}$, which fulfil the following condition:

$$\text{if} \quad a \notin A, \quad \text{then} \quad F_a \cap A = \emptyset$$
$$(\text{if} \quad a \in A, \quad \text{then} \quad F_a \cap A = \emptyset).$$

The following theorem states some fundamental relations between the introduced classes for a special choice of the function F.

Theorem 1. *Let $1 < k_1 < k_2 < \dots$ be a sequence of natural numbers and for each $d \in N$ let $F_d = \{k_1 d, k_2 d, \dots\}$. Then we have $\mathscr{S}_i \subset \mathscr{T}_i(F)$ $(i = 1, 2)$.*

Proof. Let $A \in \mathscr{S}_1$. We shall show that $A \in \mathscr{T}_1(F)$. Let $a \notin A$. If $F_a \cap A \neq \emptyset$, then there exists a $b \in F_a \cap A$. Owing to the definition of F_a there exists an n such that $b = k_n a$. But $k_n a \in A$, $a | k_n a$ and $a \notin A$. This is a contradiction to the fact that $A \in \mathscr{S}_1$. The proof of the inclusion $\mathscr{S}_2 \subset \mathscr{T}_2(F)$ is analogous.

Remark a) When the function F fulfilling the condition stated in Theorem 1 is suitably chosen, the relations

$$\mathscr{S}_i \subset \mathscr{T}_i(F), \quad \mathscr{S}_i \neq \mathscr{T}_i(F) \quad (i = 1, 2)$$

take place. E. g. if we put $k_n = 2n + 1$ $(n = 1, 2, \dots)$, then $A = \{2, 4, \dots,$

$2k, \ldots\} \notin \mathscr{S}_1$ and it is easy to see that $A \in \mathscr{T}_1(F)$. Indeed, if $d \notin A$, then d is an odd number and therefore $k_n d$ $(n = 1, 2, \ldots)$ is also an odd number. Hence $F_d \cap A = \emptyset$. If we put $k_n = 2n$ $(n = 1, 2, \ldots)$, then $A' = \{1, 3, \ldots, 2k - 1, \ldots\} \notin \mathscr{S}_2$ and it is easy to verify that $A' \in \mathscr{T}_2(F)$.

b) Using the previous considerations it can be easily seen that

1. the inclusion $\mathscr{S}_1 \subset \mathscr{T}_1(F)$ holds if and only if for each natural number d the set F_d consists of multiples of the number d;

2. the inclusion $\mathscr{S}_2 \subset \mathscr{T}_2(F)$ holds if and only if for each natural number d each element of the set F_d is different from d and is either a divisor or a multiple of d;

3. if $F_d = \{2d, 3d, 4d, \ldots\}$ for each d, then $\mathscr{S}_i = \mathscr{T}_i(F)$ $(i = 1, 2)$.

If $A \subset N$ then we put $A(n) = \sum_{a \in A, a \leqq n} 1$. The numbers $\delta_1(A) = \liminf_{n \to \infty} \frac{A(n)}{n}$ and $\delta_2(A) = \limsup_{n \to \infty} \frac{A(n)}{n}$ are called the lower and upper asymptotic density of the set A, respectively. If there exists $\lim_{n \to \infty} A(n)/n = \delta(A)$, then it is called the asymptotic density of the set A.

Every infinite subset of the set of all prime numbers belongs to the class $\mathscr{S}_2$. From this it is obvious that the class $\mathscr{S}_2$ is uncountable of the power of the continuum. An analogous result for the class $\mathscr{S}_1$ follows from the following theorem, which gives a certain more detailed view on the structure of the class $\mathscr{S}_1$.

Theorem 2. i) *For each η, $0 \leqq \eta < 1$, there exists an infinite system of the power of the continuum of sets $A \in \mathscr{S}_1$ such that $\delta(A) = \eta$.*

ii) *The set N is the only set from the class $\mathscr{S}_1$ with the asymptotic density* 1.

Proof. i) Let P denote the set of all prime numbers. If $P_1 \subset P$, $P_1 \neq \emptyset$, then denote by $M(P_1)$ the set which consists of all such natural numbers, which are not divisible by any prime number belonging to $P - P_1$. It is obvious that $M(P_1) \in \mathscr{S}_1$ for each $P_1 \subset P$, $P_1 \neq \emptyset$ and $M(P_1) \neq M(P_2)$ for $P_1 \neq P_2$, $P_i \subset P$, $P_i \neq \emptyset$ $(i = 1, 2)$.

It suffices to prove that for each η, $0 \leqq \eta < 1$, there exists an infinite number of the power of the continuum of sets $P_1 \subset P$, $P_1 \neq \emptyset$, such that $\delta(M(P_1)) = \eta$.

Let $0 \leqq \eta < 1$. We put

(1) $\quad \alpha = -\log \eta \quad$ for $\quad 0 < \eta < 1 \quad$ and $\quad \alpha = +\infty \quad$ for $\quad \eta = 0$.

Thus $\alpha > 0$ in both cases. Let $p_1 < p_2 < \ldots$ denote the sequence of all prime numbers. Then, as it is well-known, we have $\prod_{k=1}^{\infty} \left(1 - \frac{1}{p_k}\right) = 0$ and so $\sum_{k=1}^{\infty} a_k = +\infty$, where $a_k = -\log\left(1 - \frac{1}{p_k}\right)$, $a_k \geqq a_{k+1} > 0$ $(k = 1, 2, \ldots)$, $a_k \to 0$.

On account of Theorem 1,1 of the paper [4] there exists an infinite system of the power of the continuum of infinite sets

$$K = \{k_1 < k_2 < \ldots < k_n < \ldots\} \subset N, \quad K \neq N,$$

such that

$$\sum_{n=1}^{\infty} a_{k_n} = \alpha. \tag{2}$$

Put $P_1 = P - \{p_{k_1}, p_{k_2}, \ldots\}$. Then $M(P_1)$ consists of the number 1 and of all such numbers $n > 1$, which are not divisible by any prime number p_{k_j} $(j = 1, 2, \ldots)$. For the asymptotic density of $M(P_1)$

$$\delta(M(P_1)) = \prod_{j=1}^{\infty}\left(1 - \frac{1}{p_{k_j}}\right) \tag{3}$$

holds (cf. [3], II, p. 14).

From (1), (2), (3) we obtain $\delta(M(P_1)) = \eta$. Since the cardinality of the class of all

$$K = \{k_1 < k_2 < \ldots < k_n < \ldots\}, \quad K \neq N$$

with (2) is c (the power of the continuum), the cardinality of all $P_1 \subset P$, $P_1 \neq \emptyset$ with (3) is c, too.

ii) It is obvious that $N \in \mathscr{S}_1$ and $\delta(N) = 1$. If $A \in \mathscr{S}_1$ and $A \neq N$, then there exists an $a \in N$, which does not belong to A. Then $ka \notin A$ $(k = 1, 2, \ldots)$ in view of the definition of the class $\mathscr{S}_1$ and therefore $\delta_2(A) \leqq 1 - \frac{1}{a} < 1$. This ends the proof.

Remark. An analogous result to the previous theorem for the class $\mathscr{S}_2$ is not true. It is namely well-known that for each $A \in \mathscr{S}_2$ $\delta_1(\mathrm{A}) = 0$ and $\delta_2(\mathrm{A}) < \frac{1}{2}$ holds (see [1]; [2]; [3], II, p. 18—20).

In what follows we shall study the introduced classes from the metric point of view using the dyadic values of sets $A \subset N$ (cf. [3], I, p. 17, 193—195). On the system $\mathscr{U}$ of all infinite sets $A \subset N$ we define the function ϱ in the following way: $\varrho(A) = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, where $\varepsilon_k = 1$ if $k \in A$ and $\varepsilon_k = 0$ in the opposite case. The number $\varrho(A) \in (0, 1\rangle$ is called the dyadic value of the set A. The fukction ϱ is a one-to-one mapping from $\mathscr{U}$ onto $(0, 1\rangle$. If $\mathscr{W} \subset \mathscr{U}$, then $\varrho(\mathscr{W})$ stands for the set of all numbers $\varrho(\mathrm{A})$, $\mathrm{A} \in \mathscr{W}$. The study of the properties of the set $\varrho(\mathscr{W}) \subset (0, 1\rangle$ gives us a certain idea about the structure and „magnitude" of the class $\mathscr{W}$.

We express all the numbers $x \in (0, 1\rangle$ in their non-terminating dyadic expansions $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$, $\varepsilon_k(x) = 0$ or 1 and $\varepsilon_k(x) = 1$ for an infinite number of k' s.

In what follows we show that the sets $\varrho(\mathscr{T}_i(F))$ $(i = 1, 2)$ are „poor" both from the topological and metric point of view.

Theorem 3. *Let $F : N \to \mathscr{U}$. Then the set $\varrho(\mathscr{T}_i(F))$ $(i = 1, 2)$ is a non-dense set in* $(0, 1\rangle$.

Theorem 4. *Each of the sets ϱ $(\mathscr{T}_i(F))$ $(i = 1, 2)$ is a null set (in the sense of the Lebesgue measure).*

The proofs of theorems 3,4 are based on the following lemma*).

Lemma. *Let $\{\varepsilon_l\}_{l=1}^{\infty}$ be a sequence of the numbers 0, 1, let $A \in \mathscr{U}$. Denote by M_A the set of all $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k} \in (0, 1\rangle$ for which the following is true: if $k \in A$ then $\varepsilon_k(x) = \varepsilon_k$. Then M_A is a non-dense null set.*

Proof. Let $J \subset (0, 1)$ be an open interval. Choose a natural number n such that $n \in A$ and

$$I = \left(\frac{s}{2^n}, \quad \frac{s+2}{2^n}\right) \subset J$$

for a suitable non-negative integer s. Then either the interval $\left(\frac{s}{2^n}, \quad \frac{s+1}{2^n}\right)$ or the interval $\left(\frac{s+1}{2^n}, \quad \frac{s+2}{2^n}\right)$ is disjoint with the set M_A. From the last it is obvious that M_A is non-dense.

Let $A = \{n_1 < n_2 < \ldots < n_k < \ldots\}$. It can be easily calculated that the measure of the set H_m of all $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k} \in (0, 1\rangle$ with $\varepsilon_{n_j}(x) = \varepsilon_{n_j}$

*) The author is indebted to the Reviewer for the simplification of the proofs of Theorems 3, 4.

$(j = 1, 2, \ldots, m)$ is 2^{-m}. Further

$$H_1 \supset H_2 \supset H_3 \supset \ldots \quad \text{and} \quad \bigcap_{m=1}^{\infty} H_m = M_A,$$

hence M_A is a null set.

Proof of Theorem 3. Let $I \subset (0, 1)$ be an open interval. Let us choose a natural n and s even such that

$$(4) \qquad J = \left(\frac{s}{2^n}, \frac{s+1}{2^n}\right) \subset I.$$

We shall prove that

$$(5) \qquad \varrho(\mathscr{T}_1(F)) \cap J \subset M_{F_n}$$

under the choice $\varepsilon_l = 0$ $(l = 1, 2, \ldots)$ in the lemma.

Let $x \in \varrho(\mathscr{T}_1(F)) \cap J$. Then $x \in J$ and $x = \varrho(A)$, where $A \in \mathscr{T}_1(F)$. Since s is even, we have $\varepsilon_n(x) = 0$, hence $n \notin A$. According to the definition of the system $\mathscr{T}_1(F)$ we have $\varepsilon_k(x) = 0$ for each $k \in F_n$. If we use the lemma for the case $\varepsilon_l = 0$ $(l = 1, 2, \ldots)$, then (5) holds. Now the assertion follows from the lemma immediately.

The proof for $\varrho(\mathscr{T}_2(F))$ can be realized in an analogous way choosing a natural n and odd s such that (4) holds.

Proof of Theorem 4. Denote by $\mathscr{W}_k$ the system of all $A \in \mathscr{T}_i(F)$ $(i = 1, 2)$ for which the following is true: there exists a j, $1 \leqq j \leqq k$, such that $j \notin A$ (for $i = 1$), $j \in A$ (for $i = 2$), respectively. If $A \in \mathscr{W}_k$, then at least for one j, $1 \leqq j \leqq k$ we have $A \cap F_j = \emptyset$. Put $\varepsilon_l = 0$ $(l = 1, 2, \ldots)$ in Lemma. Then $\varrho(\mathscr{W}_k) \subset \bigcup_{j=1}^{k} M_{F_j}$. From this it follows on account of Lemma that $\varrho(\mathscr{W}_k)$ is a null set. Since $\mathscr{T}_i(F) = \bigcup_{k=1}^{\infty} \mathscr{W}_k$, the set $\varrho(\mathscr{T}_i(F))$ $(i = 1, 2)$ is a null set, too.

An immediate consequence of the theorems 1, 3, 4 is

Theorem 5. *Each of the sets $\varrho(\mathscr{S}_i)$ $(i = 1, 2)$ is a non-dense null set.*

The fact that the set $\varrho(\mathscr{S}_2)$ is a null set is also an easy consequence of the following theorem 6. In the sequel dim M denotes the Hausdorff dimension of the set M (cf. [3], I, p. 190, [5]). The next theorem states the exact value of the Hausdorff dimension of the set $\varrho(\mathscr{S}_2)$. The question about the magnitude of the Hausdorff dimension of each of the sets $\varrho(\mathscr{S}_1)$, $\varrho(\mathscr{T}_i(F))$ $(i = 1, 2)$ (here at least for some special choices of the function F) remains open.

Theorem 6. $\dim \varrho(\mathscr{S}_2) = 0$.

Proof. Let $\mathscr{Z}_0$ denote the class of all $A \in \mathscr{U}$ with $\delta_1(A) = 0$. Then

$$\dim \varrho(\mathscr{Z}_0) = 0 \tag{6}$$

(cf. [3], I, p. 195; [5]). For each $A \in \mathscr{S}_2$ we have $\delta_1(A) = 0$ (cf. [1]; [3], II, p. 18). Hence $\mathscr{S}_2 \subset \mathscr{Z}_0$. This together with (6) yields $\dim \varrho(\mathscr{S}_2) = 0$.

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