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NOTE ON THE THEORY OF T-PAIR OF MANIFOLDS IN THE PROJECTIVE SPACE P_n

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In paper [1] Mihailescu has discussed thoroughly the transversal map of two surfaces in P_3. In this paper we try to find some qualities of a transversal map between manifolds in the real projective space P_n

Definition 1. *Let V_k, V'_k be two k-dimensional differentiable manifolds in the projective space P_n. Let the map $f: V_k \to V'_k$ be a diffeomorphism. For any $L \in V_k$ let tangential spaces $T_L(V_k)$, $T_{f(L)}(V'_k)$ have the common $(k-1)$-dimensional lineal subspace β, which is not incident with the line $\{L, f(L)\}$.*

Let $H_1, H_2, \dots, H_{n+1}$ be points of a frame in P_n.

$$(1) \qquad \mathrm{d}H_i = \omega_i^j H_j, \quad i, j = 1, 2, \dots, n+1$$

are equations of the infenitesimal map of the frame. Pfaff forms ω_i^j suit structure equations of space P_n:

$$(2) \qquad \mathrm{d}\omega_i^j = \sum_{s=1}^{n+1} \omega_1^s \wedge \omega_s^j \quad i, j = 1, 2, \dots, n+1 .$$

Our following considerations will be local. We assume that local co-ordinates of points L and $f(L)$ of manifolds V_k and V'_k are equal. These co-ordinates will be called principal parameters and marked by $u_1, u_2, \dots, u_k$. Let us confine ourselves to the frame such that $H_1 = L \in V_k$, $H_2 = f(L)$ and $\{H_3, H_4, \dots, H_{k+2}\} = \beta$. Then the following relations

$$(3) \qquad \begin{aligned} \omega_1^{k+3} &= \omega_1^{k+4} = \dots = \omega_1^{n+1} = 0, \quad \omega_1^2 = 0 \\ \omega_2^{k+3} &= \omega_2^{k+4} = \dots = \omega_2^{n+1} = 0, \quad \omega_2^1 = 0 \end{aligned}$$

result from (1).
The forms $\omega_1^3, \omega_1^4, \dots, \omega_1^{k+2}$ are independent and principal. If ω_t^i are principal forms, then

$$(4) \qquad \omega_t^i = \sum_{s=3}^{k+2} a_{t,s}^i \, \omega_1^s .$$

For $t = 2$ we shall use: $a^i_{2,s} = a^i_s$. Let us differentiate externally the relations (4). We get for $t = 2$:

$$(4') \qquad \sum_{j=3}^{k+2} \omega_1^j \wedge \{a^i_j(\omega_1^1 - \omega_2^2) - da^i_j + \sum_{h=3}^{k+2} (a^h_j \omega^i_h - a^i_h \omega^h_j) = 0 .$$

Let us denote by τ_{H_1} (resp. τ_{H_2}) a set of tangents of the manifold V_k (resp. V'_k) at the point H_1 (resp. H_2).

The tangent $t \in \tau_{H_1}$ (resp. $\tilde{t} \in \tau_{H_2}$) is determined by

$$\omega_1^3 : \omega_1^4 : \ldots : \omega_1^{k+2} \text{ (resp. } \omega_2^3 : \omega_2^4 : \ldots : \omega_2^{k+2}) .$$

Diffeomorphism f induces a collineation $K : \tau_{H_1} \to \tau_{H_2}$. If we denote

$$K(\omega_1^3 : \omega_1^4 : \ldots : \omega_1^{k+2}) = \omega_2^3 : \omega_2^4 : \ldots : \omega_2^{k+2} ,$$

then equations (4) for $t = 2$, i. e.

$$\omega_2^i = \sum_{s=3}^{k+2} a^i_s \omega_1^s , \quad i = 3, 4, \ldots, k+2$$

and the relation (3) determine the collineation C.

The tangent $t = (\omega_1^3 : \omega_1^4 : \ldots : \omega_1^{k+2}) \in \tau_{H_1}$ (resp. $\tilde{t} = K(t)$) and the subspace β have the common point $X = h_3H_3 + h_4H_4 + \ldots + h_{k+2}H_{k+2}$ (resp. $X' = h'_3H_3 + h'_4H_4 + \ldots + h'_{k+2}H_{k+2}$).

The collineation K induces an autocollineation $C : \beta \to \beta$ so that $C(X) = X'$. That is why the autocolineation is determined by equations:

$$(5) \qquad h'_i = \sum_{s=3}^{k+2} a^i_s h_s \quad i = 3, 4, \ldots, k+2 .$$

The autocollineation C induces an autocollineation C' of hyperplanes in the subspace β.

The equations

$$(6) \qquad \begin{array}{l} (a^3_3 + \lambda)\, h_3 + a^3_4 h_4 + \ldots + a^3_{k+2} h_{k+2} = 0 , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^{k+2}_3 h_3 + a^{k+2}_4 h_4 + \ldots + (a^{k+2}_{k+2} + \lambda)\, h_{k+2} = 0 \end{array}$$

determine invariable points of the autocollineation C'.

System (6) has a solution if and only if

$$(7) \qquad \begin{vmatrix} a^3_3 + \lambda, & a^3_4, & \ldots, & a^3_{k+2} \\ \ldots & \ldots & \ldots & \ldots \\ a^{k+2}_3, & a^{k+2}_4, & \ldots, & a^{k+2}_{k+2} + \lambda \end{vmatrix} = 0 .$$

The lines $\{H_1, H_2\}$ determine a k-parametric system of lines in P_n, which we shall denote by G. Let us determine the foci of G sitting on the line $\{H_1, H_2\}$.

Definition 2. *Let r be a natural number. The point $M = h_1H_1 + h_2H_2$ will be called an ${}^rF$-focus of G if there is $(r - 1)$ — parametric system of developables $\sum\limits_M^i \subset G$ $(i = 1, 2, \ldots, r)$ so that the point M lies on the edge of regression of $\sum\limits_M^i$ for $i = 1, 2, \ldots, r$.*

Since

$dM = H_3(h_1\omega_1^3 + h_2\omega_2^3) + H_4(h_1\omega_1^4 + h_2\omega_2^4) + \ldots + H_{k+2}(h_1\omega_1^{k+2} + \omega_2^{k+2}h_2) + 0 \mod \{H_1, H_2\}$, the point M is a focus of G if the system

$$h_1\omega_1^i + h_2\omega_2^i = 0\,, \quad i = 3, 4, \ldots, k+2$$

determines a surface Σ_M.

Let us arrange this system by using relations (4) :

$$(8)\qquad \begin{array}{l} \omega_1^3(h_1 + h_2a_3^3) + \omega_1^4h_2a_4^3 + \ldots + \omega_1^{k+2}h_2a_{k+2}^3 = 0\,, \\ \omega_1^3a_3^4h_2 + \omega_1^4(h_1 + h_2a_4^4) + \ldots + \omega_1^{k+2}h_2a_{k+2}^4 = 0\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_1^3h_2a_3^{k+2} + \omega_1^4h_2a_4^{k+2} + \ldots, + \omega_1^{k+2}(h_1 + h_2a_{k+2}^{k+2}) = 0\,. \end{array}$$

The surface Σ_M is determined by this system (8) if and only if

$$(9)\qquad \begin{vmatrix} h_1 + h_2a_3^3,\ h_2a_4^3,\ \ldots,\ h_2a_{k+2}^3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ h_2a_3^{k+2},\ h_2a_4^{k+2},\ \ldots,\ h_1 + h_2a_{k+2}^{k+2} \end{vmatrix} = 0\,.$$

Definition 3. *The fundamental tangent $t \in \tau_{H_1}$ has this characteristic: Its common point with the subspace β is an invariable point of the autocolineation C The fundamental curve is an integral curve of distribution of fundamental tangents. The focal curve on the manifold V_k* (resp. V_k') *is a section $\Sigma_M \cap V_k$* (resp. $\Sigma_M \cap V_k'$) *where $\Sigma_M \subset G$ is a developable.*

From definition 3 it follows that the focal curves are determined by the system (8) if h_1, h_2 are solutions of equation (9). The following assertion results from (6) and (8):

Assertion. *Focal curves are fundamental curves, i. e. if the point H_1 moves along the fundamental curve, the lines $\{H_1, H_2\}$ create a developable.*

Equations (7) and (9) are identical. Each root of equation (7) determines one and one focus M of G on the line $\{H_1, H_2\}$ and also the set of invariable points (resp. invariable hyperplanes in β) of the autocollineation C (resp. C') which we shall denote $\varkappa(M)$ (resp. $\varkappa'(M)$).

From the assertion it follows. *If* $\dim \varkappa(M) = r - 1$ *so M is a ${}^rF$-focus* We can classify T-pairs of manifolds by types of the collineation C. We always will assume that we can confine ourselves to the frame such that the matrix of collineation C has the Jordan form.

Notation. The set o hyperplanes that are incident with the $(n \quad p - 1)$-dimensional linear subspace $\mathscr{L}$ in P_n will be called *p-bundle of hyperplanes* in P_n and the subspace $\mathscr{L}$ will be called the centre of the p-bundle.
Let the collineation C have this quality: the r-multiple real root λ_1 of equation (7) determines the focus M_1 so that $\dim \varkappa(M_1) = h + s \quad 1 \quad p$, where h is the number of independent points from $\varkappa(M_1)$ sitting in the centre $\mathscr{L}(M_1)$ of the bundle $\varkappa'(M_1)$ in β.
Let us confine ourselves to the frame such that the matrix of the collineation C has the Jordan form. Let the part of this matrix determined by the root λ_1 have the form:

a) The functions of the principal diagonal are equal, i. e.

$$\text{(A)} \quad a_3^3 \quad a_4^4 \quad \dots \quad a_{m_1}^{m} \quad \dots \quad a_{m_2}^{m_2} \quad \dots \quad a_{m_h}^{m_h} \quad \dots = a_{m_h+s}^{m_h+s} \qquad \lambda_1$$

b) For the following functions above the principal diagonal of this Jordan matrix we have:

$$\text{(B)} \quad a_{m_1+1}^{m_1} \quad a_{m_2+1}^{m_2} \quad \dots - a_{m_h+1}^{m_h} \quad a_{m_h+2}^{m_h+1} - a_{m_h+3}^{m_h+2} \quad \dots \quad a_{m_h+s}^{m_h+s\ 1} \qquad 0$$

and for the others

$$a_{i+1}^{i} \neq 0, \quad i \leq m_h - 1 \,.$$

Then

1. The points $H_3, H_{m_1\ 1}, H_{m_2+1}, \dots, H_{m_h\ 1+1}, H_{m_h+1}, H_{m_h+2}, \dots H_{m_h\ s}$ are invariable points of the collineation C and create base of the subspace $\varkappa(M_1)$.
2. The invariable by the root λ_1 determined, hyperplanes of the collineation C create the p-bundle $\varkappa'(M_1)$ with this base in β:

$$h_{m_1} \quad 0,\ h_{m_2} \quad 0, \dots, h_{m_h} \quad 0,\ h_{m_h+1} - 0, \dots h_{m_h+s} \quad 0 \,.$$

Thus

$$\mathscr{L}(M_1) \quad \{H_3, H_4, \dots, H_{m_1-1}, H_{m_1\ 1}, \dots, H_{m_h\ 1\ 1}, H_{m_h\ 1+1}, \dots, H_{m\ \ 1},$$
$$H_{r+3}, H_{r+4}, \dots, H_{k+2}\} \text{ is a centre of the } p\text{-bundle } \varkappa'(M_1)$$

Hence it follows that the independent invariale points of the collineation C $H_3, H_{m_1+1}, H_{m_2+1}, \dots, H_{m_h\ 1+1}$ sit in the centre $\mathscr{L}$ of the p-bundle $\varkappa'(M_1)$ We can write the equalities (4′) briefly:

$$(*) \qquad \omega_1^3 \wedge \Omega_3^i + \omega_1^4 \wedge \Omega_4^i + \dots + \omega_1^{r+2} \wedge \Omega_{r+2}^i + \dots + \omega_1^{k+2} \wedge \Omega_{k\ 2}^i \qquad 0$$

If the matrix of the collineation C has the Jordan form, the Pfaff forms Ω_k^i have these forms:

$$\Omega_i^i \quad a_i^i(\omega_1^1 \quad \omega_2^2) + da_i^i + a_i^{i\ 1}\omega_{i\ 1}^i \quad a_{i\ 1}^i\omega_i^{i\ 1}.$$

$$\Omega_{i\ 1}^i \quad a_{i\ 1}^i(\omega_1^1 \quad \omega_2^2) + da_{i\ 1}^i + \omega_{i+1}^i(a_{i\ 1}^{i\ 1} \quad a_i^i) \quad a_{i\ 1}^i(\omega_i^i \quad \omega^{i\ 1}_{1})$$

$$\Omega_j^i \quad a_j^{j\ 1}\omega_{j\ 1}^i + \omega_j^i(a_j^j \quad a_i^i) \quad a_{i\ 1}^i\omega_j^{i\ 1}. j \quad i, i + 1 .$$

Hence the following relations

Ω^i 0 for i $m_1, m_2, \ldots, m_h, m_h$ 1, .., m_h s and

$$j \quad 3, m_1 + 1, m_2 + 1, \ldots, m_h + 1, m_h + 2, \ldots, m_h + s,$$

$$\Omega_i^i \quad a_i^i(\omega_1^1 \quad \omega_2^2) + da_i^i \text{ for } i \quad m_h + 1, m_h + 2, \ldots, m_h \quad s$$

result form (A) and (B).

Now the equalities (*) have for i $m_h + 1$, $m_h + 2$, ..., m_h s this shape

$$\omega_1^4 \wedge \Omega_4^i \quad \ldots \quad \omega_1^{m_1} \wedge \Omega_{m_1}^i + \omega_1^{m_1+2} \wedge \Omega_{m_1\ 2}^i + \ldots \quad \omega_1^{m_2} \wedge \Omega_{m_2}^i$$
$$\omega_1^{m_2\ 2} \wedge \Omega_{m_2\ 2}^i + \ldots + \omega_1^{m_h} \wedge \Omega_{m_h}^i + \omega_1^{r+3} \wedge \Omega_{r\ 3}^i + \ldots \quad \omega_1^{k\ 2} \quad \Omega_{k\ 2}^i$$
$$+ \omega_i^i \wedge \Omega_i^i = 0 .$$

If we apply the Cartan theorem we get:

$$\Omega_i^i \quad a_i^i(\omega_1^1 \quad \omega_2^2) + da_i^i \quad 0 \bmod (\omega_1^4, \omega_1^5, \ldots, \omega_1^{m_1}, \omega_1^{m_1\ 2},$$
$$\ldots, \omega_1^{m_2}, \omega_1^{m_2\ 2}, \ \ldots, \omega_1^{m_h}, \omega_1^{r\ 3}, \ldots, \omega_1^{k\ 2}, \omega_i^i) .$$

Thus if $s > 2$, we get

$$a_i^i(\omega_1^1 \quad \omega_2^2) + da_i^i \quad 0 \bmod (\omega_1^4, \omega_1^5, \ldots, \omega_1^{m_1}, \omega_1^{m_1\ 2}, \ldots, \omega_1^{m_2},$$
$$\omega_1^{m_2\ 2}, \ldots, \omega_1^{m_h}, \omega_1^{r\ 3}, \ldots, \omega_1^{k\ 2}) .$$

The root λ_1 a_3^3 a_i^i (i 4, ..., $m_h + s$) determines the focus M_1 $a_3^3 H_1$ H_2.

$$\mathrm{d}M_1 \quad \omega_2^2 M_1 + [\mathrm{d}a_3^3 + a_3^3(\omega_1^1 \quad \omega_2^2)]H_1 \quad \omega_1^4 a_3^3 H_3 \quad \omega_1^5 a_5^4 H_4 \quad \ldots$$
$$\omega_1^{m_1} a_{m_1}^{m_1\ 1} H_{m_1\ 1} \quad \omega_1^{m_1\ 2} a_{m_1\ 2}^{m_1\ 1} H_{m_1\ 1} \quad \ldots \quad \omega_1^{m_2} a_{m_2}^{m_2\ 1} H_{m_2\ 1}$$
$$\omega_1^{m_2\ 2} a_{m_2\ 2}^{m_2\ 1} H_{m_2\ 1} \quad \omega_1^{m_h} a_{m_h}^{m_h\ 1} H_{m_h\ 1} + \omega_1^{r\ 3} B_{r\ 3} \quad \omega_1^{r\ 4} B_{r\ 4} \quad \ldots +$$
$$+ \omega_1^{k+2} B_{k\ 2}, \text{ where } B_{r\ 3}, B_{r\ 4}, \ldots, B_{k\ 2}$$

are independent points in the space $\{H_{r\ 3}, \ldots, H_{k\ 2}\}$.
From this consideration the following theorem results.

Theorem 1. *Let λ_1 be an r-multiple real root of equation* (7). *Let M_1 be a focus determined by the root λ_1. Let* dim $\varkappa(M_1)$ *be $h + s$ 1 p, where h is the number of independent points from $\varkappa(M_1)$ sitting in the centre $\mathscr{L}(M_1)$ of the p-bundle $\varkappa'(M_1)$. Then the focus M_1 moves on a j-dimensional manifold V_j, which has the contact of the* 1st *order with the linear subspace $\{H_1, H_2, \mathscr{L}(M_1)\}$. If $s > 2$, then j k h s. If s 2, then k $(h + s) \leq j \leq k$ 1 $(h$ $s)$.*

Note. If the collineation C is an identity, equation (7) has a k-multiple root, $p = k - 1$, $s = k$. Hence the map f is a centric projection of the manifold V_k on V'_k.

The T-pair (V_k, V'_k, f) determines some distributions on the manifold V_k, resp. V'_k. Every focus M of G determines the linear subspace $\varkappa(M)$ of invariable points of the collineation C. Let us denote by $^{M}\nabla(V_k)$, resp. $^{M}\nabla(V'_k)$ the following distributions on the manifold V_k, resp. V'_k:

$$^{M}\nabla(V_k) : H_1 \to \{H_1, \varkappa(M)\} ,$$
$$^{M}\nabla(V'_k) : H_2 \to \{H_2, \varkappa(M)\} .$$

Let the collineation C have the following quality:
Its Jordan matrix is diagonal, i. e. for any focus M of G the subspaces $\varkappa(M)$ and $\mathscr{L}(M)$ are not incident when $\mathscr{L}(M)$ is the common subspace of hyperplanes from $\varkappa'(M)$. Now let us confine ourselves to the frame such that the matrix of the collineation C has the Jordan form. Then $a_i^j = 0$ for $i \neq j$; $i, j = 3, 4, \dots, k + 2$. The r-multiple root λ_1 of equation (7) determines the focus M_1. Then $\varkappa(M_1)$ is an $(r - 1)$-dimensional subspace. We can assume that

$$\varkappa(M_1) = \{H_3, H_4, \dots, H_{r+2}\} .$$

The distribution

$$^{M_1}\nabla(V_k) : H_1 \to \{H_1, \varkappa(M_1)\} = \{H_1, H_3, \dots, H_{r+2}\}$$

is determined by the equations:

(D) $$\omega_1^q = 0, \quad q = r + 3, r + 4, \dots, k + 2 .$$

Let us denote by Ω the set of the quadratic external forms which we can write as follows:

$$\sum_{s=r+3}^{k+2} \omega_1^s \wedge \alpha_s .$$

Let us differentiate externally the forms on the left-hand side of equations (D):

$$d\omega_1^q = \sum_{s=3}^{r+2} \omega_1^s \wedge \omega_s^q + 0 \bmod \Omega, \text{ where } 0 \bmod \Omega \in \Omega.$$

As $a_j^i = 0$ for $i \neq j$ the equalities (4') have for $i = q = r + 3, r + 4, \dots, k + 2$ this shape:

(E) $$\sum_{s=3}^{r+2} \omega_1^s \wedge \omega_s^q (a_s^s - a_q^q) = 0 \bmod \Omega = 0 .$$

As λ_1 is an r-multiple root of (7) and $\varkappa(M_1) = \{H_3, \dots, H_{r+2}\}$ we have $a_3^3 = a_4^4 = \dots = a_{r+2}^{r+2} \neq a_q^q$.

And now we get from (E):

$$\sum_{s=3}^{r+2} \omega_1^s \wedge \omega_s^q = 0 \bmod \Omega .$$

Hence $d\omega_1^q = 0 \bmod \Omega$, $q = r+3, r+4, \ldots, k+2$. Thus the system (D) is integrable (the Frobenius theorem; see [2] p. 92). From this consideration the following theorem results.

Theorem 2. *Let the collineation C have the following quality: For any focus M of G the subspaces $\varkappa(M)$ and $\mathscr{L}(M)$ are not incident, where $\mathscr{L}(M)$ is the common subspace of all hyperplanes in β belonged to $\varkappa'(M)$. Then the distribution ${}^M\nabla(V_k)$ is integrable.*

Now we shall study the T-pair (V_{n-2}, V'_{n-2}, f) which we shall call the *T-pair of K-manifolds.* Let us confine ourselves to the frame such that

$$H_1 = L \in V_{n-2},\ H_n = f(L) \text{ and } \beta = \{H_2, H_3, \ldots, H_{n-1}\} .$$

Then $$\omega_1^n = 0,\ \omega_1^{n+1} = 0,\ \omega_n^1 = 0,\ \omega_n^{n+1} = 0 .$$

Let us differentiate these equations by the external way. We get:

$$\sum_{i=2}^{n-1} \omega_1^i \wedge \omega_i^p = 0, \quad p = n, n+1 ,$$

$$\sum_{i=2}^{n-1} \omega_n^i \wedge \omega_i^d = 0, \quad d = 1, n+1 .$$

If we apply the Cartan theorem, we get:

(10) $$\omega_i^n = \sum_{j=2}^{n-1} a_{i,j}^n \omega_1^j ,$$

$$\omega_i^{n+1} = \sum_{j=2}^{n-1} a_{i,j}^{n+1} \omega_1^j ,$$

(10) $$\omega_i^1 = \sum_{j=2}^{n-1} A_{i,j}^1 \omega_n^j ,$$

$$\omega_i^{n+1} = \sum_{j=2}^{n-1} A_{i,j}^{n+1} \omega_n^j , \quad i = 2, 3, \ldots, n-1 .$$

The lower indices of the coefficients in (10) are symmetric. As the forms $\omega_1^2, \omega_1^3, \ldots, \omega_1^{n-1}$ are independent, we can write:

(11) $$\omega_n^j = \sum_{k=2}^{n-1} a_{n,k}^j \omega_1^k , \quad j = 2, 3, \ldots, n-1 .$$

Let us substitute the relations (11) and the 2nd relations from (10) into the last relations from (10). We get

$$\sum_{j\ 2}^{n\ 1} a_{i,j}^{n+1}\omega_1^j - \sum_{j\ 2}^{n\ 1} (\sum_{k\ 2}^{n\ 1} A_{i,k}^{n+1} a_{n,j}^k)\omega_1^j .$$

Since our considerations are local, why we get by comparing:

$$a_{i,j}^{n+1} - \sum_{k\ 2}^{n\ 1} A_{i,k}^{n+1} a_{n,j}^k, \quad i, j \quad 2, 3, \ldots, n-1 . \tag{12}$$

$$\mathrm{d}H_1 \quad \omega_1^1 H_1 + \omega_1^2 H_2 + \ldots + \omega_1^{n\ 1} H_{n\ 1},$$

$$\mathrm{d}^2 H_1 \quad H_{n+1}\{\sum_{i-2}^{n\ 1} \omega_1^i \omega_i^{n+1}\} + 0 \bmod \{H_1, H_2, \ldots, H_n\} .$$

Hence the following equation

$$\sum_{i\ 2}^{n\ 1} \omega_1^i \omega_i^{n+1} = 0, \text{ or — after arrangement —}$$

$$\sum_{i\ 2}^{n\ 1} \omega_1^i \sum_{j\ 2}^{n-1} a_{i,j}^{n+1} \omega_1^j = 0 ,$$

is an equation of the curves on the manifold V_{n-2} which have the contact of the 2nd order with the hyperplane $\{T_{H_1}(V_{n-2}), H_n\}$. The tangents of these curves at H_1 create conic hypersurface in the space $T_{H_1}(V_{n\ 2})$. The cut of this conic hypersurface with the subspace β is the following hyperquadric in β:

$$\sum_{i\ 2}^{n\ 1} h_i \sum_{j\ 2}^{n\ 1} a_{i,j}^{n+1} h_j = 0 . \tag{13}$$

We get likewise:

$$\sum_{i\ 2}^{n\ 1} \omega_n^i \sum_{j\ 2}^{n-1} A_{i,j}^{n+1} \omega_n^j = 0 ,$$

an equation of the curves on the manifold $V'_{n\ 2}$, which have the contact of the 2nd order with the hyperplane $\{H_n, H_1, \beta\}$. The tangents at the H_n of these curves create a conic hypersurface in the space $T_{H_n}(V'_{n\ 2})$. The cut of this conic hypersurface with the subspace β is the following hyperquadric in β:

$$\sum_{i\ 2}^{n\ 1} h_i \sum_{j\ 2}^{n\ 1} A_{i,j}^{n+1} h_j = 0 . \tag{14}$$

It results from the relation (10) that the hyperquadric (14) is regular if and only if the hyperquadric (13) is regular, too.

Now we shall study a case of the regular hyperquadric (13). The collineation C is determined by the following equations:

$$h'_j \quad \sum_{k\ 2}^{n\ 1} a_{n,k}^j h_j, \quad j \quad 2, 3, \ldots, n- -1 . \tag{15}$$

Hyperquadrics (13) and (14) are identical if and only if such a ϱ exists tha

$$a^{n+1}_{i,j} = \varrho A^{n+1}_{i,j}, \quad i, j = 2, 3, \dots, n+1 .$$

It results from (12) that these relations are correct if and only if

$$a^{j}_{n,k} = 0 \text{ for } j \neq k \text{ and}$$

$$a^{2}_{n,2} = a^{3}_{n,3} = \dots = a^{n+1}_{n,n+1}, \text{ i. e.}$$

if and only if the collineation C is the identity.

Let the collineation C be not an identity. Then the hyperquadrics (13) and (14 determine a bundle of hyperquadrics in β. The name of this bundle will be the ,,K_f-*bundle*". Any hyperquadric from the K_f-bundle has the following equation

$$(16) \qquad \sum_{i=2}^{n+1} h_i \sum_{j=2}^{n+1} h_j (a^{n+1}_{i,j} - \lambda A^{n+1}_{i,j}) = 0 .$$

Let us confine ourselves to the case that H_2 is an invariable point of the auto collineation C. Then the following relations result from (15):

$$a^{j}_{n,2} = 0 , \quad j = 3, 4, \dots, n+1 .$$

If we substitute these relations into (12), we get

$$(17) \qquad a^{n+1}_{i,2} = A^{n+1}_{i,2} a^{2}_{n,2}, \quad i = 2, 3, \dots, n+1 .$$

Notation. The singular points of the K_f-bundle are singular points of some hyperquadratic from the K_f-bundle.

It results from (16) that singular points of the K_f-bundle are determined by the following system:

$$\sum_{j=2}^{n+1} h_j (a^{n+1}_{i,j} + \lambda A^{n+1}_{i,j}) = 0 , \quad i = 2, 3, \dots, n+1 .$$

Then it results from (17) that the point H_2 is a singular point of the K_f-bundle Let H_2 be a singular point of K_f-bundle. Then λ exists such that

$$a^{n+1}_{i,j} = \lambda A^{n+1}_{i,2}, \quad i = 2, 3, \dots, n+1 .$$

If we substract these equalities from (12) for $j = 2$, we get the following system:

$$(18) \qquad \begin{aligned} &A^{n+1}_{2,2} (a^{2}_{n,2} + \lambda) + A^{n+1}_{2,3} a^{3}_{n,2} + \dots + A^{n+1}_{2,n+1} a^{n+1}_{n,2} = 0 , \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &A^{n+1}_{n+1,2} (a^{2}_{n,2} + \lambda) + A^{n+1}_{n+1,3} a^{3}_{n,2} + \dots + A^{n+1}_{n+1,n+1} a^{n+1}_{n,2} = 0 . \end{aligned}$$

Since the hyperquadratic (12) is regular. det $(A^{n+1}_{i,j}) \neq 0$. Then the system (18 has only a zero solution, i. e.

$$a^2_{n,2} = \lambda\,, \quad a^k_{n,2} = 0\,, \quad k = 3, 4, \ldots, n-1\,.$$

Hence H_2 is an invariable point of the collineation C. From this consideration the following theorem results:

Theorem 3. *Any invariable point of the collineation C is a singular point of the K_f-bundle. If the hyperquadric* (13) *is regular, any singular point of the K_f bundle is an invariable point of the collineation C.*

Now let the hyperquadric (13) be not regular. Its singular points are determined by the system

$$(19) \qquad \sum_{j=2}^{n-1} a^{n-1}_{i,j} h_j = 0\,, \quad i = 2, 3, \ldots, n-1\,.$$

Let the rank of the system (19) be p $(0 < p < n-2)$. Then the singular points of the hyperquadric (13) create a $(n-3-p)$-dimensional subspace X. Let us confine ourselves to the frame such that

$$X = \{H_2, H_3, \ldots, H_{n-1-p}\}\,.$$

Then from (19) the following relations result:

$$a^{n-1}_{i,j} = 0\,, \quad j = 2, 3, \ldots, n-1-p\,; \; i = 2, 3, \ldots, n-1$$

Thus $\omega^{n-1}_j = 0$, $j = 2, 3, \ldots, n-1-p$. Then

$$A^{n-1}_{i,j} = 0\,, \quad j = 2, 3, \ldots, n-1-p\,; \; i = 2, 3, \ldots, n-1\,.$$

Hence the subspace X is a subspace of singular points of the hyperquadric (14), too. Equalities (12) have for $j = 2, 3, \ldots, n-1-p$; $i = n-p, n-p+1, \ldots, n-1$ the following forms:

$$\sum_{k=n-p}^{n-1} A^{n-1}_{i,k} a^k_{n,j} = 0\,.$$

There is for every fixed $j = 2, 3, \ldots, n-1-p$ an algebraic system for the unknowns $a^k_{n,j}$. The rank of this system is p. Thus

$$a_{n,j} = 0\,, \quad k = n-p, n-p+1, \ldots, n-1\,; \; j = 2, 3, \ldots, n-1-p$$

Hence already the following relation:

$$C(X) = X$$

results from (15). From this consideration the following theorem results:

Theorem 4. *If the hyperquadric* (13) *is not regular, every singular point* *of it is a singular point of the hyperquadric* (14) *and the subspace* $X \subset \beta$ *of singular points is invariant under the collineation* C.

Note: If relations $\omega_j^{n+1} = 0$, $j = 2, 3, \ldots, n - 1$ are equalities on some neighbourhood, the manifolds $V_{n\ 2}$, $V'_{n\ 2}$ lie in a hyperplane in P_n.

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