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A NOTE ON RADICALS OF SEMIGROUPS

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Let S be a semigroup. For each ideal J of S, $M(J)$, $L(J)$, $R^*(J)$, $C(J)$ and $N(J)$, respectively, will denote the McCoy (prime) radical, the Ševrin (locally nilpotent) radical, the Clifford (nil-) radical, the Jiang Luh (completely prime) radical and the set of all nilpotent elements with respect to J. J. Bosák [1] proved that

$$M(J) \subseteq L(J) \subseteq R^*(J) \subseteq N(J) \subseteq C(J) \tag{1}$$

for every ideal J of a semigroup S. R. Šulka [2] proved that if S is a commutative semigroup and J is an ideal of S, then

$$M(J) = L(J) = R^*(J) = N(J) = C(J). \tag{2}$$

Further, J. E. Kuczkowski [3] proved that if S is a C_2-semigroup then (2) holds for every ideal J of S (by a C_2-semigroup we mean a semigroup S with the property that $xyzyx = yxzxy$ for all $x, y, z \in S$). In this note we shall study the necessary and sufficient condition such that (2) holds for every ideal J of a semigroup S.

Let a be an element of a semigroup S. The principal ideal generated by a we denote by $J(a)$.

Theorem. *Let S be a semigroup. Then* (2) *holds for every ideal J of S if and only if*

$$J(a) \cap J(b) \subseteq M(J(ab)) \tag{3}$$

for all $a, b \in S$.

Proof. Let (2) hold for every ideal J of S. Let $x \in J(a) \cap J(b)$ for some $a, b \in S$. Let us assume that $x \notin M(J(ab))$. Then it follows from (2) that $x \notin C(J(ab))$. The Jiang Luh radical $C(J(ab))$ is the intersection of all completely prime ideals of S including $J(ab)$. This implies that there exists a completely prime ideal I of S such that $J(ab) \subseteq I$ and $x \notin I$. Evidently, $ab \in J(ab) \subseteq I$ and so $a \in I$ or $b \in I$. Thus $x \in J(a) \subseteq I$ or $x \in J(b) \subseteq I$. This contradicts $x \notin I$. Therefore $x \in M(J(ab))$. Hence (3) holds for all $a, b \in S$.

Let (3) hold for all $a, b \in S$. According to (1), it suffices to prove that $C(J) \subseteq$

$\subseteq M(J)$ for every ideal J of S. Let $x \in C(J)$ for some ideal J of S. Let us assume that $x \notin M(J)$. By [2] $M(J)$ consists exactly of such elements x that every m-system containing x has a non-empty intersection with J. This implies that there exists an m-system A of S such that $A \cap J = \emptyset$ and $x \in A$. Let I be the union of all ideals of S which do not meet A. The ideal I has the required maximal property. We shall prove that I is completely prime. Suppose that $a \notin I$, $b \notin I$ and $ab \in I$ for some a, $b \in S$. Then $J(a) \cap A \neq \emptyset \neq A \cap J(b)$. There exist u, $v \in S$ such that $u \in J(a) \cap A$ and $v \in J(b) \cap A$. Since A is an m-system of S, then $uzv \in A$ for some $z \in S$. Evidently, $uzv \in J(a) \cap J(b)$. It follows from (3) that $uzv \in M(J(ab))$. Since $uzv \in A \cap M(J(ab))$, then $A \cap J(ab) \neq \emptyset$. Now, $ab \in I$ implies that $J(ab) \subseteq I$ and so $A \cap I \neq \emptyset$. This contradicts $A \cap I = \emptyset$. Therefore, I is a completely prime ideal of S. Since $x \in A$, then $x \notin I$ and thus $x \notin C(J)$. This contradicts $x \in C(J)$. Therefore, we have $x \in M(J)$. Hence $C(J) \subseteq M(J)$.

Denote by $L(x)$, $R(x)$ and $Q(x)$ the principal left, right and quasi ideal of a semigroup S generated by $x \in S$, respectively. Clearly $Q(x) = L(x) \cap \cap R(x)$.

Corollary 1. *Let S be a semigroup and let for every element a of S*

$$J(a) \subseteq N(Q(a)), \tag{4}$$

$$a \in \bigcap_{n=1}^{\infty} M(J(a^n)) \tag{5}$$

hold. Then (2) *holds for every ideal J of S.*

Proof. We shall prove that (3) holds for every a, $b \in S$. Let $x \in J(a) \cap \cap J(b)$. It follows from (4) that $x \in N(Q(a)) \cap N(Q(b))$. Since $N(A)$ is the set of all nilpotent elements with respect to A, then $x^n \in Q(a) \subseteq L(a)$ and $x^m \in \in Q(b) \subseteq R(b)$ for some positive integers n, m. Then $x^{n+m} \in L(a)\,R(b) \subseteq J(ab)$. Evidently, $J(x^{n+m}) \subseteq J(ab)$. According to (5), we have $x \in M(J(x^{n+m}))$. Thus, by Lemma 7 [2], we obtain that $x \in M(J(ab))$. Hence $J(a)) \cap J(b) \subset M(J(ab)$. Theorem implies that (2) holds for every ideal J of S.

The principal biideal generated by an element x of a semigroup S we denote by $B(x)$, i.e. $B(x) = xSx \cup x^2 \cup x$. Clearly $B(x) \subseteq Q(x)$.

Corollary 2. *Let S be a semigroup and let for any elements x, y, z of S*

$$yxzxy \in B(x) \tag{6}$$

hold. Then (2) *holds for every ideal J of S.*

Proof. Let $a \in S$. We shall prove that (4) and (5) hold.

1. Let $x \in J(a)$. We shall show that $x^3 \in Q(a)$. Evidently, x may have several forms: a, sa, at or sat, where s, $t \in S$.

(i) If $x = a$ then $x^3 = a^3 \in Q(a)$.

(ii) If $x = sa$ then, by (6), we have $x^3 = (sasas)a \in B(a)a \subseteq B(a) \subseteq Q(a)$.

(iii) If $x = at$ then similarly we obtain that $x^3 \in Q(a)$.

(iv) If $x = sat$ then, by (ii) and (iii), we have $x^3 \in Q(sa) \cap Q(at) \subseteq Q(a)$.

Hence, $x \in J(a)$ implies that $x^3 \in Q(a)$ and so $x \in N(Q(a))$. Thus (4) is true.

2. Let A be an arbitrary m-system containing a. First we shall prove that $A \cap a^nSa^n \neq \emptyset$ for every positive integer n. Clearly $aza \in A$ for some $z \in S$ and so $A \cap aSa \neq \emptyset$. The statement is true for $n = 1$. Assume the statement to be true for $n = k$. Then, by the induction hypothesis, $A \cap a^kSa^k \neq \emptyset$. There exists $u \in S$ such that $a^kua^k \in A$. Since A is an m-system of S, there exists $v \in S$ such that $y = a^kua^kva^kua^k \in A$. It follows from (6) that $y \in \in a^kB(a^k)a^k$. Since $y \in A$, there exists $w \in S$ such that $ywy \in A \cap a^kB(a^k) . . a^kwa^kB(a^k)a^k \subseteq A \cap a^{k+1}Sa^{k+1}$ and so that $A \cap a^{k+1}Sa^{k+1} \neq \emptyset$. Hence the statement is valid for every positive integer n. Since $a^nSa^n \subseteq J(a^n)$, then $A \cap J(a^n) \neq \emptyset$ for any n. Thus we have $a \in \bigcap_{n=1}^{\infty} M(J(a^n))$ and (3) is true.

A semigroup S is called a *duo semigroup* if every one-sided ideal of S is a two-sided ideal. Clearly $L(x) = J(x) = R(x)$ for every $x \in S$.

Corollary 3. *Let S be a duo semigroup. Then* (2) *holds for every ideal J of S.*

Proof. We shall show that (6) holds for every $x, y, z \in S$. Evidently, we have $yxzxy \in L(x)zR(x) = R(x)zL(x) \subseteq B(x)$.

A semigroup S is called *normal* if $xS = Sx$ for every x of S. Evidently, every normal semigroup is a duo semigroup.

Corollary 4. *Let S be a normal semigroup. Then* (2) *holds for every ideal J of S.*

Corollary 5. (Cf. [3], Theorem). *Let S be a C_2-semigroup. Then* (2) *holds for every ideal J of S.*

Proof. We shall show that (6) holds for every $x, y, z \in S$. Evidently, we have $yxzxy = xyzyx \in B(x)$.

Corollary 6. (Cf. [2], Theorem 7). *Let S be a commutative semigroup. Then* (2) *holds for every ideal J of S.*

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