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ON DECOMPOSITION OF A TREE INTO THE MINIMAL NUMBER OF PATHS

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Throughout the paper we mean by a graph a non-oriented finite graph. The object of our investigation will be trees i. e. connected graphs not containing any circle (= Kreis [1]). Our considerations are necessarily based on some known lemmas, that is why we shall have to mention them (without proofs).

Lemma 1. ([1], p. 49.) *Any tree with at least one edge contains at least two vertices the first degree.*

Lemma 2. ([1], p. 21.) *The number of vertices of odd degree is in each graph even.*

Lemma 3. ([1], p. 22.) *Let G be any connected graph and let $2n$ (where $n > 0$) be the number of its vertices of odd degree, then there exists a decomposition of the graph G into n open moves* (move = Kantenzug [1]) *and any decomposition of the graph G into open moves contains at least n moves.*

Apart from the above, we have:

Lemma 4. *Let G be any tree, then there does not exist in G any closed move with at least one edge and every open move in G is a path* (= Weg [1]).

Proof. The validity of the lemma is evident from the fact that any tree does not contain a circle.

Lemma 5. *Let G be any tree and let $2n$ $(n > 0)$ be the number of its vertices of odd degree, then G may be decomposed into n paths and any decomposition of the graph G into paths contains at least n paths.*

Proof. From Lemma 1 it follows that $n \geqq 1$. Hence it follows from Lemma 3 that G may be decomposed into n open moves and from Lemma 4 it follows that each such open move is a path. From the above the validity of the first assertion of the lemma is evident. From Lemma 3 and from the fact that $n > 0$ it follows that each decomposition of the graph G into paths contains at least n paths. The proof lemma is accomplished.

Let us now put the following question: How many different decompositions of the given tree into the minimal number of paths do there exist? The following theorem solves the problem:

Theorem. *Let G be any tree with at least one edge and let $d(i)$ be the number of its vertices of i-th degree. Let $2n$ be the number of the vertices from G that are of odd degree* (i. e. $2n = d(1) + d(3) + \dots$) *and let r be the number of different decompositions of G into n paths, then we have*

$$r = \prod_{i=1}^{\infty} g(i)^{d(i)},$$

where for every natural i we put $g(2i-1) = g(2i) = 1 . 3 . 5 . \dots . (2i-1)$.

Proof. Let $V = \{v_1, v_2, \dots, v_m\}$ be the set of vertices of G. By H_i we denote the set of all edges from G incident at v_i ($i = 1, 2, \dots, m$). Let R_i be any decomposition of the set H_i with following property: if $|H_i| \equiv 0 \pmod 2$, then each class from R_i has exactly two elements and if $|H_i| \equiv 1 \pmod 2$, then one of the classes of R_i contains an only edge (it will be called the significant edge with respect to R_i; if H_i contains an odd number of edges then none of its edges is significant to R_i) and the other classes of the decomposition contain two elements each.

With regard to the system $\bar{R} = \{R_1, R_2, \dots, R_m\}$ of decompositions with the above property the following evidently holds: each vertex and only vertex of odd degree v_j is incident at such an edge and only at one such an edge that is significant with respect to $R_j \in \bar{R}$.

Let us travel along the elements of G according to the following rules:

(1) If in a travel we arrive along an edge f at is end (= vertex v_x), then we proceed along that edge which with f forms a 2-element class of $R_x \in \bar{R}$. If, however, the edge f is significant with respect to R_x, we finish our travelling in v_x.

(2) We start each of our travels in a vertex v_u of odd degree along such an edge from H_u that is significant with respect to $R_u \in \bar{R}$.

It is evident that the elements covered at any of these travels form a path of G, whereby the starting (as well as the final) vertex is a vertex of the odd degree.

Any edge from G belongs evidently to one of the n paths describing all such travels (if, of course, we do not take into consideration in which of the two possible directions we travel).

Hence: To each system $\bar{R} = \{R_1, R_2, \dots, R_m\}$ of decompositions with the required property there corresponds (uniquely) a decomposition of the graph G into n paths. To the different systems there correspond different decompositions of G into n paths. It follows that r is equal to the number of different systems $\bar{R}$ with the required property. Then the validity of the theorem becomes evident from the fact that $g(s)$ is the number of the different decompositions of the set of all edges incident at the given vertex of s-th degree with the required property as well as from the fact that the decomposition of such a set may be chosen for the individual vertices quite independently.

REFERENCE

[1] König D., *Theorie der endlichen und unendlichen Graphen*, Leipzig 1936.

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