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NOTE ON ERGODICITY

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A measurable transformation T on a measure space (X, S, m) is ergodic, iff for any almost invariant set $E \in S$ (i. e. such set E that $m(T^{-1} E \triangle E) = 0$) it is $m(E) = 0$ or $m(X - E) = 0$. (We do not suppose that T is measure preserving.)

Our note deals with a criterion of ergodicity from paper [2]. We shall prove that in the criterion the assumption that T is measure preserving can be replaced by the weaker assumption that T is incompressible.

First we shall formulate our propositions algebraically. We shall suppose that a Boolean σ-algebra S and a σ-isomorphism T of this algebra are given Further a σ-ideal $N \subset S$ is given and $TN = N$.

A transformation T of S into S will be called incompressible, iff from the relation $T^{-1} E \subset E$ it follows that $E - T^{-1} E \in N$. A σ-isomorphism T is incompressible iff from the relation $T^{-1} E - E \in N$ it follows that $E - T^{-1} E \in N$ or else iff from the relation $E - T^{-1} E \in N$ it follows that $T^{-1} E - E \in N$

If T is an incompressible transformation, then $E - \bigcup_{n=1}^{\infty} T^{-n} E \in N$ (see [3])

Let (X, S, m) be a measure space, S a σ-algebra, T an invertible transformation X into X (i. e. T is one-to-one, onto and the transformations T, T^{-1} are measurable). If in addition T is non singular (i. e. $m(E) = 0$ iff $m(T^{-1} E) = 0$) then all the assumptions of our algebraic formulation are satisfied. We have $TE = \{Tx : x \in E\}$. Besides, if T is incompressible and invertible, then T is also non singular.

Theorem 1. *Let T be an incompressible σ-isomorphism of a Boolean σ algebra S onto itself, N be a σ-ideal in S, $TN = N$. For any $E \in S$ put $E_1 = E \cap T^{-1} E$*

$$E_n = E \cap (T^{-n}E - \bigcup_{i=1}^{n-1} T^{-i} E) \quad (n = 2, 3, \ldots).$$

$P = \{T^i E_j : 1 \leq i < j, j > 1\}$, $G = E \cup \bigcup \{L : L \in P\}$, $F = G'$.(1)

Then the set $R = \{E_i\} \cup P \cup \{F\}$ is a partition of the greatest element X of

(1) G' is the complement of the element G.

the Boolean σ-algebra S and the elements G, F are almost invariant under T (i.e. $T^{-1}G \triangle G \in N$, $T^{-1}F \triangle F \in N$).

Proof. Evidently $E \cap \bigcup_{n=1}^{\infty} T^{-n}E = \bigcup_{n=1}^{\infty} E_n$. Since T is incompressible, it is $E - \bigcup_{n=1}^{\infty} E_n \in N$. Notice that E_n are pairwise disjoint. Besides, for $i < j$ we have $T^iE_j \subset E'$, but $T^jE_j \subset E$. Hence $E \cap D = O$ for all $D \in P$. Let $T^iE_j \in P$, $T^kE_n \in P$ and $(i, j) \neq (k, n)$. If $i = k$, then $T^iE_j \cap T^kE_n = T^i(E_j \cap E_n) = T^iO = O$. If $i \neq k$, hence e. g. $i < k$, then $T^iE_j \cap T^kE_n = T^i(E_j \cap T^{k-i}E_n)$. But $k - i < n$, hence $T^{k-i}E_n \subset E'$, while $E_j \subset E$. Hence any two elements from the set P are disjoint.

It remains to be proved that the elements F and G are almost invariant. Clearly $G = C \cup \bigcup_{i=1}^{\infty} E_i \cup \bigcup_{i<j} T^iE_j$, where $C \in N$. Prove that $TG \subset D \cup E \cup \bigcup_{i<j} T^iE_j$, where $D \in N$. First of all $TC \in N$. Further

$$T \bigcup_{i=1}^{\infty} E_i = (T \bigcup_{i=1}^{\infty} E_i \cap E) \cup (T \bigcup_{i=1}^{\infty} E_i - E) \subset E \cup (T \bigcup_{i=1}^{\infty} E_i - E).$$

But $T \bigcup_{i=1}^{\infty} E_i - E = \bigcup_{k=2}^{\infty} (TE_k - E)$, since $TE_1 - E = T(E \cap T^{-1}E) - E = TE \cap E - E = O$. From this it follows

$$T \bigcup_{i=1}^{\infty} E_i - E \subset \bigcup_{k=2}^{\infty} TE_k \subset \bigcup_{i<j} T^iE_j.$$

Finally

$$T(\bigcup_{i<j} T^iE_j) \subset \bigcup_{i<j} T^iE_j \cup \bigcup_{j=2}^{\infty} T^jE_j \subset E \cup \bigcup_{i<j} T^iE_j.$$

We have proved that $TG \subset D \cup G$, where $D \in N$, hence $G - T^{-1}G \in N$. Since T is incompressible, we have $T^{-1}G - G \in N$, hence $G \triangle T^{-1}G \in N$ and G is almost invariant. Now it is obvious that F is almost invariant too.

Note 1. From Theorem 1 the recurrence-partition theorem from article [2] easily follows. That theorem can result from Theorem 1 by the special choice of S, T, N introduced above. In [2] it is assumed besides that X has a finite measure and T is measure preserving.

For an algebraic formulation of the next theorem we need to modify the notion of ergodic transformation. An isomorphism T of the algebra S onto S is ergodic iff from the relation $T^{-1}E \triangle E \in N$ it follows that $E \in N$ or $E' \in N$. We want to define another notion. An element $H \in S$ has a recurrent part iff there is $D \subset H$, $D \notin N$ and a positive integer k such that $T^kD - H \in N$.

Theorem 2. *Let under the assumptions of Theorem 1 be $F \in N$. A sufficient condition that T be ergodic is that E contains no element $H \subset E$, $E - H \notin N$ with a recurrent part.*(2)

Proof. If T is not ergodic, then there are $H_1, H_2 \in S$ such that $G = H_1 \cup \cup H_2$, H_1, H_2 are almost invariant, $H_1 \cap H_2 \in N$, $H_1 \notin N$, $H_2 \notin N$. If $H_1 \cap \cap E \in N$, then $T^i E_j = N_1 \cup N_2$, where $N_1 \in N$, $N_2 \subset H_2$. Then also $G = N_1 \cup N_2$, where $N_1 \in N$, $N_2 \subset H_2$, but it is in contradiction to the assumption. Hence $H_1 \cap E \notin N$ and also $H_2 \cap E \notin N$.

Put $H = H_1 \cap E$. From the above $H \notin N$, $E - H \notin N$. Since $H = N_1 \cup \cup \bigcup_{n=1}^{\infty} (H \cap E_n)$, where $N_1 \in N$ and N is a σ-ideal, there is such an n that $H \cap E_n \notin N$. But then H has a recurrent part $D = H \cap E_n$, since $T^n(H \cap \cap E_n) \subset T^n E_n \subset E$.

Theorem 3. *Let (X, S, m) be a measure space with a completely finite measure, T be an incompressible and invertible transformation on X. Let $E \in S$. Denote by E_i the set of all $x \in E$ for which $T^i x \in E$, but $T^j x \notin E$ for $i > j$. Let $m(X - E \cup \cup \bigcup \{T^i E_j : i < j, j > 1\}) = 0$.*

A suffitient condition that T be ergodic is that E contains no proper subsets with recurrent parts (i. e. that there do not exist sets, $D, H \in S$, $D \subset H \subset E$, $m(E) > m(H) > 0$, $T^n D \subset E$ for some n).

Proof. S is a Boolean σ-algebra, T a σ-isomorphism. If we put $N = \{E : : m(E) = 0\}$, then all assumptions of Theorem 2 are satisfied.

Note 2. From Theorem 3 the ergodicity theorem from article [2] follows. In [2] it is supposed in addition that T is measure preserving. But we know an example of a space (X, S, m) and an incompressible and invertible transformation T such that there is no invariant measure equivalent to m.(3)

Theorem 3 can be formulated also in another way. A set B is called the least almost invariant set over E, if $B \supset E$, B is almost invariant and for any almost invariant set $C \supset E$ we have $B - C \in N$.

Theorem 4. *Let (X, S, m) be a measure space with a completely finite measure, T be an incompressible and invertible transformation on X. Let $E \in S$ be an arbitrary set and X be the least almost invariant set over E. If E contains no proper subsets with recurrent parts then T is ergodic.*

Proof. If X is the least almost invariant set over E, then, since $E \cup \cup \bigcup \{T^i E_j : i < j, j > 1\}$ is almost invariant, we have $m(X - E \cup \bigcup \{T^i E_j : : i < j, j > 1\}) = 0$, hence we can use Theorem 3.

(2) E is an arbitrary but fixed element.

(3) See e. g. [1], p. 116 of the Russian translation.

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