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ABSTRACT FORMULATION OF THE INDIVIDUAL ERGODIC THEOREM

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Let $(X, \mathscr{S}, m)$ be a measure space, $\mathscr{S}$ be a σ-algebra, m be a σ-finite measure on $\mathscr{S}$ and $T : X \to X$ be a measure preserving transformation. For $T^n = T \circ T \circ \ldots \circ T$, the composition T with itself n times, and $T^\circ(x) = x$ the individual ergodic theorem asserts: If f is integrable, then $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$ exists almost everywhere, the limit function f^* is integrable and invariant under T. If $m(X) < +\infty$, then $\int f^* dm = \int f dm$.

In this paper we generalize the individual ergodic theorem. Instead of L_1 — the system of m-integrable functions, we shall consider some system $\mathscr{P}$ of functions which are defined on the measurable space $(X, \mathscr{S})$. The system $\mathscr{P}$ satisfies some system of axioms. The present paper consists of two parts. In the first part we shall construct the "theory of integration" i. e. with the help of axioms we shall prove theorems analogous to Beppo-Levi's theorem, Lebesque's theorem, etc. In the second part we shall formulate and prove the „individual ergodic theorem". Similar axiomatic methods were used in papers [3], [4], [5].

1.

Let $(X,\mathscr{S})$ be a measurable space.
Let $\mathscr{R} \subset \mathscr{S}$ be such a ring that $A \in \mathscr{R}$, $B \in \mathscr{S} \Rightarrow A \cap B \in \mathscr{R}$.
Let $\mathscr{C} \subset \mathscr{R}$ be such a σ-ring that $A \in \mathscr{C}$, $B \in \mathscr{S} \Rightarrow A \cap B \in \mathscr{C}$.
Let us denote $\mathscr{P}^\circ = \{f; \{x \in X; f(x) \neq 0\} \in \mathscr{C}\}$,

$$\mathscr{P}^j = \{f; f = \sum_{k=1}^{n} \alpha_k \chi_{A_k}, A_k \in \mathscr{R}, n \text{ is an integer}, |\alpha_k| < \infty\}$$

$\mathscr{P}^1 = \bigcup_{s\in E} \mathscr{N}_s$, where E is some non-empty set, $\{\mathscr{N}_s\}$ are non-empty classes of non-negative measurable functions, which satisfy

1. $\{f \in \mathscr{P}^j; f \geq 0\} \subset \mathscr{P}^1$.

2. $f \in \mathscr{N}_s$ if and only if there exists a non-decreasing sequence $\{f_n\}$ of non-negative functions belonging to $\mathscr{P}^j$ such that for every n, $f_n \in \mathscr{N}_s$ and $f = \lim_{n\to\infty} f_n$.

3. If $f, g \in \mathscr{P}^j \cap \mathscr{P}^1$ and $\{x \in X; f(x) < g(x)\} \in \mathscr{C}$, then $f \in \mathscr{N}_s \Rightarrow g \in \mathscr{N}_s$.

Let us denote $\mathscr{P} = \{f; f = h - k\ h, k \in \mathscr{P}^1, h - k$ is defined$\}$.

The ring $\mathscr{R}$, the σ-ring $\mathscr{C}$, the class $\mathscr{P}^j$ and the class $\mathscr{N}_s$ are abstract analogues of all sets of finite measure, of all sets of measure zero, of integrable simple functions, of non-negative measurable functions for which is $\int f dm \leq s$, respectively.

In the next considerations we shall use this notation: $\{f_n\} \subset \mathscr{N}_s$ if $f_n \in \mathscr{N}_s$ for every n,

$f_n \nearrow f$ if $\{f_n\}$ is a non-decreasing sequence of functions which satisfies $f = \lim_{n\to\infty} f_n$,

$N(f) = \{x \in X; f(x) \neq 0\}$.

It is easy to prove that if α, β are real numbers and $f, g \in \mathscr{P}^j$, then $f \cdot g \in \mathscr{P}^j$, $(\alpha f + \beta g) \in \mathscr{P}^j$. This statement is used in the next lemma.

Lemma 1. *Let $\{g_n\}$ be a non-decreasing sequence of non-negative functions belonging to $\mathscr{P}^j$. If f is a simple function and $0 \leq f \leq \lim_{n\to\infty} g_n$, then $\{g_n\} \subset \mathscr{N}_s \Rightarrow f \in \mathscr{N}_s$.*

Proof. Let us denote $f_n = \min(f, g_n)$. Since f_n is a simple function and $N(f_n) \subset N(g_n)$, where $N(g_n) \in \mathscr{R}$, we have $f_n \in \mathscr{P}^j$. Let $\{g_n\} \subset \mathscr{N}_s$. Then both Axiom 3 and $f_n = \min(f, g_n) \leq g_n$ yield $\{f_n\} \subset \mathscr{N}_s$. Since $0 \leq f_n \nearrow f$, we have $f \in \mathscr{N}_s$.

Lemma 2. *Let $\{f_n\}$, $\{g_n\}$ be a non-decreasing sequences of non-negative functions belonging to $\mathscr{P}^j$.*

(i) *If $f_n \nearrow f$, $g_n \nearrow g$ and $f \leq g$, then $\{g_n\} \subset \mathscr{N}_s \Rightarrow \{f_n\} \subset \mathscr{N}s$;*
(ii) *If $f_n \nearrow f$, $g_n \nearrow f$, then $\{g_n\} \subset \mathscr{N}_s \Leftrightarrow \{f_n\} \subset \mathscr{N}_s$;*
(iii) *If $f_n \nearrow f$, then $f \in \mathscr{N}_s \Leftrightarrow \{f_n\} \subset \mathscr{N}_s$.*

Proof. (i) If m is an arbitrary fixed integer, then $0 \leq f_m \leq g = \lim_{n\to\infty} g_n$ and by using Lemma 1 we have $\{g_n\} \subset \mathscr{N}_s \Rightarrow f_m \in \mathscr{N}_s$. Since m is arbitrary, $\{g_n\} \subset \mathscr{N}_s \Rightarrow \{f_n\} \subset \mathscr{N}_s$.

(ii) If $0 \leq f_n \nearrow f$, $0 \leq g_n \nearrow f$ is valid, by (i) we have $\{g_n\} \subset \mathscr{N}_s \Rightarrow \{f_n\} \subset \mathscr{N}_s$ and also $\{f_n\} \subset \mathscr{N}_s \Rightarrow \{g_n\} \subset \mathscr{N}_s$.

(iii) If $\{f_n\} \subset \mathscr{N}_s$, then from the definition of $\mathscr{P}^1$ obviously $f \in \mathscr{N}_s$. Let

$f \in \mathcal{N}_s$, i. e. there is a sequence $\{g_n\}$ of functions belonging to $\mathcal{P}^j$ such that $\{g_n\} \subset \mathcal{N}_s$ and $0 \leq g_n \nearrow f$, but then by (ii) the relation $\{f_n\} \subset \mathcal{N}_s$ holds.

Lemma 3. *Let f, $g \in \mathcal{P}^1$. If $f \leq g$, then $g \in \mathcal{N}_s \Rightarrow f \in \mathcal{N}_s$.*

Proof. Since f, $g \in \mathcal{P}^1$, there are sequences $\{f_n\}$, $\{g_n\}$ of functions belonging to $\mathcal{P}^j$ such that $0 \leq f_n \nearrow f$, $0 \leq g_n \nearrow g$.

Let $g \in \mathcal{N}_s$. By (iii) of Lemma 2 $\{g_n\} \subset \mathcal{N}_s$. Since $f \leq g$, by (i) of Lemma 2 $\{f_n\} \subset \mathcal{N}_s$ and it implies $f \in \mathcal{N}_s$.

Theorem 1. *If $f \in \mathcal{P}$ and f is non-negative, then $f \in \mathcal{P}^1$.*

Proof. $f = h - k$, h, $k \in \mathcal{P}^1$ and $h \geq k$.

From the non-negativity of the measurable function f it follows that there is a sequence $\{f_n\}$ of simple functions for which $0 \leq f_n \nearrow f$. Since $h \in \mathcal{P}^1$, there is a sequence $\{h_n\}$ of functions belonging to $\mathcal{P}^j$, $0 \leq h_n \nearrow h$. Let us denote $f_n^* = \min(h_n, f_n)$, i. e. f_n^* is a non-negative and belongs to $\mathcal{P}^j$. Since $0 \leq f_n^* \nearrow \min(h, f) = f$ and $f_n^* \leq h_n$, we have $\{h_n\} \subset \mathcal{N}_s \Rightarrow \{f_n^*\} \subset \mathcal{N}_s$. This means that $f \in \mathcal{N}_s$ and thus $f \in \mathcal{P}^1$.

Theorem 2. *If $0 \leq g \leq f$, where $f \in \mathcal{P}^1$ and g is measurable, then $g \in \mathcal{P}^1$.*

Proof. Since $f \in \mathcal{P}^1$ there is a sequence $\{f_n\}$ of functions belonging to $\mathcal{P}^j$, $0 \leq f_n \nearrow f$ and $\{f_n\} \subset \mathcal{N}_s$ for some $s \in E$.

Since $g \geq 0$ is measurable, there is a sequence $\{g_n\}$ of simple functions such that $0 \leq g_n \nearrow g$. If we denote $g_n^* = \min(f_n, g_n)$, then g_n^* is simple. The inclusion $N(g_n^*) \subset N(f_n)$ implies $g_n^* \in \mathcal{P}^j$. The last relation together with $0 \leq g_n^* \leq f_n$ give $\{g_n^*\} \subset \mathcal{N}_s$.

Theorem 3. *If f is a measurable function, then*

(i) $f \in \mathcal{P} \Leftrightarrow f^+, f^- \in \mathcal{P}$;

(ii) $|f| \in \mathcal{P} \Rightarrow f^+, f^- \in \mathcal{P}$.

Proof. (i) If $f^+, f^- \in \mathcal{P}$, then $f^+, f^- \in \mathcal{P}^1$ by Theorem 1 and therefore $f = (f^+ - f^-) \in \mathcal{P}$. Let $f \in \mathcal{P}$, i.e. $f = h - k$, where h, $k \in \mathcal{P}^1$. If the number $f(x) \geqslant 0$, then $f^+(x) = h(x) - k(x) \leq h(x)$ and it follows that $0 \leq f^+ \leq h$. Since f^+ is measurable, by Theorem 2 $f^+ \in \mathcal{P}^1 \subset \mathcal{P}$. Since also $-f \in \mathcal{P}$ and $(-f)^+ = f^-$, we have $f \in \mathcal{P} \Rightarrow f^+, f^- \in \mathcal{P}$.

(ii) $|f| \in \mathcal{P} \Rightarrow |f| \in \mathcal{P}^1$ and since f^+, f^- are measurable and $0 \leq f^+ \leq |f|$, $0 \leq f^- \leq |f|$, we have $f^+, f^- \in \mathcal{P}^1$.

If f is measurable, then $B \in \mathcal{S}$ implies $f^+ . \chi_B$ and $f^- . \chi_B$ are measurable. Moreover the obvious inequalities $0 \leq f^+ . \chi_B \leq f^+$ and $0 \leq f^- . \chi_B \leq f^-$ hold. Therefore using Theorems 2 and 3 we can assert.

Theorem 4. *If $f \in \mathcal{P}$ and $B \in \mathcal{S}$, then $f . \chi_B \in \mathcal{P}$.*

Theorem 5. *If f, $g \in \mathcal{P}$ and $f \leq h \leq g$ where h is measurable, then $h \in \mathcal{P}$.*

Proof. Since $f \leq h \leq g$, we have $0 \leq h^+ \leq g^+$, $0 \leq h^- \leq f^-$. Since g^+, $f^- \in \mathscr{P}^1$ and h^+, h^- are measurable, it follows from Theorem 2 that $h^+, h^- \in \mathscr{P}^1$, hence $h \in \mathscr{P}$ by Theorem 3.

Theorem 6. *Let $f \in \mathscr{P}$ and g be a measurable function. If $\{x \in X; f(x) \neq g(x)\} \in \in \mathscr{C}$, then $g \in \mathscr{P}$ and moreover $f \in \mathscr{M}_{s,t} = \{f \in \mathscr{P};\ f^+ \in \mathscr{N}_s,\ f^- \in \mathscr{N}_t\}$ if and only if $g \in \mathscr{M}_{s,t}$.*

Proof. Since

$\{x \in X;\ f(x) \neq g(x)\} = \{x \in X;\ f^+(x) \neq g^+(x)\} \cup \{x \in X;\ f^-(x) \neq g^-(x)\}$, it is sufficient to prove: If $f \in \mathscr{N}_s$ and $g \geqslant 0$ is measurable, then $\{x \in X;\ f(x) \neq \neq g(x)\} \in \mathscr{C}$ implies that $g \in \mathscr{P}^1$ and moreover $f \in \mathscr{N}_s \Leftrightarrow g \in \mathscr{N}_s$. Let $f \in \mathscr{N}_s$. The set $A = \{x \in X;\ f(x) \geqslant g(x) > 0\}$ is measurable. From Theorem 2 and Lemma 3 it is obvious that $g\,.\,\chi_A \in \mathscr{N}_s$, i.e. there is a sequence $\{g_n\}$ of functions belonging to $\mathscr{P}^j \cap \mathscr{P}^1$ such that $\{g_n\} \subset \mathscr{N}_s$ and $0 \leq g_n \nearrow g\,.\,\chi_A$. Since $B = = \{x \in X;\ 0 < f(x) < g(x)\} \in \mathscr{C} \subset \mathscr{R}$, the function $g\,.\,\chi_B$ is measurable and there is a sequence $\{h_n\}$ of functions which are elements of $\mathscr{P}^\circ \cap \mathscr{P}^j$ such that $0 \leq h_n \nearrow g\,.\,\chi_B$. If we denote $g_n^* = g_n + h_n$, then $g_n^* \in \mathscr{P}^j \cap \mathscr{P}^1$. Since $\{x \in X;\ g_n^*(x) > g_n(x)\}$ belongs to $\mathscr{C}$, we have $\{g_n^*\} \subset \mathscr{N}_s$ and for $0 \leq g_n^* \nearrow g\,.\,.\,(\chi_A + \chi_B) = g$ we have $g \in \mathscr{N}_s$. The proof of the converse implication is analogous.

Lemma. *Let $a = \lim_{n\to\infty} a_n$, $a_n = \lim_{m\to\infty} a_{nm}$, where $\{a_n\}_{n=1}^\infty$, $\{a_{nm}\}_{m=1}^\infty$ are non--decreasing sequences of real numbers. If we denote $b_k = \max(a_{1k}, a_{2k}, \ldots, a_{kk})$, then $b_k \nearrow a$.*

Theorem 7. *Let $\{f_n\}$ be a non-decreasing sequence of functions belonging to $\mathscr{P}$ such that $f_n \nearrow f$. If $\{f_n\} \subset \mathscr{M}_{s,t} = \{f \in \mathscr{P};\ f^+ \in \mathscr{N}_s,\ f^- \in \mathscr{N}_t\}$, then $f \in \mathscr{P}$ and $f \in \mathscr{M}_{s,t}$.*

Proof. I. Let $\{f_n\}$ be a non-decreasing sequence of functions belonging to $\mathscr{P}^1$. For each $n = 1, 2, \ldots$ let $\{f_{nm}\}_{m=1}^\infty$ be a non-decreasing sequence of functions belonging to $\mathscr{P}^j$ for which $0 \leq f_{nm} \nearrow f_n$. If we put $g_k = = \max(f_{1k}, f_{2k}, \ldots, f_{kk})$, then $\{g_n\}$ is a non-decreasing sequence of non-negative simple functions (maximum of two simple functions is a simple function). Since $g_k = \max(f_{1k}, \ldots, f_{kk}) \leq \max(f_1, \ldots, f_k) = f_k$, we have $0 \leq g_k \leq f_k$ and therefore $\{f_n\} \subset \mathscr{N}_s \Rightarrow \{g_n\} \subset \mathscr{N}_s$. As $\{f_n\} \subset \mathscr{N}_s$ and by the lemma $0 \leq g_n \nearrow f$, we have $f \in \mathscr{N}_s$.

II. Let $\{f_n\}$ be a non-decreasing sequence of measurable functions belonging to $\mathscr{P}$. Then $0 \leq f_n^+ \nearrow f^+$, $f_n^- \searrow f^- \geqslant 0$. Since $\{f_n^+\} \subset \mathscr{N}_s$, then by I $f^+ \in \mathscr{N}_s$. Since a limit of a sequence of measurable functions is measurable, f^- is a measurable function. Since $0 \leq f^- \leq f_1^-$, where $f_1^- \in \mathscr{N}_t$, using Theorem 2 and Lemma 3 we obtain $f^- \in \mathscr{N}_t$, i.e. $f = (f^+ - f^-) \in \mathscr{P}$ and $f \in \mathscr{M}_{s,t}$.

Since $f_n \searrow f$ implies $-f_n \nearrow -f$ and $\{f_n\} \subset \mathscr{M}_{s,t}$ implies $\{-f_n\} \subset \mathscr{M}_{t,s}$, by means of Theorem 7 we obtain.

Theorem 8. *Let $\{f_n\}$ be a sequence of functions belonging to $\mathscr{P}$ and $f_n \searrow f$. If $\{f_n\} \subset \mathscr{M}_{s,t}$, then $f \in \mathscr{P}$ and $f \in \mathscr{M}_{s,t}$.*

Theorem 9. *Let $\{f_n\}$ be a sequence of functions belonging to $\mathscr{P}$ and $f = \lim_{n\to\infty} f_n$. If $\{f_n\} \subset \mathscr{M}_{s,t}$, then $f \in \mathscr{P}$ and $f \in \mathscr{M}_{s,t}$.*

Proof. Let us denote $g_k = \inf_{n\geq k} f_n^+$, $h_k = \inf_{n\geq k} f_n^-$. Since $\{g_n\}$ is a non-decreasing sequence of measurable functions and $\lim_{k\to\infty} g_k = \sup_{k\geq 1} (\inf_{n\geq k} f_n^+) = \lim_{k\to\infty} \inf f_k^+$, we have $g_k \nearrow f^+$. From $0 \leq g_k \leq f_k^+$ follows that $\{g_k\} \subset \mathscr{N}_s$, i.e. $f^+ \in \mathscr{N}_s$ (by means of Theorem 7). Similarly for the non-decreasing sequence $\{h_k\}_{k=1}^{\infty}$ we have $\{h_k\} \subset \mathscr{N}_t$, $0 \leq h_k \nearrow f^-$ and therefore $f^- \in \mathscr{N}_t$.

Theorem 10. *Let $\{f_n\}$ be a sequence of functions belonging to $\mathscr{P}$ and $f = \lim_{n\to\infty} f_n$. If there is $g \in \mathscr{P}^1$ such that $|f_n| \leq g$ for each n, then $f \in \mathscr{P}$.*

Proof. Since $0 \leq f_n^+ \leq g$, $0 \leq f_n^- \leq g$, where $g \in \mathscr{N}_s$ for some $s \in E$, it follows from Lemma 3 that $\{f_n\} \subset \mathscr{M}_{s,s}$, hence $f = \lim_{n\to\infty} f_n$ belongs to $\mathscr{P}$ by Theorem 9.

2.

Let us denote $\mathscr{P}^+ = \{f \in \mathscr{P};\ f^+ \in \mathscr{N}_s \Rightarrow f^- \in \mathscr{N}_s\}$. In this part we shall suppose that

4. If f, $g \in \mathscr{P}^+$ are such that the sum $f + g$ is defined, then $(f + g) \in \mathscr{P}^+$.

If in the axioms of part 1 we take instead of E the set of all non-negative rational numbers, we see that $\mathscr{P}^+$ substitutes for the class of integrable functions for which $\int f dm \geq 0$.

Theorem 11. *If, $g \in \mathscr{P}$ and $(\alpha f + \beta g)$ where α, β are real numbers is defined, then $(\alpha f + \beta g) \in \mathscr{P}$.*

Proof. Clearly (by means of Theorem 3 (i)) it is sufficient to consider $\alpha, \beta \geq 0$ and f, $g \in \mathscr{P}^1$. If $f \in \mathscr{P}^1$, then by Axiom 4 and by the induction $nf \in \mathscr{P}^1$ for any non-negative integer n, hence the non-negative function $\alpha f \leq ([\alpha] + 1) . f$ belongs to P^1 by Theorem 2. If f, $g \in \mathscr{P}^1$ and α, β are non-negative real numbers, then αf, βg belong to $\mathscr{P}^1 \subset \mathscr{P}^+$. Thus, $(\alpha f + \beta g) \in \mathscr{P}^+ \subset \mathscr{P}$ and $(\alpha f + \beta g)$ belongs to $\mathscr{P}^1$ by Theorem 1.

Now we put the definition of the m-leader.

Let us suppose that $a_1, \ldots, a_n$ is a finite sequence of real numbers and that m is a positive integer. A term a_k of the sequence will be called an m-leader,

if there exists a positive integer p such that $1 \leq p \leq m$ and such **that** $\sum_{j=0} a_{k+j} \geq$
≥ 0.

The following lemma holds.

Lemma. *The sum of the m-leaders of a finite sequence of real numbers is non-negative.* (For proof cf. [1].).

Definition. *We shall denote* $T \in \mathscr{Z}$ *if and only if* $T : X \to X$, *T is measurable and satisfies*:

(i) *If* $f, g \in \mathscr{P}$ *are such that the sum* $f + g$ *is defined, then* $(f + g(T)) \in \mathscr{P}^+ \Rightarrow$
$\Rightarrow (f + g) \in \mathscr{P}^+$;

(ii) $f \in \mathscr{P} \Rightarrow f(T) \in \mathscr{P}$.

Now we want to formulate the "maximal ergodic theorem".

Theorem I. *Let* $\mathscr{S}$ *be a* σ*-algebra and let* $f \in \mathscr{P}$ *be such that* $x \in X \Rightarrow f(x) \neq -\infty$.

Let us denote by A the set $\{x \in X;\ \exists n : \sum_{j=0}^{n-1} f(T^j x) \geq 0\}$. *If* $T \in \mathscr{Z}$ *and* $M \in \mathscr{S}$ *is such that* $T^{-1}M = M$, *then* $f \cdot \chi_{A \cap M} \in \mathscr{P}^+$.

Proof. Let us denote

$$f_j(x) = f(T^j x),\ A_j = \{x \in X;\ \exists p\ 1 \leq p \leq j : f_\circ(x) + f_1(x) \ldots + f_{p-1}(x) \geq 0\}.$$

Evidently $A_j \in \mathscr{S}$, $A_j \subset A_{j+1}$ and $A = \bigcup_{j=1}^{\infty} A_j$.

Let $n \geq 1$ be an arbitrary fixed integer. Let us take the sequence $f_0, f_1, \ldots, f_{n-1}$ and let us denote
$A(x) = \{m;\ f_m(x)$ is the n-leader of the sequence $f_0(x), \ldots, f_{n-1}(x)\}$,
$N_j = \{x \in X;\ j \in A(x)\}$.
Since the sum of the m-leaders of a finite sequence is non-negative,

$$0 \leq \sum_{j \in A(x)} f_j(x) = \sum_{j=0}^{n-1} f_j(x) \cdot \chi_{N_j}(x).$$

Since
$$\begin{aligned} T^{-j}A_{n-j} &= T^{-j}\{x \in X;\ \exists p\ 1 \leq p \leq n - j : f_0(x) + \ldots + f_{p-1}(x) \geq 0\} = \\ &= \{x \in X;\ \exists p\ 1 \leq p \leq n - j : f_j(x) + \ldots + f_{j+p-1}(x) \geq 0\} = \\ &= \{x \in X;\ f_j(x) \text{ is } n\text{-leader of } f_0(x), \ldots, f_{n-1}(x)\} = \\ &= \{x \in X;\ j \in A(x)\} = N_j \\ A_n &= N_0 \end{aligned}$$
we have

$$0 \leq \sum_{j=0}^{n-1} f_j(x) \cdot \chi_{T^{-j}A_{n-j}}(x) = \sum_{j=0}^{n-1} f_j(x) \cdot \chi_{A_{n-j}}(T^j x).$$

The equality $T^{-j}M = M$ implies $\chi_M(T^j x) = \chi_M(x)$ and this gives

$$0 \leq \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \, . \, \chi_M(T^j x) \, . \, \chi_{A_{n-j}}(T^j x) .$$

Since $f \, . \, \chi_M \, . \, \chi_{A_{n-j}} \in \mathscr{P}$ (see Theorem 4), we have that functions $f(T^j) \, . \, \chi_M(T^j) \, . \, \chi_{A_{n-j}}(T^j)$ belong to $\mathscr{P}$. Using Theorem 11 and with respect to the nonnegativity of the last sum we obtain

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j) \, . \, \chi_M(T^j) \, . \, \chi_{A_{n-j}}(T^j) \in \mathscr{P}^+.$$

By the asumptions concerning T we have $f\chi_M \frac{1}{n} \sum_{j=1}^{n} \chi_{A_j} \in \mathscr{P}^+$. Let $x \in X$. If $n_0 \geq 1$ is the smallest integer such that $x \in A_{n_0}$, then for $n > n_0$ the equalities $n = n_0 + m$ and $\frac{1}{n} \sum_{j=1}^{n} \chi_{A_j}(x) = \frac{m+1}{n_0 + m}$ hold. Hence if $n \to \infty$ (i.e. if $m \to \infty$), we have $\frac{1}{n} \sum_{j-1}^{n} \chi_{A_j} \nearrow \chi_A$ and therefore

$$0 \leq f^+ \, . \, \chi_M \frac{1}{n} \sum_{j=1}^{n} \chi_{A_j} \nearrow f^+ \, . \, \chi_{A \cap M}$$

$$0 \leq f^- \, . \, \chi_M \frac{1}{n} \sum_{j=1}^{n} \chi_{A_j} \nearrow f^- \, . \, \chi_{A \cap M} \, .$$

By Theorem 4 $f \, . \, \chi_{A \cap M} \in \mathscr{P}$. Now we want to show that $f \, . \, \chi_{A \cap M}$ belongs to P^+. If $f^+ \, . \, \chi_{A \cap M} \in \mathscr{N}_s$, then by means of Theorem 2 and Lemma 3 $f^+ \, . \, \chi_M \times \times \frac{1}{n} \sum_{j=1}^{n} \chi_{A_j} \in \mathscr{N}_s$ for each $n = 1, 2, \ldots$. Since for each $n > 1 f \, . \, \chi_M \frac{1}{n} \sum_{j=1}^{n} \chi_{Aj} \in$

$\in \mathscr{P}^+$ it follows that $f^- . \chi_M \frac{1}{n} \sum_{j=1}^{n} \chi_{A_j} \in \mathscr{N}_s$ for each $n > 1$. By Theorem 7 $f^- \cdot \chi_{A \cap M} \in \mathscr{N}_s$ and thus $f\chi_{A\cap M} \in \mathscr{P}^+$.

Lemma. *If $f_n \searrow f$ and $\{f_n\}$ are functions belonging to $\mathscr{P}^+$, then $f \in \mathscr{P}$.*

Proof. $f_n^+ - f_n^- \leq f_1^+ - f_1^-$ implies $f_n^+ \leq f_1^+$ and therefore $f_1^+ \in \mathscr{N}_s \Rightarrow f_n^+ \in \mathscr{N}_s$. Since $\{f_n\} \subset \mathscr{P}^+$, it follows that $\{f_n\} \subset \mathscr{M}_{s,s}$ for some $s \in E$. Hence by Theorem 8 we have $f \in \mathscr{P}$.

If the following axiom is true,

5. If $f \in \mathscr{N}_s$ for each $s \in E$, then $f \in \mathscr{P}^0$,

we can prove

Lemma. *If $T \in \mathscr{Z}$, then $T^{-1}(\mathscr{C}) \subset \mathscr{C}$.*

Proof. Let $A \in \mathscr{C}$ and $B = T^{-1}A$. Then $-\chi_B(x) + \chi_A(Tx) = -\chi_B(x) + \chi T^{-1}A(x) = 0 \in \mathscr{P}^1$, i.e. $-\chi_B + \chi_A(T) \in \mathscr{P}^+$. Both $-\chi_B + \chi_A(T) \in \mathscr{P}^+$ and $\chi_A \in \mathscr{P}$ imply $(\chi_A - \chi_B) \in \mathscr{P}^+$. Since $(\chi_A - \chi_B)^+ = \chi_A - \chi_{A\cap B}$ we have $(\chi_A - \chi_{A\cap B}) \in \mathscr{N}_s \Rightarrow (\chi_B - \chi_{A\cap B}) \in \mathscr{N}_s$. Since $A \cap B \in \mathscr{C}$, by means of Theorem 6 we have $\chi_A \in \mathscr{N}_s \Rightarrow \chi_B \in \mathscr{N}_s$. Since $\{x \in X; \chi_A(x) \neq \chi_\emptyset(x) = 0\} \in \mathscr{C}$ and $f \in \mathscr{N}_s \Rightarrow 0 = \chi_\emptyset \leq f$, we obtain $\chi_A \in \mathscr{N}_s$ for each $s \in E$. This means $\chi_B \in \mathscr{N}_s$ for each $s \in E$ and by Axiom 5, $B \in \mathscr{C}$.

Theorem II. *Let $\mathscr{S}$ be such a σ-algebra that $X = \bigcup_{n-1}^{\infty} A_n$, where $A_n \in \mathscr{R}$ and let $f \in \mathscr{P}$.*

(i) *Let $T \in \mathscr{Z}$. If $\{x \in X;\ f(x) = -\infty\} \in \mathscr{C}$, then $\lim_{n\to\infty} 1/n \sum_{j=0}^{n-1} f(T^j x)$ exists outside a set belonging to $\mathscr{C}$. If $\{x \in X;\ |f(x)| = \infty\} \in \mathscr{C}$, then the limit function f^* is invariant under T* (i.e. $f^*(Tx) = f^*(x)$ *outside a set belonging to* $\mathscr{C}$);

(ii) *If $T : X \to X$ is such a measurable transformation that g_1, g_2 are non-negative, $(g_1 + g_2) \in \mathscr{N}_s \Rightarrow (g_1 + g_2(T)) \in \mathscr{N}_s$ and if the limit function f^* exists outside a set belonging to $\mathscr{C}$, then $f^* \in \mathscr{P}$.*

Proof. (i) Let $\{x \in X; f(x) = -\infty\} = \emptyset$.
Suppose $\alpha < \beta$ are rational numbers and $B = B(\alpha, \beta)$ is the set of points x for which

$$\liminf_{n\geq 1} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) < \alpha < \beta < \limsup_{n\geq 1} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x).$$

We may assume without loss of generality that $\beta > 0$, for otherwise the argument can be carried out with f replaced by $-f$.
Let $C \subset B(\alpha, \beta)$ and $C \in \mathscr{R}$.

$$\text{Let us denote } A = \{x \in X;\ \exists n: \sum_{j=0}^{n-1} (f(T^j x) - \beta \cdot \chi_C(T^j x)) \geq 0\}.$$

Since $(f - \beta \cdot \chi_C) \in \mathscr{P}$ (because of $C \in \mathscr{R}$), by means of Theorem I we have $(f - \beta \cdot \chi_C) \cdot \chi_A \in \mathscr{P}^+$. If $x \in B$, then there is n such that $\frac{1}{n} \sum_{j=0}^{n-1} f_j(x) > \beta$.

Since β is non-negative, we have

$$\sum_{j=0}^{n-1} (fT^j x) - \beta \cdot \chi_C(T^j x) \geq \sum_{j=0}^{n-1} f(T^j x) - \beta > 0.$$

This means that $B(\alpha, \beta) \subset A$ and therefore $(f \cdot \chi_A - \beta \cdot \chi_C) \in \mathscr{P}^+$. If $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of sets belonging to $\mathscr{R}$ for which $\bigcup_{n=1}^{\infty} C_n = B$, then $(f \cdot \chi_A - \beta \cdot \chi_{C_n}) \searrow (f \cdot \chi_A - \beta \cdot \chi_B)$. The functions $\{(f \cdot \chi_A - \beta \cdot \chi_{C_n})\}$ belong to $\mathscr{P}^+$ and therefore $(f \cdot \chi_A - \beta \cdot \chi_B) \in \mathscr{P}$. If we denote $A_o = f^{-1}(\{+\infty\})$, then $\chi_{B-A_o} = \beta^{-1} \cdot (f\chi_A - (f\chi_A - \beta\chi_B))\chi_{X-A_o}$ belongs to $\mathscr{P}$ by Theorem 11. Since $0 \leq \chi_{B \cap A_o} \leq f\chi_{A_o}$, it follows from Theorem 2 that $\chi_{B \cap A_o} \in \mathscr{P}$ and therefore χ_B, $(f - \beta)\chi_B$, $(\alpha - f)\chi_B$ belong to $\mathscr{P}$.

If $x \in B$, then $\sum_{j=0}^{n-1}(f(T^j x) - \beta)\chi_B(T^j x) = \sum_{j=0}^{n-1}(f(T^j x) - \beta) > 0$ for some positive integer n, hence $(f - \beta)\chi_B \in \mathscr{P}^+$. If $x \in B$, then $\sum_{j=0}^{n-1}(\alpha - f(T^j x)) > 0$ for some positive integer n, hence $(\alpha - f) \cdot \chi_B \in \mathscr{P}^+$ by $T^{-1}B = B$ and Theorem I. Thus, $(\alpha - \beta)\chi_B = (\alpha - f) \cdot \chi_B + (f - \beta) \cdot \chi_B$ belongs to $\mathscr{P}^+$ by Axiom 4 and $A_o \cap B = \emptyset$.
We have $\chi_\emptyset \leq f$ for any $f \geq 0$ and therefore $0 \in \mathscr{N}_s$ for each $s \in E$. Since $((\alpha - \beta) \cdot \chi_B)^- = (\beta - \alpha) \cdot \chi_B$, by the definition of $\mathscr{P}^+$ we obtain $(\beta - \alpha) \cdot \chi_B \in \in \mathscr{N}_s$ for each $s \in E$ i.e. $(\beta - \alpha) \cdot \chi_B \in \mathscr{P}^\circ$. The last relation implies $B \in \mathscr{C}$. Hence

$$\left\{x \in X;\ \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \text{ does not exist}\right\} = \bigcup_{(\alpha,\beta) \text{ rational}} B(\alpha, \beta) \in \mathscr{C}.$$

If $D = \{x \in X;\ f(x) = -\infty\} \in \mathscr{C}$, then $T^{-n}D \in \mathscr{C}$ for each $n = 0, 1, \ldots$. For $g = f\,.\,(1 - \chi_D)$ the inclusion

$$\{x \in X;\ g^*(x) \neq f^*(x)\} \subset \bigcup_{n=0}^{\infty} (\{x \in X; f(T^n x) \neq g(T^n x)\} = \bigcup_{n=0}^{\infty} T^{-n}D$$

holds, i.e. $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$ exists outside a set belonging to $\mathscr{C}$. Let $f : X \to R$.

Since $|f(x)| < \infty$, it is easy to prove that $f^*(x) = f^*(Tx)$ outside a set belonging to $\mathscr{C}$. It is a trivial consequence of the elementary properties of Cesaro's convergence. Let $D = \{x \in X;\ |f(x)| = \infty\} \in \mathscr{C}$ and $g = f\,.\,(1 - \chi_D)$. Clearly $f^* = g^*$ outside a set belonging to $\mathscr{C}$ and therefore f^* is invariant under T.

(ii) The limit of a sequence of measurable functions is a measurable function and therefore f^* is measurable (we can put $f^*(x) = 0$ if the limit f^* does not exist in x).

Since $f \in \mathscr{P}$, for some $s \in E$ we have $|f| \in \mathscr{N}_s$, whence it follows that $\frac{n}{n}\,.\,|f| = \frac{1}{n}\,.\,(|f| + \ldots + |f|) \in \mathscr{N}_s$ and by the assumption concerning T

$$\frac{1}{n}\,.\,(|f| + |f(T)| + \ldots + |f(T^{n-1})|) \in \mathscr{N}_s \text{ for each } n = 1, 2, \ldots.$$

Since $\left|\frac{1}{n} \sum_{j=0}^{n-1} f(T^j)\right| \leq \frac{1}{n} \sum_{j=0}^{n-1} |f(T^j)|$, by means of Theorem 9 $|f^*| \in \mathscr{N}_s$ and thus $f^* \in \mathscr{P}$.

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