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CUBIC MOORE GRAPHS

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By a *tied graph of type* (d, k) we understand — in accordance with [1] — a regular graph with a (finite or infinite) degree d and with a finite diameter k, not containing any circuit of length $\leqslant 2k$. Finite tied graphs (i. e., tied graphs of finite degree — so-called *Moore graphs*) were studied in [1], [2], [3]. In the present paper — except in the last § 4 — we shall consider only tied graphs of type $(3, k)$, that is *cubic Moore graphs*. Obviously, there is no Moore graph of type $(3, 0)$ and there exists up to isomorphism exactly one Moore graph of type $(3, 1)$ (tetrahedron). It is known [2] that there exists — up to isomorphism — just one Moore graph of type $(3, 2)$ (the Petersen graph) and no Moore graph of type $(3, 3)$. In this paper we prove the non-existence of Moore graphs of type $(3, k)$, where $3 \leqslant k \leqslant 8$. (1) For $k \geqslant 9$ the question of the existence of Moore graphs of type $(3, k)$ remains open. In § 4 we give a survey of known results on the existence and the uniqueness of tied graphs of a given type.

§ 1. BASIC PROPERTIES OF CUBIC MOORE GRAPHS

Let G_k be a Moore graph of type $(3, k)$ where $k \geqslant 3$. Pick a vertex w of G_k. As G_k is a cubic graph, w is adjacent to three vertices a, b and c of G_k (Fig. 1). The distance of vertices x and y in G_k will be denoted by $r(x, y)$. Vertices x such that $r(x, w) = k$, will be called *w-vertices* of G_k, edges joining such vertices — *w-edges* of G_k. As $r(x, w) = k$, the vertex x is adjacent to a vertex y such that $r(y, w) = k - 1$. Considering the fact that G_k does not contain any circuit of length $\leqslant 2k$, the remaining two vertices, adjacent to x, are w-vertices. Therefore the w-vertices and the w-edges form a quadratic subgraph of G_k, the circuits of which it consists are called *w-circuits* of G_k. Evidently, G_k contains exactly 3.2^{k-1} w-vertices and the same number of w-edges. Further, G_k has

(1) This result was presented at the Colloquium on Graph Theory in Manebach (G.D.R.) in May 1967.

$$1 + \sum_{i \;\, 1}^{k \;\, 1} 3.2^i - 3.2^k - 2$$

vertices and

$$\tfrac{3}{2}\,(3.2^k - 2) \qquad 3(3.2^{k \;\, 1} \qquad 1)$$

edges. If we omit all w-edges from G_k, the graph $T(w)$ obtained in this way will be also connected (from every vertex there is a path to w in $T(w)$). As $T(w)$ has $3.2^k \quad 2$ vertices and $3(3.2^{k \;\, 1} \quad 1) \quad 3.2^{k \;\, 1} - 3.2^k - 3$ edges, $T(w)$ is a tree, namely a spanning tree of G_k. The symbol $r_w(x, y)$ denotes the distance of vertices x and y in $T(w)$ and the symbol $x, :, y$ denotes the unique path connecting x and y in $T(w)$. Obviously, $r_w(x, y) \geqslant r(x, y)$ and $r(w, x) \quad r_w(w, x)$.

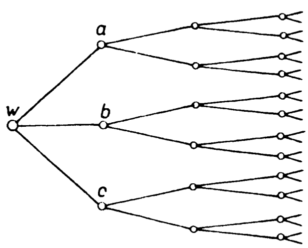

Fig. 1

Suppose $r(x, w) \geqslant 2$. Evidently, there exists a unique vertex y such that $r(x, w) \quad r(y, w)$ and $r_w(x, y) - 2$. This vertex will be denoted by $y \quad \alpha x$ Obviously, $\alpha^2 x \quad x$.

Let $i \in \{0, 1, 2, \ldots, k\}$ and $r(w, x) \geqslant i$. Then there is exactly one vertex y for which $r(w, y) \quad i$, $r(w, y) + r(y, x) \quad r(w, x)$. It will be denoted by $y \quad \beta_i x$ Instead of β_3 we shall write briefly β. Evidently, if $k \geqslant 4$ and x is a w-vertex of G_k, then $\beta\alpha x \quad \beta x$. G_k is a tied graph, therefore it contains no multiple edges. Thus we may denote the edge joining vertices x and y by (x, y) and the path with vertices $x_1, x_2, \ldots, x_n$ by $[x_1, x_2, \ldots, x_n]$.

Lemma 1. *Let* (x, y_1) *and* (x, y_2), *where* $y_1 \neq y_2$, *are* w-*edges of* G_k. *Then the following equality of sets holds:*

$$\{\beta_1 x, \beta_1 y_1, \beta_1 y_2\} \quad \{a, b, c\}\,.$$

Proof. Evidently, each of the elements $\beta_1 x$, $\beta_1 y_1$, $\beta_1 y_2$ belongs to the set $\{a, b, c\}$. If the assertion of Lemma 1 were false, two of elements $\beta_1 x$, $\beta_1 y_1$, $\beta_1 y_2$ would coincide. If $\beta_1 x \quad \beta_1 y_i$ $(i \in \{1, 2\})$, there exists in G_k a circuit $[x, :, y_i, x]$ of length $\leqslant 2k \quad 1$, which is in contradiction to the definition of a tied graph. If $\beta_1 y_1 \quad \beta_1 y_2$, there exists in G_k a circuit $[y_1, :, y_2, x, y_1]$ of length $\leqslant 2k$, a contradiction again. The lemma follows.

Now we can assign to every w-vertex x of G_k a w-vertex $y = \varphi x$ adjacent to x so that

$$\text{if } \beta_1 x = a, \text{ then } \beta_1 y = b,$$
$$\text{if } \beta_1 x = b, \text{ then } \beta_1 y = c,$$
$$\text{if } \beta_1 x = c, \text{ then } \beta_1 y = a.$$

Lemma 1 guarantees the existence and uniqueness of φx.

§ 2. AUXILIARY RESULTS

Henceforth we shall use notation introduced in § 1.

Lemma 2. *Let x be a w-vertex of G_k. We have:*

(a) $\beta_1\varphi^i x = \beta_1\varphi^j x$ *if and only if* $i \equiv j \pmod 3$.
(b) *The elements* βx, $\alpha\beta x$, $\beta\varphi^3 x$, $\alpha\beta\varphi^3 x$ *are mutually different.*
(c) $\beta\varphi^6 x = \alpha\beta x$, $\beta\varphi^9 x = \alpha\beta\varphi^3 x$, $\beta\varphi^{12} x = \beta x$.
(d) *The elements* βx, $\beta\varphi x$, $\beta\varphi^2 x$, $\beta\varphi^3 x$, $\ldots$, $\beta\varphi^{11} x$ *are mutually different.*

Proof. (a) follows from Lemma 1.

(b) From the definition of α it follows that $\beta x \neq \alpha\beta x$, and $\beta\varphi^3 x \neq \alpha\beta\varphi^3 x$. If $\beta x = \beta\varphi^3 x$, then there exists a circuit $[x, \varphi x, \varphi^2 x, \varphi^3 x, :, x]$ in G_k of length $< 2k - 3$, which is in contradiction to the definition of a tied graph. If $\beta\varphi^3 x = \alpha\beta x$, we have a circuit $[x, \varphi x, \varphi^2 x, \varphi^3 x, :, x]$ of length $2k - 1$, a contradiction again. If $\beta x = \alpha\beta\varphi^3 x$, then $\alpha\beta x = \alpha^2\beta\varphi^3 x = \beta\varphi^3 x$, and we have the case treated above. If $\alpha\beta x = \alpha\beta\varphi^3 x$, then $\alpha^2\beta x = \alpha^2\beta\varphi^3 x$, i. e. $\beta x = \beta\varphi^3 x$, which is also impossible.

(c) According to (b) the elements βx, $\alpha\beta x$, $\beta\varphi^3 x$, $\alpha\beta\varphi^3 x$ are mutually different. But from Lemma 1 it follows that $\beta_1\beta x = \beta_1\beta\varphi^3 x = \beta_1\alpha\beta x = \beta_1\alpha\beta\varphi^3 x = \beta_1\beta\varphi^6 x$. Therefore $\beta\varphi^6 x \in \{\beta x, \beta\varphi^3 x, \alpha\beta x, \alpha\beta\varphi^3 x\}$. If $\beta\varphi^6 x = \beta x$, then a circuit $[x, \varphi x, \varphi^2 x, \varphi^3 x, \varphi^4 x, \varphi^5 x, \varphi^6 x, :, x]$ of length $\leqslant 2k$ would exist in G_k, which is a contradiction. If $\beta\varphi^6 x = \beta\varphi^3 x$, for $y = \varphi^3 x$ we should have $\beta\varphi^3 y = \beta y$, which contradicts (b). If $\beta\varphi^6 x = \alpha\beta\varphi^3 x$, then analogously we have $\beta\varphi^3 y = \alpha\beta y$, again in contradiction to (b). Therefore $\beta\varphi^6 x = \alpha\beta x$. Using this relation we obtain $\beta\varphi^9 x = \beta\varphi^6(\varphi^3 x) = \beta\varphi^6 y = \alpha\beta y = \alpha\beta\varphi^3 x$. Further, $\beta\varphi^{12} x = \beta\varphi^6(\varphi^6 x) = \alpha\beta(\varphi^6 x) = \alpha^2\beta x = \beta x$.

(d) Let $\beta\varphi^i x = \beta\varphi^j x$, $i, j \in \{0, 1, 2, \ldots, 11\}$, $i < j$. Evidently, $\beta_1\varphi^i x = \beta_1\varphi^j x$; according to (a), we have $i \equiv j \pmod 3$, i. e. we can write $j = i + 3t$, $t \in \{1, 2, 3\}$. Put $y = \varphi^i x$. We have: $\beta y = \beta\varphi^i x = \beta\varphi^j x = \beta\varphi^{i+3t} x = \beta\varphi^{3t} y$. But from (b) and (c) it follows that $\beta y \neq \beta\varphi^{3t} y$, which is impossible.

Lemma 3. *The length of every w-circuit of G_k is a multiple of 12.*

Proof follows from (c) and (d) of Lemma 2.

Lemma 4. *Let M be a set of w-vertices of G_k, $k \geqslant 5$. If M has more than 2^{k-5} elements and for every $y_1, y_2 \in M$ we have $\beta y_1 = \beta y_2$, then there exist $x_1, x_2 \in M$, $x_1 \neq x_2$ such that $r_w(x_1, x_2) \leqslant 4$.*

Proof. Form the set $N = \{\beta_{k-2} x\}_{x \in M}$. The set N evidently cannot have more than 2^{k-5} elements; therefore for some $x_1, x_2 \in M$, $x_1 \neq x_2$ we have $\beta_{k-2} x_1 = \beta_{k-2} x_2$, i. e. $r_w(x_1, x_2) \leqslant 4$.

Lemma 5. *Let x and y be w-vertices of G_k. If $\beta x = \beta y$, then $r_u(x, y) \leqslant 2k - 6$. If $\beta x = \alpha\beta y$, then $r_u(x, y) = 2k - 4$.*

Proof. The path $[x, :, y]$ has evidently the length $\leqslant 2k - 6$ in the first case and the length $2k - 4$ in the second case.

Lemma 6. *If $x \neq y$ are such w-vertices of G_k that $\beta x = \beta y$ and $\beta\varphi x = \beta\varphi y$, then $r_w(x, y) > 6$.*

Proof. If the assertion of the lemma were not true, then $r_w(x, y) < 4$. By Lemma 5 we have $r_w(\varphi x, \varphi y) \leqslant 2k - 6$. But then $[x, \varphi x, :, \varphi y, y, :, x]$ would be a circuit of length $\leqslant 2k$, which is impossible.

Lemma 7. *Let x be a w-vertex of G_k, $k \geqslant 4$. Then we have.*

(1) $$\beta\varphi^{-2}\alpha x = \alpha\beta\varphi x,$$
(2) $$\beta\varphi^{-1}\alpha x = \alpha\beta\varphi^2 x,$$
(3) $$\beta\varphi\alpha x = \alpha\beta\varphi^{-2} x,$$
(4) $$\beta\varphi^2\alpha x = \alpha\beta\varphi^{-1} x.$$

Proof. First we prove (3). As $\beta x = \beta\alpha x$, consequently $\beta_1 x = \beta_1\alpha x$, and also $\beta_1\varphi x = \beta_1\varphi\alpha x$. According to (d) of Lemma 2 the elements $\beta(\varphi\alpha x)$, $\beta\varphi^3(\varphi\alpha x)$, $\beta\varphi^6(\varphi\alpha x)$, $\beta\varphi^9(\varphi\alpha x)$ are mutually different. By (a) of Lemma 2 we have $\beta_1(\varphi\alpha x) = \beta_1\varphi^3(\varphi\alpha x) = \beta_1\varphi^6(\varphi\alpha x) = \beta_1\varphi^9(\varphi\alpha x)$. Since $\beta_1(\varphi\alpha x) = \beta_1(\varphi x)$, the element $\beta(\varphi\alpha x)$ equals one of the elements $\beta(\varphi x)$, $\beta\varphi^3(\varphi x) = \beta\varphi^6\varphi^{-2}x$, $\beta\varphi^6(\varphi x)$, $\beta\varphi^9(\varphi x) = \beta\varphi^{12}\varphi^{-2}x$, hence — with respect to (c) of Lemma 2 — $\beta(\varphi\alpha x)$ is equal to some of the elements $\beta\varphi x$, $\alpha\beta\varphi^{-2}x$, $\alpha\beta\varphi x$, $\beta\varphi^{-2}x$.

If $\beta\varphi\alpha x = \beta\varphi x$, then the circuit $[\varphi x, x, :, \alpha x, \varphi\alpha x, :, \varphi x]$ has the length $\leqslant 2k - 2$, because $r_u(x, \alpha x) = 2$ and according to Lemma 5 $r_w(\varphi\alpha x, \varphi x) \leqslant 2k - 6$. If $\beta\varphi\alpha x = \alpha\beta\varphi x$, the circuit $[\varphi x, x, :, \alpha x, \varphi\alpha x, :, \varphi x]$ has the length $2k$, for Lemma 5 yields $r_w(\varphi\alpha x, \varphi x) = 2k - 4$. If $\beta\varphi\alpha x = \beta\varphi^{-2}x$, the circuit $[\varphi^{-2}x, \varphi^{-1}x, x, :, \alpha x, \varphi\alpha x, :, \varphi^{-2}x]$ has the length $\leqslant 2k - 1$, because Lemma 5 implies $r_w(\varphi^{-2}x, \varphi\alpha x) < 2k - 6$. Therefore only the last possibility, i. e. (3), can be valid.

The proof of (2) is „dual" to that of (3): it is sufficient to replace φ^2, φ, φ^{-1} and φ^{-2} by φ^{-2}, φ^{-1}, φ and φ^2, respectively.

If in (3) we replace x by αx, we obtain

$$\beta\varphi\alpha^2 x = \alpha\beta\varphi^{-2}\alpha x,$$

whence, as α^2 is an identical mapping, it follows that

$$\beta\varphi^{-2}\alpha x = \alpha^2\beta\varphi^{-2}\alpha x = \alpha\beta\varphi\alpha^2 x = \alpha\beta\varphi x,$$

that is, the relation (1).

The proof of (4) is „dual“ to that of (1).

Lemma 8. *Let x be a w-vertex of G_k, where $k \geqslant 4$. Then we have*:

$$\begin{aligned}
\beta\varphi^4\alpha x &= \beta\varphi x,\\
\beta\varphi^5\alpha x &= \beta\varphi^2 x,\\
\beta\varphi^6\alpha x &= \alpha\beta x,\\
\beta\varphi^7\alpha x &= \beta\varphi^{-2} x,\\
\beta\varphi^8\alpha x &= \beta\varphi^{-1} x,\\
\beta\varphi^{10}\alpha x &= \alpha\beta\varphi x,\\
\beta\varphi^{11}\alpha x &= \alpha\beta\varphi^2 x,\\
\beta\varphi^{12}\alpha x &= \beta x,\\
\beta\varphi^{13}\alpha x &= \alpha\beta\varphi^{-2} x.
\end{aligned}$$

The proof follows from (c) of Lemma 2 and Lemma 7, for instance:

$$\beta\varphi^4\alpha x = \beta\varphi^6(\varphi^{-2}\alpha x) = \alpha\beta(\varphi^{-2}\alpha x) = \alpha(\beta\varphi^{-2}\alpha x) = \alpha(\alpha\beta\varphi x) = \beta\varphi x,$$

$$\beta\varphi^5\alpha x = \beta\varphi^6(\varphi^{-1}\alpha x) = \alpha\beta(\varphi^{-1}\alpha x) = \alpha(\beta\varphi^{-1}\alpha x) = \beta\varphi^2 x,$$

$$\beta\varphi^6(\alpha x) = \alpha\beta(\alpha x) = \alpha\beta x, \text{ etc.}$$

§ 3. MAIN RESULTS

Lemma 9. *There is no Moore graph of type* (3, 3). (2)

Proof. Let G_3 be a Moore graph of type (3, 3). Then for any w-vertex x of G_3 we have $\beta x = x$. (c) of Lemma 2 yields $\alpha x = \alpha\beta x = \beta\varphi^6 x = \varphi^6 x$, $\alpha\varphi x = \alpha\beta(\varphi x) = \beta\varphi^6(\varphi x) = \varphi^7 x$. Therefore G_3 contains a hexagon $[x, \varphi x, :, \varphi^7 x, \varphi^6 x, :, x]$ which contradicts the definition of a Moore graph.

Lemma 10. *There is no Moore graph of type* (3, 4).

Proof. Let G_4 be a Moore graph of type (3, 4). Let x be a w-vertex in G_4. Evidently G_4 has just 24 w-vertices, so that, according to Lemma 6, in G_4 there is either one single w-circuit with 24 vertices or two w-circuits, each with 12 vertices. In the first case G_4 contains a hexagon $[x, \varphi x, :, \varphi^{13}x, \varphi^{12}x, :, x]$, and we have a contradiction. In the second case from (c) of Lemma 2 and Lemma 7 it follows that

(2) This result follows also from [2].

$$\beta\varphi^8 x = \beta\varphi^6(\varphi^2 x) = \alpha\beta\varphi^2 x = \beta\varphi^{-1}\alpha x\,,$$
$$\beta\varphi^7 x = \beta\varphi^6(\varphi x) = \alpha\beta\varphi x = \beta\varphi^{-2}\alpha x\,,$$

therefore G_4 contains a hexagon $[\varphi^7 x, \varphi^8 x, :, \varphi^{-1}\alpha x, \varphi^{-2}\alpha x, :, \varphi^7 x]$, thus we have arrived at a contradiction again.

Lemma 11. *The length of any w-circuit in G_k ($k \geqslant 5$) is at most 3.2^{k-5}.*

Proof. Let C be a w-circuit in G_k of the length $12s$ (see Lemma 3). Pick a vertex v of C. Denote $\beta\varphi^2 v = d$, $\beta\varphi^6 v = e$. Let Z be the set of all vertices of C of the form $\varphi^{12n} v$, where $n = 0, 1, 2, \ldots, s-1$. Let $z \in Z$. From (c) of Lemma 2 it easily follows that $\beta\varphi^2 z = d$, $\alpha\beta z = e$.

Define the functions $\delta_1, \delta_2, \delta_3, \delta_4$ thus (x runs through the set of all w-vertices):

$$\begin{aligned}
\delta_1(x) &= \varphi^5\alpha x,\\
\delta_2(x) &= \varphi\alpha\varphi\alpha x,\\
\delta_3(x) &= \alpha\varphi^2 x,\\
\delta_4(x) &= \varphi^{10}\alpha\varphi^5\alpha\varphi^2 x.
\end{aligned}$$

Let us prove that $\beta\delta_i(z) = d$, $\beta\varphi\delta_i(z) = e$ for $i = 1, 2, 3$ and 4. By systematic using of (c) of Lemma 2 and of Lemmas 7 and 8 we obtain:

$$\begin{aligned}
\beta\delta_1(z) &= \beta\varphi^5\alpha z = \beta\varphi^2 z = d,\\
\beta\delta_2(z) &= \beta(\varphi\alpha\varphi\alpha z) = \beta\varphi\alpha(\varphi\alpha z) = \alpha\beta\varphi^{-2}(\varphi\alpha z) = \alpha(\beta\varphi^{-1}\alpha z) = \alpha(\alpha\beta\varphi^2 z) =\\
&\quad \beta\varphi^2 z = d,\\
\beta\delta_3(z) &= \beta\alpha(\varphi^2 z) = \beta\varphi^2 z = d,\\
\beta\delta_4(z) &= \beta\varphi^{10}\alpha(\varphi^4\alpha\varphi^2 z) = \alpha\beta\varphi(\varphi^5\alpha\varphi^2 z) = \alpha(\beta\varphi^6\alpha(\varphi^2 z)) = \alpha(\alpha\beta(\varphi^2 z)) =\\
&\quad \beta\varphi^2 z = d,\\
\beta\varphi\delta_1(z) &= \beta\varphi^6\alpha z = \alpha\beta z = e,\\
\beta\varphi\delta_2(z) &= \beta\varphi^2\alpha(\varphi\alpha z) = \alpha\beta\varphi^{-1}(\varphi\alpha z) = \alpha\beta(\alpha z) = \alpha\beta z = e,\\
\beta\varphi\delta_3(z) &= \beta\varphi\alpha(\varphi^2 z) = \alpha\beta\varphi^{-2}(\varphi^2 z) = \alpha\beta z = e,\\
\beta\varphi\delta_4(z) &= \beta\varphi^{11}\alpha(\varphi^5\alpha\varphi^2 z) = \alpha\beta\varphi^2(\varphi^5\alpha\varphi^2 z) = \alpha\beta\varphi^7\alpha(\varphi^2 z) = \alpha\beta\varphi^{-2}(\varphi^2 z) =\\
&\quad \alpha\beta z = e.
\end{aligned}$$

Evidently, for every $z \in Z$ and $i \in \{1, 2, 3, 4\}$ the edge $[\delta_i(z), \varphi\delta_i(z)]$ is a w-edge of G_k. We shall prove that all such edges are mutually different. Suppose that $[\delta_{i_1}(z_1), \varphi\delta_{i_1}(z_1)] = [\delta_{i_2}(z_2), \varphi\delta_{i_2}(z_2)]$, where $i_1, i_2 \in \{1, 2, 3, 4\} : z_1, z_2 \in Z$. There are two possibilities:

I. $\delta_{i_1}(z_1) = \varphi\delta_{i_2}(z_2)$. But then we have $\beta\varphi^2 v = d = \beta\delta_{i_1}(z_1) = \beta\varphi\delta_{i_2}(z_2) = e = \beta\varphi^6 v$, which contradicts (d) of Lemma 2.

II. $\delta_{i_1}(z_1) = \delta_{i_2}(z_2)$. We first prove that $i_1 = i_2$. By using (c) of Lemma 2, Lemma 7 and Lemma 8 we obtain for any w-vertex x

$$\beta\varphi^{-1}\delta_1(x) = \beta\varphi^4\alpha x = \beta\varphi x,$$

$\beta\varphi^{\ 1}\delta_2(x) \quad \beta\alpha\varphi\alpha x = \beta\varphi\alpha x - \alpha\beta(\varphi^{\ 2}x) \quad \beta\varphi^6(\varphi^{\ 2}x) \quad \beta\varphi^4x,$
$\beta\varphi^{\ 1}\delta_3(x) \quad \beta\varphi^{\ 1}\alpha\varphi^2x \quad \alpha\beta\varphi^4x \quad \beta\varphi^{10}x,$
$\beta\varphi^2\delta_1(x) \quad \beta\varphi^7\alpha x \quad \beta\varphi^{\ 2}x \quad \beta\varphi^{10}x,$
$\beta\varphi^2\delta_3(x) \quad \beta\varphi^2\alpha\varphi^2x - \alpha\beta\varphi x \quad \beta\varphi^7x,$
$\beta\varphi^2\delta_4(x) \quad \beta\varphi^{12}\alpha(\varphi^5\alpha\varphi^2x) \quad \beta\varphi^5\alpha(\varphi^2x) \quad \beta\varphi^2(\varphi^2\alpha) \quad \beta\varphi^4x.$

According to (d) of Lemma 2 the elements $\beta\varphi x$, $\beta\varphi^4x$, $\beta\varphi^7x$, $\beta\varphi^{10}x$ are mutually different. From the equality $\delta_{i_1}(z_1) \quad \delta_{i_2}(z_2)$ it follows that $\beta\varphi^{\ 1}\delta_{i_1}(z_1)$ $\beta\varphi^{\ 1}\delta_{i_2}(z_2)$ and $\beta\varphi^2\delta_{i_1}(z_1) \quad \beta\varphi^2\delta_{i_2}(z_2)$. Bdt this is possible only if $i_1 \quad i_2$ or if $\{i_1, i_2\} \quad \{2, 4\}$. First analyse the second possibility. Let, e. g., $i_1 - 2$, $i_2 \quad 4$ i. e., $\delta_2(z_1) - \delta_4(z_2)$. Put $y \quad \alpha\varphi\alpha z_1$. We have: $\beta y \quad \beta\alpha\varphi\alpha z_1 \quad \beta\varphi\alpha z_1 \quad \alpha\beta\varphi^{\ 2}z_1$ $\beta\varphi^4 z_1 - \beta\varphi^4 v$, $\beta\varphi^3 y \quad \beta\varphi^2(\varphi\alpha\varphi\alpha z_1) \quad \beta\varphi^2\delta_2(z_1) - \beta\varphi^2\delta_4(z_2) \quad \beta\varphi^4 z_2 - \beta\varphi^4 v$ Thus we obtain that $\beta y \quad \beta\varphi^3 y$, which contradicts (d) of Lemma 2. Therefore only the possibility $i_1 \quad i_2$ remains. Put $i \quad i_1 - i_2$ so that $\delta_i(z_1) \quad \delta(z_2$ α and φ are one-to-one functions. Consequently also every δ_i is a one to one function and from the equality $\delta_i(z_1) \quad \delta_i(z_2)$ it follows that $z_1 \quad z_2$.

Thus we proved that all edges of a form $[\delta_i(z), \varphi\delta_i(z)]$, where $i \in \{1, 2, 3, 4\}$ $z \in \{v, \varphi^{12}v, \varphi^{24}v, \ldots, \varphi^{12(s\ 1)}v\}$ are mutually different. Hence we have $4s$ such edges, and always $\beta\delta_i(z) \quad d$, $\beta\varphi\delta_i(z) \quad e$. According to Lemma 6 any two of the vertices $\delta_i(z)$ have their distance r_w at least 6. But from Lemma 4 it follows that we can have at most $2^{k\ 5}$ such vertices. Therefore $4s \leqslant 2^{k\ 5}$ i. e. the length of C is $12s \leqslant 3.2^{k\ 5}$.

Theorem. *There is no Moore graph of type* $(3, k)$, *where* $3 \leqslant k \leqslant 8$.

Proof. Let G_k be a Moore graph of type $(3, k)$, $3 \leqslant k \leqslant 8$. Lemmas 9 and 10 imply that $k \geqslant 5$. From Lemma 3 we know that the length of any w-circuit in G_k is a multiple of 12. According to Lemma 11 this is possible only if $k \quad 7$ But G_k contains no circuits of length $\leqslant 14$, especially no 12-gons. From Lemma 11 it follows that $k - 8$ and all w-circuits in G_k are 24-gons. Choose a w-circuit C, a vertex v of C and construct by the method from the proof of Lemma 11 (for $s \quad 2$) 8 w-edges of a form $(\delta_i(z), \varphi\delta_i(z))$, where $\beta\delta_i(z) \quad d$ $\beta\varphi\delta_i(z) \quad e$. Consider the 9th edge $(\varphi^{\ 1}\alpha\varphi^6v, \alpha\varphi^6v)$. By Lemma 2, (c), Lemma 7 and Lemma 8 it is easy to prove that $\beta\varphi^{\ 1}\alpha\varphi^6v - d$, $\beta\alpha\varphi^6v \quad e$, $\beta\varphi^{\ 2}\alpha\varphi^6v$ $\beta\varphi v$, $\beta\varphi\alpha\varphi^6v \quad \beta\varphi^{10}v$. From the proof of Lemma 11 it follows that if this edge equals one of the former 8 edges, we necessarily have $i \quad 1$, i. e. $\varphi^{\ 1}\alpha\varphi^6v$ $\delta_1(z)$. As C is a 24-gon, either $z - v$ or $z \quad \varphi^{12}v$. In the first case in G_k there exists a path $[v, \varphi v, \varphi^2v, \varphi^3v, \varphi^4v, \varphi^5v, \varphi^6v, :, \alpha\varphi^6v \quad \varphi^6\alpha v, \varphi^5\alpha v, \varphi^4\alpha v, \varphi^3\alpha v,$ $\varphi^2\alpha v, \varphi\alpha v, \alpha v, :, v]$; in the second case there is in G_k a path $[\varphi^6v, \varphi^7v, \varphi^8v, \varphi^9v,$ $\varphi^{10}v, \varphi^{11}v, \varphi^{12}v, :, \alpha\varphi^{12}v, \varphi\alpha\varphi^{12}v, \varphi^2\alpha\varphi^{12}v, \varphi^3\alpha\varphi^{12}v, \varphi^4\alpha\varphi^{12}v, \varphi^5\alpha\varphi^{12}v, \varphi^6\alpha\varphi^{12}v$ $\alpha\varphi^6v, :, \varphi^6v]$. Both these paths contain a circuit of length $\leqslant 16$, which is in G^8 impossible. Therefore in G^8 there exist 9 edges of type (δ, ε), where $\beta\delta \quad d$

$\beta\varepsilon \ \ \varepsilon, \varepsilon \ \ q\delta$. According to Lemma 4 at least two of the vertices of type δ say δ' and δ'' have the distance $r_w(\delta', \delta'') \leqslant 4$. But this contradicts Lemma 6. The theorem follows.

§ 4. A SURVEY OF TIED GRAPHS

Results of [1], [2] and our Theorem make it possible to summarize the known results on the existence and uniqueness of tied graphs of type (d, k) into Table 1.

Table 1

tied graphs of type (d, k)		diameter $k = 0$	$k = 1$	$k = 2$	$k = 3$	$4 < k < 8$	$k \geq 9$
	$d = 0$	$K_1 = R_0$		/	/		
	$d = 1$		K_2		/		
	$d = 2$	$C_1 = R_1$	$C_3 = K_3$	C_5	C_7	C_{2k+1}	C
	$d = 3$	/	K_4	P	/		/
degree	$d > 4$, even	$R_{\frac{1}{2}d}$	K_{d+1}		/	?	/
	$d \geq 5$, odd, $\neq 7, 57$		K_{d+1}		/	?	
	$d = 7$		K_8	HS	/	?	/
	$d = 57$		K_{58}	?		?	/
	$d = \aleph_0$	R_d	K_d	E	E	E	/

Here the symbol ? means that neither the existence nor the uniqueness of a tied graph of type (d, k) has been proved. The symbol / means that there is no tied graph of the corresponding type, the symbol E denotes that so far only the existence (but not the uniqueness) for a given type has been proved. In the remaining cases there exists (up to isomorphism) exactly one tied graph as indicated in the table, where K_n is the complete graph with n vertices, C_n is the circuit with n vertices, R_n is the graph consisting of one vertex and n loops, P is the Petersen graph and HS denotes the Moore graph of type (7, 2) with 50 vertices constructed by Hoffman and Singleton in [2]. The „non trivial" part of the table is strongly framed.

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