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Renáta Hrmová

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# RELATIVE IDEALS IN SEMIGROUPS

RENÁTA HRMOVÁ, Bratislava

In the papers [9] and [2] the notion of a left (right, two-sided) $B$-ideal of a semigroup has been introduced as follows: Let $S$ be a semigroup, $B \subset S$, $B \neq \emptyset$. A left $B$-ideal of $S$ is a non void set $A \subset S$ such that $BA \subset A$. Similarly one defines a right $B$-ideal and a two-sided $B$-ideal of $S$.

It turns out that it is possible to generalize the notion of a $B$-ideal of $S$. The generalization is given by introducing the notion of a $(B_1, B_2)$-ideal of a semigroup $S$, $B_1$, $B_2$ being subsets of $S$. Using this notion some results of [2] and [10] are generalized in this paper.

## 1

Let $S$ be a semigroup, $A_1$, $A_2$ subsets of $S$. We define:
If $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, then $A_1A_2 = \{a_1a_2 : a_1 \in A_1, a_2 \in A_2\}$.
If $A_1 = \emptyset$, then $A_1A_2 = A_2$. If $A_2 = \emptyset$, then $A_1A_2 = A_1$.

In the following $S$ will denote a semigroup.

**Definition 1,1.** *Let $B_1 \subset S$, $B_2 \subset S$. Let $I(B_1,B_2) = \{A \subset S$: $B_1A \subset A$, $AB_2 \subset A\}$ and $I = \{I(B_1, B_2) : B_1 \subset S, B_2 \subset S\}$. The elements $A \in I(B_1, B_2)$ will be called $(B_1, B_2)$-ideals of $S$. The elements $A \in \bigcup I$ will be called relative ideals of $S$. By a one-sided relative ideal we mean any $(B_1, B_2)$-ideal for which either $B_1 = \emptyset$ or $B_2 = \emptyset$. Any $(B_1, B_2)$-ideal of $S$ is said to be a two-sided relative ideal of $S$ if $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$.*

Our definition implies:

1) $I(\emptyset, \emptyset) = \{A : A \subset S\}$.
2) $\emptyset \in I(B_1, B_2)$ if and only if $B_1 = \emptyset$ and $B_2 = \emptyset$.
3) If $B_1 \subset B_1'$, $B_2 \subset B_2'$, then $I(B_1', B_2') \subset I(B_1, B_2)$.
4) $I(B_1, B_2) = I(B_1, \emptyset) \cap I(\emptyset, B_2)$.

Remark. The notion of a $(B_1, B_2)$-ideal is, evidently, not only a generalization of a left, right and two-sided ideal of $S$ but also a generalization of a left, right and two-sided $B$-ideal of $S$ defined in [2] and [9].

In [2] examples of $(B, \emptyset)$-ideals, $(\emptyset, B)$-ideals and $(B, B)$-ideals have been given. In the following we give some examples for the notion introduced above.

Example 1,1. Let $H_1$, $H_2$ be subsemigroups of $S$ and $B_1, B_2$ subsets of $S$ such that $B_1 \subset H_1$, $B_2 \subset H_2$. Then for every $a \in S$ we have $a \cup H_1 a \cup \cup aH_2 \cup H_1 a H_2 = A \in I(B_1, B_2)$, hence $A \in I(H_1, H_2)$.

Example 1,2. Let $G$ be a group, $H_1, H_2$ subgroups of $G$. Then for any left coset $H_1 a$ we have $H_1 a \in I(H_1, \emptyset)$, for any right coset $aH_2$ we have $aH_2 \in \in I(\emptyset, H_2)$, and for any double coset $H_1 a H_2$ we have $H_1 a H_2 \in I(H_1, H_2)$.

Example 1,3. Let $A \subset S$ be a biideal of $S$, i. e. a subsemigroup of $S$ such that $ASA \subset A$. Then $A \in I(AS, SA)$.

Example 1,4. Let $A \subset S$ be a $(m, n)$-ideal of $S$, i. e. a subsemigroup of $S$ such that $A^m S A^n \subset A$, for some integers $m > 1$, $n > 1$. Then $A \in I(A^m S A^{n-1}, A^{m-1} S A^n)$.

Clearly the following lemma holds:

**Lemma 1,1.** *Let* $B_{11} \subset S$, $B_{21} \subset S$, $B_{12} \subset S$, $B_{22} \subset S$, $B_{11} \cap B_{12} = B_1$, $B_{21} \cap B_{22} = B_2$, $A_1 \in I(B_{11}, B_{21})$, $A_2 \in I(B_{12}, B_{22})$, *Then*:

1) $A_1 \cup A_2 \in I(B_1, B_2)$.
2) *If* $A_1 \cap A_2 \neq \emptyset$, *then* $A_1 \cap A_2 \in I(B_1, B_2)$.
3) $A_1 A_2 \in I(B_{11}, B_{22})$.

The next two theorems show the importance of the set $\bigcup I_H$ where $I_H = \{I(H_1, H_2) : H_1, H_2$ are subsemigroups of $S\}$.

In the following we shall cosider the empty set $\emptyset$ as a subsemigroup of $S$. It is easy to prove

**Theorem 1,1.** $I(B_1, B_2) = I(H_1, H_2)$, *where* $H_1 = B_1 \cup B_1^2 \cup B_1^3 \cup \ldots.$, $H_2 = B_2 \cup B_2^2 \cup B_2^3 \cup \ldots..$

We shall need the following

**Definition 1,2.** *Let* $A \in I(B_1, B_2)$, *for a given* $B_1 \subset S$, $B_2 \subset S$. *A set* $\bar{B}_1 \supset B_1$ *will be called the first saturation set of* $A$ *if* $A \in I(\bar{B}_1, B_2)$ *and there is no subset* $B_1'$, $B_1' \supsetneqq \bar{B}_1$ *such that* $A \in I(B_1', B_2)$ *holds. Analogously the second saturation set* $\bar{B}_2$ *of* $A$ *is defined. If* $\bar{B}_1 = B_1$, $\bar{B}_2 = B_2$, *then* $A$ *will be called a saturated* $(B_1, B_2)$*-ideal.*

Evidently the couple $\bar{B}_1, \bar{B}_2$ is uniquely defined (for given $A$, $B_1$, $B_2$).

**Theorem 1,2.** *The saturation sets of any* $A \in I(B_1, B_2)$ *are subsemigroups of* $S$.

Proof. Since we consider the empty set as a subsemigroup of $S$, it is sufficient to prove it for non-empty saturation sets. Let, for instance, $\bar{B}_1 \neq \emptyset$. Let be $a \in \bar{B}_1$, $b \in \bar{B}_1$, i. e. $aA \subset A$, $bA \subset A$. Since $baA \subset bA \subset A$ and $abA \subset aA \subset \subset A$, we have $ab \in \bar{B}_1$, $ba \in \bar{B}_1$. Analogously for the second saturation set.

The following example shows that the saturation sets $\bar{B}_1$ or $\bar{B}_2$ of a $(B_1, B_2)$-ideal of $S$ can be empty.

Example 1,5. Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $c$ | $a$ |
| $d$ | $a$ | $b$ | $a$ | $d$ |

and $S' = \{a, b, c\}$ the semigroup with the multiplication table

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $a$ |
| $c$ | $a$ | $a$ | $c$ |

The set $\{b\} \subset S$ is a saturated $(B_1, B_2)$-ideal of $S$, $B_1 = \{d\}, B_2 = \emptyset$. (The set $\{b\} \subset S$ is a right antiideal of $S$ since $\{b\}S \cap \{b\} = \emptyset$.) The set $\{b\} \subset S'$ is a saturated $(\emptyset, \emptyset)$-ideal of $S'$.

Example 1,6. Any subgroup of a group $G$ is a saturated $(G, G)$-ideal of $G$.

Example 1,7. Let $S$ contain the unit element $e$ and $e \notin B_1$, $e \notin B_2$. Then no $(B_1, B_2)$-ideal of $S$ is saturated.

The following example shows that the subsemigroups $H_1, H_2$ of Theorem 1.1 need not be saturation sets of a $(B_1, B_2)$-ideal of $S$.

Example 1.8. Let $S$ be the multiplicative semigroup of all residue classes mod 12, which will be denoted by 0, 1, ..... 11. If we choose $B = \{2\}$, then $A = \{2, 4, 8\}$ is a $(B, B)$-ideal of $S$. Evidently the saturation sets of $A$ coincide, $\bar{B}_1 = \bar{B}_2 = \bar{B} = \{2, 4, 8, 1, 7, 10\}$. But $H_1 = H_2 = B \cup B^2 \cup \ldots = \{2,4,8\} \subsetneqq \bar{B}$.

This example shows also that the subsemigroups $H_1, H_2$ considered in Theorem 1.1 are in general only proper subsets of the intersection of the saturation sets of all $A \in I(B_1, B_2)$. In fact the intersection of the saturation sets of all $A \in I(\{2\}, \{2\})$ contains the element 1 while $H_1 = H_2 = \{2, 4, 8\}$.

It can be shown further by means of this example that the saturation sets of two $(B_1, B_2)$-ideals $A$ and $A'$ need not be the same. For instance, the sets $A = \{2, 4, 8\}$, $A' = \{7, 2, 4, 8, 10\}$ are $(\{2\}, \{2\})$-ideals of $S$ but the saturation sets $\bar{B}_A$ of $A$ and $\bar{B}_{A'}$ of $A'$ are distinct. In fact $7 \in \bar{B}_A$, but $7 \notin \bar{B}_{A'}$.

It is easily to see that the notion of a relative ideal of $S$ may be used to develop the theory in two ways:

1) Given a set $A \subset S$ to find $B_1, B_2$ such that $A \in I(B_1, B_2)$.

2) To study the elements of the set $I(B_1, B_2)$, for given $B_1, B_2 \subset S$ (satisfying eventually the required properties).

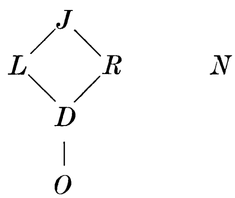
With regard to 1) it will be useful to introduce the following

**Definition 1.3.** *We say that a set $P \subset S$ can be properly idealized in $S$ if there exists a set $B \subset S$, $B \neq \emptyset$ such that $P \in I(B, \emptyset)$ or $P \in I(\emptyset, B)$.* Denote by $J$ the set of all subsets of $S$ which can be properly idealized. Denote further by

$$\begin{aligned}
D &= \{P \in J : P \in I(B_1, B_2) \text{ for some } B_1 \neq B_2, B_1 \neq \emptyset, B_2 \neq \emptyset\}.\\
O &= \{P \in J : P \in I(B_1, B_2) \text{ for some } B_1 = B_2 \neq \emptyset\}.\\
L &= \{P \in J : P \in I(B_1, \emptyset) \text{ for some } B_1 \neq \emptyset\}.\\
R &= \{P \in J : P \in I(\emptyset, B_2) \text{ for some } B_2 \neq \emptyset\}.\\
N &= \{P \subset S : P \notin J\}.
\end{aligned}$$

*We shall say that the subsets $P \subset S$, $P \in D$ or $P \in O$ can be two-sidedly idealized and the subsets $P \subset S$ such that $P \in L$ or $P \in R$ can be one-sidedly idealized.*

Evidently the set $V = \{J, D, O, L, R, N\}$ is partially ordered by set theoretical inclusions according to the following diagram:




Remark. The sets $N, J$ are non-empty because $\emptyset \in N$, $S \in J$. However, it follows from Example 1,5 that there exist semigroups containing proper subsets which cannot be properly idealized. The following example shows that there exists a subset $P \subset S$ such that $P \in D$ but $P \notin O$.

Example 1,9. Let $S = \{a, b, c, d, e, f, g, h\}$ be a semigroup with the following multiplication table:

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $a$ | $b$ | $a$ | $b$ | $e$ | $f$ | $f$ | $e$ |
| $b$ | $b$ | $a$ | $b$ | $a$ | $f$ | $e$ | $e$ | $f$ |
| $c$ | $c$ | $d$ | $c$ | $d$ | $h$ | $g$ | $g$ | $h$ |
| $d$ | $d$ | $c$ | $d$ | $c$ | $g$ | $h$ | $h$ | $g$ |
| $e$ | $a$ | $b$ | $b$ | $a$ | $e$ | $f$ | $e$ | $f$ |
| $f$ | $b$ | $a$ | $a$ | $b$ | $f$ | $e$ | $f$ | $e$ |
| $g$ | $d$ | $c$ | $c$ | $d$ | $g$ | $h$ | $g$ | $h$ |
| $h$ | $c$ | $d$ | $d$ | $c$ | $h$ | $g$ | $h$ | $g$ |

Choose $B_1 = \{a, c\}$, $B_2 = \{e, g\}$. Then the subset $P = \{f, g\}$ is a saturated $(B_1, B_2)$-ideal of $S$ and therefore $P \in D$ but $P \notin O$.

In the following sections we shall study relative ideals from the standpoint of [2]. With regard to Theorem 1,1 and 1,2 we shall study only the sets $I(B_1, B_2)$, where $B_1$, $B_2$ are subsemigroups of $S$. In the following we shall denote them by $H_1$ and $H_2$. The results obtained will generalize some known results concerning ideals of semigroups and some results of [2], [9], [10].

## 2

## Minimal relative ideals in semigroups

**Definition 2,1.** *Let $H_1$, $H_2$ be subsemigroups of $S$ (including the case of empty subsemigroups). We shall say that a set $A \subset S$, $A \in I(H_1, H_2)$ is a minimal $(H_1, H_2)$-ideal of $S$ if there is no $A' \subset S$, $A' \subsetneqq A$ such that $A' \in I(H_1, H_2)$. The set of all minimal $(H_1, H_2)$-ideals of $S$ will be denoted by $I_m(H_1, H_2)$.*

Example 2,1. If $S$ contains the zero element 0, then $\{0\} \in I_m(H_1, H_2)$ for each couple $H_1, H_2$.

Remark. If $S$ contains the zero element 0 and $H_1 \neq S$, $H_2 \neq S$, then the set $\{0\}$ is in general not contained in every $(H_1, H_2)$-ideal of $S$. But if at least one of the subsemigroups $H_1$, $H_2$ contains 0, then the set $\{0\}$ is contained in every $A \in I(H_1, H_2)$. To obtain non trivial results concerning minimal $(H_1, H_2)$-ideals of $S$ containing the zero element 0, it is necessary to assume that none of the subsemigroups $H_1$, $H_2$ contains 0.

From Lemma 1,1 and Definition 2,1 there follows

**Theorem 2,1.** *Let $A_1 \in I_m(H_1, H_2)$, $A_2 \in I_m(H_1, H_2)$, $A_1 \neq A_2$. Then $A_1 \cap A_2 = \emptyset$.*

**Theorem 2,2.** *Let $L \in I(H_1, \emptyset)$, $L \subset H_1$, $R \in I(\emptyset, H_2)$, $R \subset H_2$, and $A \in I_m(H_1, H_2)$. Then $A = LaR$, for every $a \in A$.*

Proof. Evidently $LaR \in I(H_1, H_2)$. Further for every $a \in A$ we have $LaR \subset LAR \subset H_1AH_2 \subset A$. Since $A \in I_m(H_1, H_2)$, we have $LaR = A$.

Notice that for $H_1 = \emptyset$ ($H_2 = \emptyset$; $H_1 = \emptyset$ and $H_2 = \emptyset$) we have $L = \emptyset$ ($R = \emptyset$; $L = \emptyset$ and $R = \emptyset$) and in the sense of our definition the set $LaR$ is of the form $aR$($La$; $\{a\}$).

**Corollary.** *If $A \in I_m(H_1, H_2)$, then $A = H_1aH_2$ for every $a \in A$.*

Remark. The supposition of Theorem 2,2 that $L$ and $R$ are subsets of $H_1$, $H_2$ (in the case $H_1 \neq \emptyset$, $H_2 \neq \emptyset$) is an essential one. By means of the one sided relative ideals not contained in $H_1$ and $H_2$ it is in general not possible to describe a minimal $(H_1, H_2)$-ideal even in the case when $L$ is a minimal $(H_1, \emptyset)$-ideal of $S$ and $R$ is a minimal $(\emptyset, H_2)$-ideal of $S$. This can be shown on Example

1,8 if we choose $H = \{1, 5, 7, 11\}$ and consider $A = H \in I_m(H, \emptyset)$ and we choose $L = \{2, 10\} \in I_m(H, \emptyset)$. Then we easily establish that $A = La$ does not hold for any $a \in S$.

**Lemma 2,1.** 1. *Let $L \in I_m(H_1, \emptyset)$. Then $L c \in I_m(H_1, \emptyset)$ for every $c \in S$.*
2. *Let $R \in I_m(\emptyset, H_2)$. Then $cR \in I_m(\emptyset, H_2)$ for every $c \in S$.*

Proof. 1. If $H_1 = \emptyset$, then either $L = \emptyset$ or $L$ is a one point set $L = \{a\}$. In both cases we have $Lc \in I_m(\emptyset, \emptyset)$ for every $c \in S$. If $H_1 \neq \emptyset$, by Theorem 2,2 $L \quad H_1a$ for every $a \in L$, hence $Lc = H_1ac \in I(H_1, \emptyset)$. Let now $B \subset Lc$, $B \in I(H_1, \emptyset)$ and $b \in B \subset Lc$, i.e. $b = a_1c$ for some $a_1 \in L$. Then $H_1a_1c = Lc = H_1b \subset B$ and therefore $Lc \in I_m(H_1, \emptyset)$.

The second case can be treated analogously.

**Corollary.** *Let $L$ be a minimal $(H_1, \emptyset)$-ideal of $S$ and $R$ a minimal $(\emptyset, H_2)$-ideal of $S$. Then the set $LaR$ is for every $a \in S$ an $(H_1, H_2)$-ideal of $S$, which is a set-theoretical union of some minimal $(H_1, \emptyset)$-ideals of $S$ and also a union of some minimal $(\emptyset, H_2)$-ideals of $S$.*

We have namely $LaR = \bigcup \{Lar : r \in R\} = \bigcup \{laR : l \in L\}$, and in accord with Lemma 2,1 the set $La$ and therefore also the set $Lar$ is a minimal $(H_1, \emptyset)$-ideal. Analogously the set $laR$ is a minimal $(\emptyset, H_2)$-ideal of $S$.

**Theorem 2,3.** *Let $L_0$ be a minimal $(H_1, \emptyset)$-ideal contained in $H_1$, and $R_0$ a minimal $(\emptyset, H_2)$-ideal of $S$ contained in $H_2$. Then the set $L_0cR_0$ is a minimal $(H_1, H_2)$-ideal of $S$ for every $c \in S$.*

Proof. If $H_1 = \emptyset$ or $H_2 = \emptyset$, then the set $L_0cR_0$ has one of the following forms: $cR_0$, $L_0c$, $\{c\}$. By Lemma 2,1 we have $cR_0 \in I_m(\emptyset, H_2)$, $L_0c \in I_m(H_1, \emptyset)$, $\{c\} \in I_m(\emptyset, \emptyset)$.

Let $H_1 \neq \emptyset$, $H_2 \neq \emptyset$. Suppose that for some $c \in S$ there exists a set $B \subset L_0cR_0$ such that $B \in I(H_1, H_2)$. Let $b \in B$. Then $b = l_0cr_0$, $l_0 \in L_0$, $r_0 \in R_0$. By Theorem 2,2, $L_0cR_0 = H_1l_0cr_0H_2 = H_1bH_2 \subset H_1BH_2 \subset B$. Hence $B = L_0cR_0$. This implies $L_0cR_0 \in I_m(H_1, H_2)$ for every $c \in S$.

**Corollary 1.** *Let $H_1$ contain at least one minimal $(H_1, \emptyset)$-ideal of $S$ and $H_2$ contain at least a minimal $(\emptyset, H_2)$-ideal of $S$. Let $L$ be a minimal $(H_1, \emptyset)$-ideal and $R$ a minimal $(\emptyset, H_2)$-ideal of $S$. Then the set $LcR$ is for every $c \in S$ a minimal $(H_1, H_2)$-ideal of $S$.*

With respect to the foregoing it is sufficient to prove it in the case $H_1 \neq \emptyset$, $H_2 \neq \emptyset$. Let by supposition $L_0 \subset H_1$, $L_0 \in I_m(H_1, \emptyset)$, $R_0 \subset H_2$, $R_0 \in I_m(\emptyset, H_2)$. By Theorem 2,2 we have $L = L_0a$, $R = bR_0$ for some $a \in S$, $b \in S$. Hence $LcR - L_0acbR_0 \in I_m(H_1, H_2)$.

**Corollary 2.** *Under the same assumptions as in the foregoing Corollary 1 we have for every $a \in S$, $b \in S$ either $LaR \cap LbR = \emptyset$ or $LaR = LbR$.*

By summarizing we get:

**Theorem 2,4.** *Let $H_1$ contain a minimal $(H_1, \emptyset)$-ideal $L_0$, and $H_2$ contain a minimal $(\emptyset, H_2)$-ideal $R_0$. Then:*

1) *Every minimal $(H_1, H_2)$-ideal $A$ of $S$ is of the form: $A = L_0 a R_0$ with some $a \in S$.*

2) *The set $L_0 a R_0$ is for every $a \in S$ a minimal $(H_1, H_2)$-ideal of $S$.*

Remark. This Theorem generalizes the wellknown theorems concerning semigroups containing minimal one-sided ideals.

In the following we generalize some results concerning minimal two-sided ideals of a semigroup.

**Lemma 2,2.** *Let $L_0 \in I_m(H_1, \emptyset)$, $L_0 \subset H_1$. Then $L_0 H_1 = \bigcup \{L_0 h : h \in H_1\} \in I_m(H_1, H_1)$. Analogously, if $R_0 \in I_m(\emptyset, H_2)$, $R_0 \subset H_2$, then $H_2 R_0 = \bigcup \{h R_0 : h \in H_2\} \in I_m(H_2, H_2)$.*

The proof follows from the known results in the theory of semigroups containing minimal one-sided ideals.

For brevity we denote in the following $L_0 H_1 = N_0^1$, $H_2 R_0 = N_0^2$. Evidently we have $N_0^1 \subset H_1$, $N_0^2 \subset H_2$.

**Theorem 2,5.** *Let $H_1$ contain at least one minimal $(H_1, \emptyset)$-ideal and $H_2$ contain at least one minimal $(\emptyset, H_2)$-ideal of $S$. Let $N_1$ be a minimal $(H_1, H_1)$-ideal and $N_2$ a minimal $(H_2, H_2)$-ideal of $S$. Then the set $N_1 a N_2$ is for every $a \in S$ an $(H_1, H_2)$-ideal, which is a set-theoretic union of some minimal $(H_1, H_2)$-ideals of $S$.*

Proof. Let $L_0 \in I_m(H_1, \emptyset)$, $L_0 \subset H_1$, $R_0 \in I_m(\emptyset, H_2)$, $R_0 \subset H_2$. By Theorem 2,2 for every $n_1 \in N_1$ we have $N_1 = N_0^1 n_1 N_0^1 = L_0 H_1 n_1 L_0 H_1 = L_0 B_1$, where $B_1 = H_1 n_1 L_0 H_1$. Analogously for every $n_2 \in N_2$ we have $N_2 = N_0^2 n_2 N_0^2 = H_2 R_0 n_2 H_2 R_0 = B_2 R_0$, where $B_2 = H_2 R_0 n_2 H_2$. Hence for every $a \in S$ we have $N_1 a N_2 = \bigcup \{L_0 c R_0 : c \in B_1 a B_2\}$.

Notice that if we replace in the case of $H_1 = \emptyset$ ($H_2 = \emptyset$; $H_1 = \emptyset$ and $H_2 = \emptyset$) $N_0^1$ by $\emptyset$ ($N_0^2$ by $\emptyset$; $N_0^1$ by $\emptyset$ and $N_0^2$ by $\emptyset$), then the corresponding sets in our Theorem are set-theoretic unions of minimal $(\emptyset, H_2)$-ideals ($(H_1, \emptyset)$-ideals; $(\emptyset, \emptyset)$-ideals).

Example 2,2. The following example shows that a set $N_1 a N_2$, $N_1 \in I_m(H_1, H_1)$, $N_2 \in I_m(H_2, H_2)$ need not be itself a minimal $(H_1, H_2)$-ideal of $S$ even in the case of $N_1 \subset H_1$, $N_2 \subset H_2$.

Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:

| | *a* | *b* | *c* | *d* | *e* |
|---|---|---|---|---|---|
| *a* | *a* | *a* | *a* | *a* | *a* |
| *b* | *a* | *a* | *b* | *a* | *d* |
| *c* | *a* | *a* | *c* | *a* | *e* |
| *d* | *a* | *b* | *b* | *d* | *d* |
| *e* | *a* | *c* | *c* | *e* | *e* |

Choose $H_1 = \{c, e\}$, $H_2 = \{d\}$. Then $N_0^1 = H_1$, $N_0^2 = H_2$. The set $N_0^1 d N_0^2$ is the union of two minimal $(H_1, H_2)$-ideals of $S$, namely $N_0^1 d N_0^2 = \{a\} \cup \{e\}$.

In contradistinction to Corollary 2, if $N_1 \in I_m(H_1, H_1)$, $N_2 \in I_m(H_2, H_2)$, then $N_1 a N_2 \cap N_1 b N_2 \neq \emptyset$ does not imply $N_1 a N_2 = N_1 b N_2$. This can be shown on Example 2,2 if we consider the sets $H_1 d H_2$ and $H_1 a H_2$.

## 3

## Relative socles in semigroups

In this section we again assume that $H_1$, $H_2$ are subsemigroups of $S$ (including the case of the empty subsemigroups).

**Definition 3,1.** *Suppose that $I_m(H_1, H_2)$ is non-void. The set-theoretic union $\bigcup \{A : A \in I_m(H_1, H_2)\}$ will be called the $(H_1, H_2)$-socle of $S$ and will be denoted by $\mathfrak{S}(H_1, H_2)$.*

Remark. The notion of the $(H_1, H_2)$-socle is a generalization of the left, right and two-sided $H$-socle introduced in [2].

**Theorem 3,1.** *Let $H_1$ contain at least one minimal $(H_1, \emptyset)$-ideal and $H_2$ contain at least one minimal $(\emptyset, H_2)$-ideal of $S$. Then*

$$\mathfrak{S}(H_1, H_2) = \mathfrak{S}(H_1, \emptyset) \cap \mathfrak{S}(\emptyset, H_2)$$

Proof. If $H_1 = \emptyset$ or $H_2 = \emptyset$, our statement trivially holds since $\mathfrak{S}(\emptyset, \emptyset) = S$.

Let $H_1 \neq \emptyset$, $H_2 \neq \emptyset$. By suppostion there exist $L_0 \subset H_1$, $L_0 \in I_m(H_1, \emptyset)$, $R_0 \subset H_2$, $R_0 \in I_m(\emptyset, H_2)$. By Theorem 2,4, $\mathfrak{S}(H_1, \emptyset) = L_0 S$, $\mathfrak{S}(\emptyset, H_2) = S R_0$, $\mathfrak{S}(H_1, H_2) = L_0 S R_0 \subset L_0 S = \mathfrak{S}(H_1, \emptyset)$. Analogously $\mathfrak{S}(H_1, H_2) \subset S R_0 =$, $= \mathfrak{S}(\emptyset, H_2)$, and therefore $\mathfrak{S}(H_1, H_2) \subset \mathfrak{S}(H_1, \emptyset) \cap \mathfrak{S}(\emptyset, H_2)$. Conversely let $a \in \mathfrak{S}(H_1, \emptyset) \cap \mathfrak{S}(\emptyset, H_2)$ for some $a \in S$. Then there exists some $L \in I_m(H_1, \emptyset)$ and $R \in I_m(\emptyset, H_2)$ such that $a \in L$ and $a \in R$. Moreover $a \in L_0 a$ since $L_0 a \subset \subset L_0 L \subset L$ implies $L = L_0 a$. Analogously $a \in a R_0$, hence $a = l_0 a = a r_0$ for some $l_0 \in L_0$ and some $r_0 \in R_0$. This implies $a = l_0 a r_0 \in L_0 S R_0 \subset \mathfrak{S}(H_1, H_2)$.

Example 3,1. The following example shows that a two-sided relative socle can be a proper subset of a one-sided relative socle even in the case of $H_1 = H_2$.

Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ | $d$ |
| $b$ | $b$ | $a$ | $d$ | $c$ |
| $c$ | $a$ | $b$ | $c$ | $d$ |
| $d$ | $b$ | $a$ | $d$ | $c$ |

If we choose $H_1 = H_2 = \{a, b\} = H$, then $\mathfrak{S}(H, \emptyset) = S$, while $\mathfrak{S}(H, H) = \{a, b\}$.

It is useful to notice the following. If $H_1 = H_2 = S$ and there exists $N_0 \subset S$, $N_0 \in I_m(S, S)$, then $\mathfrak{S}(H_1, H_2) = N_0$. It is known that $N_0$ exists if there exists at least one minimal $(S, \emptyset)$-ideal or one minimal $(\emptyset, S)$-ideal of $S$. For instance if there exists one $L \in I_m(S, \emptyset)$, then we have $LS = \mathfrak{S}(S, \emptyset) = N_0 = N_0SN_0 = \mathfrak{S}(S, S)$.

But in the case of $H_1 = H_2$, $H_1 \neq \emptyset$, $H_1 \neq S$, $H_2 \neq \emptyset$, $H_2 \neq S$, for describing $\mathfrak{S}(H, H)$ it is (in general) not sufficient to know a single relative one-sided minimal ideal of $S$. We have seen namely that in general we only have $\mathfrak{S}(H, H) \subset \mathfrak{S}(H, \emptyset)$, and not necessarily $\mathfrak{S}(H, H) = \mathfrak{S}(H, \emptyset)$.

In the case of $H_1 \neq H_2$, $H_1 \neq \emptyset$, $H_2 \neq \emptyset$, we may obtain an analogy with the case of the $(S, S)$-socle. In this case it is sufficient to suppose for describing $\mathfrak{S}(H_1, H_2)$ the existence of only one one-sided relative ideal in each of the semigroups $H_1$, $H_2$, namely the existence of a minimal $(H_1, \emptyset)$-ideal in $H_1$ and the existence of a minimal $(\emptyset, H_2)$-ideal in $H_2$.

In the following theorem we shall give the conditions under which the sets $\mathfrak{S}(H_1, \emptyset)$ and $\mathfrak{S}(\emptyset, H_2)$ coincide.

**Lemma 3,1.** *Let $H_1$ contain at least one minimal $(H_1, \emptyset)$-ideal and $H_2$ contain at least one minimal $(\emptyset, H_2)$-ideal of $S$. Then* $\mathfrak{S}(H_1, H_2) = N_0^1SN_0^2$, $N_0^1 \in I_m(H_1, H_1)$, $N_0^1 \subset H_1$, $N_0^2 \in I_m(H_2, H_2)$, $N_0^2 \subset H_2$.

Proof. Let $L_0 \subset H_1$, $L_0 \in I_m(H_1, \emptyset)$, $R_0 \subset H_2$, $R_0 \in I_m(\emptyset, H_2)$. Then we have $\mathfrak{S}(H_1, H_2) = L_0SR_0 \subset N_0^1SN_0^2$. Conversely by Theorem 2,5, $N_0^1SN_0^2 \subset \mathfrak{S}(H_1, H_2)$.

**Corollary.** *Under the suppositions of the foregoing Lemma the relative socles of a semigroup $S$ are subsemigroups of $S$.*

**Lemma 3,2.** *Under the same suppositions as in Lemma* 3,1 $\mathfrak{S}(H_1, \emptyset) = \mathfrak{S}(\emptyset, H_2)$ *if and only if for every $L \in I_m(H_1, \emptyset)$ and every $R \in I_m(\emptyset, H_2)$ we have $L \subset \mathfrak{S}(H_1, H_2)$ and $R \subset \mathfrak{S}(H_1, H_2)$.*

The proof follows from Theorem 3,1.

**Theorem 3,2.** *Under the suppositions of Lemma* 3,1 *suppose moreover $H_1 = H_2 = H$. Then $\mathfrak{S}(H_1, \emptyset) = \mathfrak{S}(\emptyset, H_2)$ if and only if $\mathfrak{S}(H_1, H_2)$ is an $(S, S)$-ideal of $S$.*

Proof. For $H = \emptyset$ the proof is trivial. Let $H \neq \emptyset$. If $\mathfrak{S}(H, \emptyset) = \mathfrak{S}(\emptyset, H)$, then by Theorem 3,1 we have $\mathfrak{S}(H, H) = \mathfrak{S}(H, \emptyset) = \mathfrak{S}(\emptyset, H) = L_0S = SR_0$ and so $\mathfrak{S}(H, H) \in I(S, S)$. Conversely, let $\mathfrak{S}(H, H) \in I(S, S)$. We shall prove that for every $L \in I_m(H, \emptyset)$ and for every $R \in I_m(\emptyset, H)$ we have $L \subset \mathfrak{S}(H, H)$ and $R \subset \mathfrak{S}(H, H)$. If $L \in I_m(H, \emptyset)$, then by Theorem 2,2 there exists an element $a \in S$ such that $L = L_0a$. Further $L_0a \subset N_0a$, $N_0 \in I_m(H, H)$, $N_0 \subset H$. Since for every $n \in N_0$ we have $N_0nN_0 = N_0$, we conclude $L_0a \subset N_0nN_0a \subset \subset N_0SN_0S \subset N_0SN_0 = \mathfrak{S}(H, H)$. Similarly we can show that for every $R \in I_m(\emptyset, H)$ we have $R \subset \mathfrak{S}(H, H)$. By Lemma 3,2 $\mathfrak{S}(H, \emptyset) = \mathfrak{S}(\emptyset, H)$.

Remark. It follows from the proof of Theorem 3,2 that the condition $\mathfrak{S}(H_1, H_2) \in I(S, S)$ is necessary for the validity of the relation $\mathfrak{S}(H_1, \emptyset) = \mathfrak{S}(\emptyset, H_2)$ even in the case when $H_1 \neq H_2$. However, if $\mathfrak{S}(H_1, H_2) \neq S$, $H_1 \neq H_2$, this condition is not sufficient. This can be shown on the following example:

Example 3,2. Let $S = \{a, b, c, d, e\}$ be a semigroup with the following multiplication table:

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $d$ | $d$ |
| $b$ | $a$ | $b$ | $c$ | $d$ | $d$ |
| $c$ | $a$ | $c$ | $b$ | $d$ | $d$ |
| $d$ | $d$ | $d$ | $d$ | $a$ | $a$ |
| $e$ | $d$ | $e$ | $e$ | $a$ | $a$ |

Choose $H_1 = \{b\}$, $H_2 = \{a, d\}$. Then $L_0 = \{b\}$, $\mathfrak{S}(H_1, \emptyset) = L_0S = \{a, b, c, d\}$, $R_0 = \{a, d\}$, $\mathfrak{S}(\emptyset, H_2) = SR_0 = \{a, d\}$, $\mathfrak{S}(H_1, H_2) = L_0SR_0 = \{a, d\}$. $\mathfrak{S}(H_1, H_2) \in \in I(S, S)$, but $\mathfrak{S}(H_1, \emptyset) \neq \mathfrak{S}(\emptyset, H_2)$.

**Theorem 3,3.** *Let $H_1$ contain at least one minimal $(H_1, \emptyset)$-ideal and a minimal $(\emptyset, H_1)$-ideal of $S$. Let $H_2$ contain at least one minimal $(H_2, \emptyset)$-ideal and a minimal $(\emptyset, H_2)$-ideal of $S$. Then*

$$\mathfrak{S}(H_1, H_2) \cap \mathfrak{S}(H_2, H_1) = \mathfrak{S}(H_1, H_1) \cap \mathfrak{S}(H_2, H_2).$$

The proof follows from Theorem 3,1.

Remark. The intersection of the relative socles in the foregoing formula can be the void set. This, e.g. is the case if we choose in Example 3,1, $H_1 = \{a, b\}$ and $H_2 = \{c\}$.

## 4

## Principal relative ideals of semigroups

In this section some notions and some results of [10] are generalized. Moreover the notion of the simplicity of a semigroup is generalized in various ways.

We assume again that $H_1$, $H_2$ are subsemigroups of $S$ (including the case of the empty subsemigroups).

**Definition 4,1.** *Let $a \in S$. The set $A = a \cup H_1 a \cup aH_2 \cup H_1 a H_2$ will be called the principal $(H_1, H_2)$-ideal of $S$ generated by the element $a$. It will be denoted by ${}_{H_1}(a)_{H_2}$.*

This definition evidently generalizes not only the notion of a principal left, right and two-sided ideal but also the notion of a principal $T$-ideal defined in [10].

**Theorem 4,1.** *Let $A$ be a $(H_1, H_2)$-ideal of $S$. Then $A = \bigcup \{{}_{H_1}(a)_{H_2} \colon a \in A\}$.*

Proof. If $A \in I(H_1, H_2)$, then for every $a \in A$ we have ${}_{H_1}(a)_{H_2} \subset A$. Conversely, $a \in {}_{H_1}(a)_{H_2}$ for every $a \in A$.

Using the notion of a principal $(H_1, H_2)$-ideal of $S$ we can generalize the notion of Green's relations.

**Definition 4,2.** *Let for $a \in S$, $b \in S$* be ${}_{H_1}(a)_{H_2} = {}_{H_1}(b)_{H_2}$. *Then we shall write $(a, b) \in {}_{H_1}\mathscr{J}_{H_2}$ and shall say that the elements $a$ and $b$ are ${}_{H_1}\mathscr{J}_{H_2}$-equivalent.*

Remark. The relation ${}_{H_1}\mathscr{J}_{H_2}$ is clearly an equivalence relation on $S$, and it is a generalization not only of Green's relations on $S$ but also of the relations introduced in [10].

We shall denote the classes corresponding to this equivalence relation by ${}_{H_1}F_{H_2}$.

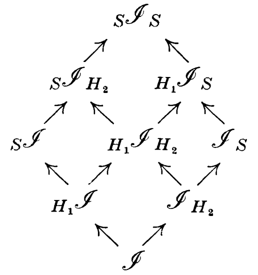
In the following (for typographical reasons) the relations ${}_{H_1}\mathscr{J}_{\emptyset}$, ${}_{\emptyset}\mathscr{J}_{H_2}$, ${}_{\emptyset}\mathscr{J}_{\emptyset}$ and the classes ${}_{H_1}F_{\emptyset}$, ${}_{\emptyset}F_{H_2}$, ${}_{\emptyset}F_{\emptyset}$ will be briefly denoted by ${}_{H_1}\mathscr{J}$, $\mathscr{J}_{H_2}$, $\mathscr{J}$, and ${}_{H_1}F$, $F_{H_2}$, $F$ respectively.

Recall that for $H_1 = \emptyset$ and $H_2 = \emptyset$, the relation ${}_{H_1}\mathscr{J}_{H_2} = \mathscr{J}$ is the equality relation on $S$ and the corresponding classes ${}_{H_1}F_{H_2} = F$ are one point sets.

From the preceding definition there follows

**Theorem 4,2.** *Let $T_1, H_1, T_2, H_2$ be subsemigroups of $S$ such that $H_1 \subset T_1$, $H_2 \subset T_2$. Then ${}_{H_1}\mathscr{J}_{H_2} \subset {}_{T_1}\mathscr{J}_{T_2}$.*

Remark. The known relations between one-sided and two-sided „classical" Green's relations follow from Theorem 4,2 if we take $H_1 = S$, $H_2 = \emptyset$, $T_1 = S$, $T_2 = S$ and $H_1 = \emptyset$, $H_2 = S$, $T_1 = S$, $T_2 = S$ respectively. Further if we replace (for typographical reasons) the symbol of inclusion $\subset$ by the symbol $\rightarrow$, we get from Theorem 4,2 the following diagram:



**Definition 4,3.** *Denote the class* ${}_{H_1}F_{H_2}$ *containing the element* $a$ *by* ${}_{H_1}F^a_{H_2}$. *We shall write* ${}_{H_1}F^a_{H_2} \leqq {}_{H_1}F^b_{H_2}$ *if and only if* ${}_{H_1}(a)_{H_2} \subset {}_{H_1}(b)_{H_2}$.

**Theorem 4,3.** *For each* ${}_{H_1}(a)_{H_2}$ *we have*

$$ {}_{H_1}(a)_{H_2} = \bigcup \{ {}_{H_1}F_{H_2} : {}_{H_1}F_{H_2} \leqq {}_{H_1}F^a_{H_2} \}. $$

Proof. If for some $x \in S$ and some class ${}_{H_1}F_{H_2} \leqq {}_{H_1}F^a_{H_2}$, $x \in {}_{H_1}F_{H_2}$ holds, then by the definition of our partial ordering $x \in {}_{H_1}(a)_{H_2}$. If for $x \in {}_{H_1}F^x_{H_2}$ we have $x \in {}_{H_1}(a)_{H_2}$, then ${}_{H_1}(x)_{H_2} \subset {}_{H_1}(a)_{H_2}$, hence ${}_{H_1}F^x_{H_2} \leqq {}_{H_1}F^a_{H_2}$.

Evidently the equivalence relation ${}_{H_1}\mathscr{I}$ is a right congruence and the equivalence relation $\mathscr{I}_{H_2}$ is a left congruence.

In the following we suppose the familiarity with the notion of the product of two relations.

Notation: The product of the relations ${}_{H_1}\mathscr{I}$ and $\mathscr{I}_{H_2}$ : ${}_{H_1}\mathscr{I} \,.\, \mathscr{I}_{H_2}$ will be denoted by ${}_{H_1}\mathscr{D}_{H_2}$.

Evidently:

For $H_1 = \emptyset$, $H_2 \neq \emptyset$ we have ${}_{H_1}\mathscr{D}_{H_2} = \mathscr{I}_{H_2}$.

For $H_1 \neq \emptyset$, $H_2 = \emptyset$ we have ${}_{H_1}\mathscr{D}_{H_2} = {}_{H_1}\mathscr{I}$.

For $H_1 = \emptyset$, $H_2 = \emptyset$ we have ${}_{H_1}\mathscr{D}_{H_2} = \mathscr{I}$ (the equality relation on $S$).

**Lemma 4,1.** ${}_{H_1}\mathscr{I} \,.\, \mathscr{I}_{H_2} = \mathscr{I}_{H_2} \,.\, {}_{H_1}\mathscr{I}$.

Proof. Since ${}_{H_1}\mathscr{I}$, $\mathscr{I}_{H_2}$ are symmetric relations, it is sufficient to prove that ${}_{H_1}\mathscr{I} \,.\, \mathscr{I}_{H_2} \subset \mathscr{I}_{H_2} \,.\, {}_{H_1}\mathscr{I}$.

Let $(a, b) \in {}_{H_1}\mathscr{I} \,.\, \mathscr{I}_{H_2}$. Then there exists $c \in S$ such that $(a, c) \in {}_{H_1}\mathscr{I}$, $(c, b) \in \mathscr{I}_{H_2}$.

The following cases are possible:

1) $a = b = c$. In this case evidently $(a, b) \in \mathscr{I}_{H_2} \,.\, {}_{H_1}\mathscr{I}$.
2) $b = c \neq a$. Since $(a, a) \in \mathscr{I}_{H_2}$, $(a, c) \in {}_{H_1}\mathscr{I}$, we have $(a, b) \in \mathscr{I}_{H_2} \,.\, {}_{H_1}\mathscr{I}$.
3) $a = b \neq c$. Then $(a, c) \in {}_{H_1}\mathscr{I}$, $(c, a) \in \mathscr{I}_{H_2}$ implies $(a, a) = (a, b) \in \mathscr{I}_{H_2} \,.\, {}_{H_1}\mathscr{I}$.
4) $a = c \neq b$. We have $(a, b) \in \mathscr{I}_{H_2}$, and since $(b, b) \in \mathscr{I}_{H_1}$, we conclude $(a, b) \in \mathscr{I}_{H_2} \,.\, {}_{H_1}\mathscr{I}$.
5) $a \neq c$, $b \neq c$. Then there exist $h_1 \in H_1$, $h_2 \in H_2$ such that $a = h_1c$, $b = ch_2$. Since $\mathscr{I}_{H_2}$ is a left congruence, we have $(h_1c, h_1b) \in \mathscr{I}_{H_2}$, i. e. $(a, h_1b) \in \mathscr{I}_{H_2}$. Analogously $(ah_2, ch_2) \in {}_{H_1}\mathscr{I}$. This implies $(h_1ch_2, ch_2) = (h_1b, b) \in {}_{H_1}\mathscr{I}$ hence $(a, b) \in \mathscr{I}_{H_2} \,.\, {}_{H_1}\mathscr{I}$. This completes our proof.

**Theorem 4,4.** *The relation* ${}_{H_1}\mathscr{D}_{H_2}$ *is an equivalence relation.*

Proof. The reflexivity of ${}_{H_1}\mathscr{D}_{H_2}$ follows from the reflexivity of ${}_{H_1}\mathscr{I}$ and $\mathscr{I}_{H_2}$.

The symmetry follows from Lemma 4,1. The transitivity follows from Lemma 4,1 and from the transitivity of ${}_{H_1}\mathscr{I}$ and $\mathscr{I}_{H_2}$.

Denote by ${}_{H_1}\mathscr{H}_{H_2}$ the equivalence relation ${}_{H_1}\mathscr{I} \cap \mathscr{I}_{H_2}$.

It is easy to prove the following

**Theorem 4,5.** *The following inclusions hold:*

$$ {}_{H_1}\mathscr{H}_{H_2} \subset {}_{H_1}\mathscr{I} \cup \mathscr{I}_{H_2} \subset {}_{H_1}\mathscr{D}_{H_2} \subset {}_{H_1}\mathscr{I}_{H_2} $$

Notation: Let $T \subset S$. The equivalence induced on $T$ by the equivalence ${}_{H_1}\mathscr{H}_{H_2}$, ${}_{H_1}\mathscr{D}_{H_2}$, and ${}_{H_1}\mathscr{I}_{H_2}$ respectively will be denoted by ${}_{H_1}\mathscr{H}^T_{H_2}$, ${}_{H_1}\mathscr{D}^T_{H_2}$, and ${}_{H_1}\mathscr{I}^T_{H_2}$ respectively. Denote further ${}_{H_1}F_{H_2} \cap T = {}_{H_1}F^T_{H_2}$.

**Definition 4,6.** *Let $T \subset S$. Then we shall say that the subset $T$ of $S$ is ${}_{H_1}\mathscr{I}_{H_2}$-simple if $T$ consists exactly of one class ${}_{H_1}F^T_{H_2}$. Similarly one may define the ${}_{H_1}\mathscr{H}_{H_2}$-simplicity and ${}_{H_1}\mathscr{D}_{H_2}$-simplicity of a subset $T$ of $S$.*

**Theorem 4,6.** *An ${}_{H_1}\mathscr{I}_{H_2}$-simple subset $T$ of $S$ does not contain any proper $(H_1, H_2)$-ideal of $S$.*

Proof. Let $N \subsetneqq T$, $N \in I(H_1, H_2)$. Then there exists $b \in T$, $b \notin N$. Let $a \in N$. Then $H_1 a \subset H_1 N \subset N$, $aH_2 \subset NH_2 \subset N$, $H_1 a H_2 \subset H_1 N H_2 \subset N$. Therefore ${}_{H_1}(a)_{H_2} \subset N$. Definition 4,6 implies ${}_{H_1}(a)_{H_2} = {}_{H_1}(b)_{H_2}$, hence $b \in N$, contrary to the supposition.

Evidently an ${}_{H_1}\mathscr{I}_{H_2}$-simple subset $T$ of $S$ is an $(H_1, H_2)$-ideal of $S$ if and only if $T$ is a minimal $(H_1, H_2)$-ideal of $S$.

**Theorem 4,7.** *Every minimal $(H_1, H_2)$-ideal of $S$ coincides with some class ${}_{H_1}F_{H_2}$.*

Proof. If $N \in I_m(H_1, H_2)$, then ${}_{H_1}(a)_{H_2} = {}_{H_1}(b)_{H_2}$ for every $a, b \in N$, and thus all elements of $N$ are contained in the same class. Further, if $c$ is any element contained in that class ${}_{H_1}F_{H_2}$ which contains $N$, then by Theorem 4,6 $c \in N$.

Remark. Under the assumptions of Theorem 3,1 the sets $\mathfrak{S}(H_1, H_2)$ are subsemigroups of $S$, which are disjoint unions of classes ${}_{H_1}F_{H_2}$.

**Theorem 4,8.** *A semigroup $S$ is ${}_{H_1}\mathscr{I}_{H_2}$-simple if and only if $S$ does not contain any proper $(H_1, H_2)$-ideal of $S$.*

Proof. Since $S \in I(H_1, H_2)$, the statement follows from Theorem 4,6 and Theorem 4,7.

Remark. The notion of the ${}_{H_1}\mathscr{I}_{H_2}$-simple semigroup $S$ coincides in the case of $H_1 = S$, $H_2 = \emptyset$ ($H_1 = \emptyset, H_2 = S$; $H_1 = S$ and $H_2 = S$) with the known notion of a left simple (right simple; simple) semigroup $S$.

But if $H_1 \neq S$, $H_2 \neq S$, it is not true that any set $T \subset S$ not containing any proper $(H_1, H_2)$-ideal of $S$ is necessarily ${}_{H_1}\mathscr{I}_{H_2}$-simple, even in the case when $T$ is a subsemigroup of $S$. We can see it on Example 1,9 if we choose $H_1 = \{a, e\}$, $H_2 = \{c, d\}$. The subsemigroup $T = \{a, c\}$ does not contain any proper $(H_1, H_2)$-ideal of $S$, but the elements $a, c$ generate principal $(H_1, H_2)$-ideals, which do not coincide.

## 5

In this section we shall use the notions defined in the previous sections for the theory of groups and completely simple semigroups without zero. The results obtained will complete some results of [2].

In [2] it was already remarked that a group $G$ does not contain any proper $(G, \emptyset)$-ideal, $(\emptyset, G)$-ideal and $(G, G)$-ideal of $G$ but important subsets of a group, cosets, e. g. are relative ideals of $G$.

In the following ${}_H(a)_\emptyset$, ${}_\emptyset(a)_H$ will be briefly denoted by ${}_H(a)$, $(a)_H$.

From Definition 2,1, Theorem 2,4 and Theorem 4,8 there follows

**Theorem 5,1.** *Let $G$ be a group, $H$ a subgroup. Then for every $a \in G$ the set $Ha$ is a minimal $(H, \emptyset)$-ideal, the set $aH$ is a minimal $(\emptyset, H)$-ideal and the set $HaH$ is a minimal $(H, H)$-ideal of $G$. Moreover for every $a \in G$ we have* $Ha = {}_HF^a = {}_H(a)$, $aH = F^a_H = (a)_H$, $HaH = {}_HF^a_H = {}_H(a)_H$.

Denote the right congruence on a group $G$ corresponding to the right coset decomposition of $H$ by $\mathscr{K}^R$, and the analogous left congruence by $\mathscr{K}^L$. Then the following theorem holds:

**Theorem 5,2.** *Let $H$ be a subgroup of a group $G$. Then* $\mathscr{K}^R = {}_H\mathscr{I}$, $\mathscr{K}^L = \mathscr{I}_H$.

Proof. Let us for $a, b \in S$ have $(a, b) \in {}_H\mathscr{I}$, i. e. $Ha = Hb$. Then $ab^{-1} \in H$, i. e. $(a, b) \in \mathscr{K}^R$. Analogously for $\mathscr{K}^L$. Conversely, if $ab^{-1} \in H$, then $Ha = Hab^{-1}b = Hb$, hence $(a, b) \in {}_H\mathscr{I}$. Analogously for $\mathscr{K}^L$.

**Theorem 5,3.** *Let $G$ be a group, $H$ a subgroup of $G$. Then $H$ is a normal subgroup of $G$ if and only if every minimal $(H, \emptyset)$-ideal is a minimal $(\emptyset, H)$-ideal of $G$ and conversely every minimal $(\emptyset, H)$-ideal is a minimal $(H, \emptyset)$-ideal of $G$.*

Proof. If $H$ is a normal subgroup of $G$, then for every $a \in G$ we have $Ha = aH = HaH$. By Theorem 2,4 for every $N \in I_m(H, \emptyset)$ we have $N \in I_m(\emptyset, H)$ (also $N \in I_m(H, H)$) and conversely. Let $a$ be any element of $G$. By the supposition and Theorem 2,4, if $N = Ha$, $Ha \in I_m(H, \emptyset)$, then for some $b \in G$ we have $Ha = bH$. Hence it follows that $Ha = HbH$, $HaH = bH^2 = bH = Ha$. Consider a minimal $(\emptyset, H)$-ideal $aH$. Then for some $c \in G$ we have $aH = Hc$ and analogously $aH = HaH$. Therefore $aH = Ha$ for every $a \in G$.

Remark. In accord with Theorem 5,1 one can state the preceding Theorem as follows: $H$ is a normal subgroup of $G$ if and only if the principal relative ideals ${}_H(a)$, $(a)_H$, ${}_H(a)_H$ coincide for every $a \in G$. It further follows from the foregoing Theorem that ${}_H\mathscr{I} = \mathscr{I}_H = {}_H\mathscr{I}_H$ if and only if $H$ is a normal subgroup of $G$.

Let $H_1, H_2$ be subgroups of a group $G$. By Theorem 2,4 the sets $H_1aH_2$ are minimal $(H_1, H_2)$-ideals of $G$ for every $a \in G$. Moreover these sets are principal $(H_1, H_2)$-ideals of $G$ generated by $a$. Therefore the known decomposition $G = H_1H_2 \cup H_1h'H_2 \cup \dots$ is a decomposition of $G$ into minimal $(H_1, H_2)$-ideals of $G$.

**Theorem 5,4.** *Let $H_1, H_2$ be subgroups of a group $G$. Then every minimal $(H_1, \emptyset)$-ideal is a minimal $(\emptyset, H_2)$-ideal of $G$ and conversely every minimal $(\emptyset, H_2)$-ideal is a minimal $(H_1, \emptyset)$-ideal of $G$ if and only if $H_1 = H_2 = H$, and $H$ is a normal subgroup of $G$.*

Proof. It is sufficient to prove the necessity of the condition. Let $a$ be any element of $G$. It follows from the supposition that $H_1a = bH_2$ for some $b \in G$. This implies $H_1a = H_1bH_2 = H_1aH_2$. For the same element $a \in G$ we also have $aH_2 = H_1aH_2$. This implies $H_1[\bigcup_{a\in H_1} a] = H_1[\bigcup_{a\in H_1} a]H_2$, i. e. $H_1 = H_1H_2$. Analogously we get $H_2 = H_1H_2$. Hence $H_1 = H_2$. Moreover $H_1a = aH_2$ for every $a \in G$. Hence $H_1 = H_2 = H$ is a normal subgroup of $G$.

The following results will complete to a certain extent the results of section 4 of [2] concerning completely simple semigroups without zero. Some results have been found by Š. Schwarz in [6] without the use of the notion of a relative ideal of a semigroup.

We shall use the following theorem proved in [6]:

Let $S$ be a completely simple semigroup without zero. This is in our terminology a ${}_S\mathscr{I}_S$-simple semigroup containing at least one minimal $(S, \emptyset)$-ideal and at least one minimal $(\emptyset, S)$-ideal of $S$. It is known that: $S = \bigcup\{R_\alpha : R_\alpha \in I_m(\emptyset, S)\} = \bigcup\{L_\beta : L_\beta \in I_m(S, \emptyset)\} = \bigcup\{G_{\alpha\beta} : G_{\alpha\beta} = R_\alpha \cap L_\beta\}$, $G_{\alpha\beta}$ are disjoint maximal isomorphic groups. Let $H$ be a subsemigroup of $S$, which is ${}_H\mathscr{I}_H$-simple and contains at least one idempotent. Then 1) $H = \bigcup\{R'_\alpha : R'_\alpha = R_\alpha \cap H\} = \bigcup \{L'_\beta : L'_\beta = L_\beta \cap H\} = \bigcup \{G'_{\alpha\beta} : G'_{\alpha\beta} = R'_\alpha \cap L'_\beta\}$, $R'_\alpha \in I_m(\emptyset, H)$, $L'_\beta \in I_m(H, \emptyset)$. 2) The set $\bar{H} = \bigcup\{R_\alpha : R_\alpha \cap H \neq \emptyset\} = \bigcup\{L_\beta : L_\beta \cap H\} \neq \emptyset\} = \bigcup\{G_{\alpha\beta} = R_\alpha \cap L_\beta : R_\alpha \cap H \neq \emptyset,\ L_\beta \cap H \neq \emptyset\}$ is a maximal subsemigroup of $S$ containing the same idempotents as $H$.

In [2] we have proved

**Theorem 5,5.** *Let $S$ be a completely simple semigroup without zero, $H$ a subsemigroup of $S$ containing an idempotent. Then*:

$$\mathfrak{S}(H, \emptyset) = \bigcup\{R_\alpha : R_\alpha \in I_m(\emptyset, S), R_\alpha \cap H \neq \emptyset\},$$

$$\mathfrak{S}(\emptyset, H) = \bigcup\{L_\beta : L_\beta \in I_m(S, \emptyset), L_\beta \cap H \neq \emptyset\}.$$

**Corollary.** *If $H$ contains all idempotents of $S$, then* $\mathfrak{S}(H, \emptyset) = \bigcup\{R_\alpha : R_\alpha \in I_m(\emptyset, S)\} = \bigcup\{L_\beta : L_\beta \in I_m(S, \emptyset)\} = \mathfrak{S}(\emptyset, H)$. *Further* $\mathfrak{S}(H, \emptyset) = HS$, $\mathfrak{S}(\emptyset, H) = SH$, *hence in this case* $S = \bigcup\{Ha : a \in S\} = \bigcup\{aH : a \in S\}$ *holds.*

In the following $S$ means a completely simple semigroup without zero.

**Theorem 5,6.** *Suppose that an ${}_H\mathscr{I}_H$-simple subsemigroup of $S$ contains all idempotents of $S$. Then the set $Ha$ is for every $a \in S$ a minimal $(H, \emptyset)$-ideal and the set $aH$ a minimal $(\emptyset, H)$-ideal of $S$. Also $Ha = {}_HF^a$, $Ha = {}_H(a)$, and $aH = F^a_H$, $aH = (a)_H$.*

Proof. Let $h \in H$. Then $h \in L'_\beta$ and $Hh = L'_\beta$. If $e_\beta$ is an idempotent, $e_\beta \in L_\beta$, $L_\beta \cap H = L'_\beta$, then by the supposition $e_\beta \in H$ and therefore $e \in L'_\beta$. Also $He_\beta = L'_\beta$. Let $s \in S$. Denote the unit element of the group containing $s$ by $e_\sigma$. Then we have $Hs = He_\sigma s = L'_\sigma s$. By Theorem 2,4 we have $L'_\sigma s \in I_m(H, \emptyset)$ for every $s \in S$. Analogously $sH \in I_m(\emptyset, H)$ for every $s \in S$. The last part of the statement follows from Theorem 4,7 and from the fact that $s \in Hs$ and $s \in sH$, for every $s \in S$.

**Corollary.** *Under the assumptions of the preceding Theorem for any $a, b \in S$ we have either $Ha \cap Hb = \emptyset$ or $Ha = Hb$. Also $aH \cap bH = \emptyset$ or $aH = bH$.*

**Theorem 5,7.** *Let $H$ be an ${}_H\mathscr{I}_H$-simple subsemigroup of $S$. Let $H$ contain at least one idempotent. Then the two-sided socle $\mathfrak{S}(H, H)$ is the maximal subsemigroup of $S$ containing the same idempotents as $H$.*

The proof has been given in [2].

**Corollary.** *If $H$ contains all idempotents of $S$, then $\mathfrak{S}(H, H) = HSH = S$ and hence $S = \bigcup\{HaH : a \in S\}$.*

**Theorem 5,8.** *Suppose that under the suppositions of the preceding Theorem $H$ contains all idempotents of $S$. Then the set $HaH$ is for every $a \in S$ a minimal $(H, H)$-ideal of $S$. Moreover $HaH = {}_HF^a_H$, $HaH = {}_H(a)_H$.*

Proof. It follows from the proof of Theorem 5,6 that for every $s \in S$ we have $Hs = L'_\sigma s$, $L'_\sigma \subset H$, $L'_\sigma \in I_m(H, \emptyset)$ and $L'_\sigma$ contains an idempotent $e_\sigma$ which is the unit element for $s$. By the assumption $e_\sigma \in H$. By the analogy with the proof of the same Theorem concerning $(\emptyset, H)$-ideals of $S$ we get $e_\sigma H = R'_{\sigma_0}$, where $R'_{\sigma_0} \in I_m(\emptyset, H)$, $R'_{\sigma_0} \subset H$, and $R'_{\sigma_0}$ contains the idempotent $e_\sigma$. This implies $Hse_\sigma H = HsH = L'_\sigma se_\sigma R'_{\sigma_0} = L'_\sigma sR'_{\sigma_0}$. By Theorem 2,4 we have $L'_\sigma sR'_{\sigma_0} \in I_m(H_1, H_2)$ for every $s \in S$ and for every $L'_\sigma$, $R'_{\sigma_0}$.

**Corollary.** *Under the assumptions of Theorem 5,6 in the decomposition $S \; \bigcup\{HaH : a \in S\}$ we have either $HaH = HbH$ or $HaH \cap HbH = \emptyset$.*

**Theorem 5,9.** *Let $H$ be an ${}_H\mathscr{I}_H$-simple subsemigroup of $S$ containing at least one idempotent. Then $S = \bigcup\{Ha : a \in S\} = \bigcup\{aH : a \in S\} = \bigcup\{HaH : a \in S\}$ if and only if $H$ contains all idempotents of $S$.*

Proof. It is sufficient to prove the necessity of the condition. Evidently, if $S = \bigcup\{Ha : a \in S\} = HS$, and $S = \bigcup\{aH : a \in S\} = SH$, then we have $S = HSH = \bigcup\{HaH : a \in S\}$. The end of the proof follows from Theorem 14 of [2].

Remark. If $H$ does not contain all idempotents of $S$, then $S = \bigcup\{Ha : a \in S\}$ and $S = \{aH : a \in S\}$ cannot hold. However in this case it may be either $S = \bigcup\{Ha : a \in S\}$, or $S = \bigcup\{aH : a \in S\}$. This is shown on Example 1,9 if we choose $H = \{a, c\}$.

**Theorem 5,10.** *Let $H_1$ be an ${}_{H_1}\mathscr{I}_{H_1}$-simple subsemigroup of $S$, $H_2$ a ${}_{H_2}\mathscr{I}_{H_2}$-simple subsemigroup of $S$, and suppose that each of these subsemigroups contains at least one idempotent. Then*

$$\mathfrak{S}(H_1, H_2) = R \cap L,\ R = \bigcup\{R_\alpha : R_\alpha \in I_m(\emptyset, S),\ R_\alpha \cap H_1 \neq \emptyset\},$$
$$L = \bigcup\{L_\beta : L_\alpha \in I_m(S, \emptyset),\ L_\beta \cap H_2 \neq \emptyset\}.$$

Proof. It follows from Theorem 3,1 and 5,5 that $\mathfrak{S}(H_1, H_2) = H_1SH_2 = H_1S \cap SH_2$ and $H_1S$, $SH_2$ have the properties mentioned in our Theorem.

Remark. If $H_1$ and $H_2$ contain all idempotents of $S$, then by Theorem 5,9 we have $H_1S = SH_2 = S$ and $H_1SH_2 = S = \bigcup\{H_1aH_2 : a \in S\}$. Analogously as in Theorem 5,8 it can be proved that the set $H_1aH_2$ is for every $a \in S$ a minimal $(H_1, H_2)$-ideal of $S$. Moreover $H_1aH_2 = {}_{H_1}F^a_{H_2}$, and $H_1aH_2 = {}_{H_1}(a)_{H_2}$. Hence the sets in the decomposition of $S$ considered above are either disjoint or coincide.

It is, of course, possible that there exists a decomposition of $S$ into disjoint summands: $S = \bigcup\{H_1aH_2 : a \in S\}$ where $H_1$ and $H_2$ do not contain all idempotents of $S$. This can be shown on Example 1,9 if we choose $H_1 = \{a, c\}$, $H_2 = \{a, e\}$.

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*Katedra matematiky a deskriptívnej geometrie*
*Elektrotechnickej fakulty*
*Slovenskej vysokej školy technickej*
*Bratislava*