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INTERSECTION GRAPHS OF LATTICES

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The intersection graph of an algebra A is by definition the graph whose vertices are proper subalgebras of A and in which two vertices are joined by an edge if and only if the corresponding subalgebras have a non-empty intersection.

Intersection graphs of semigroups were studied mainly by J. Bosák [2]. He also suggested to study intersection graphs of other algebras, including lattices. The latter is the subject of this paper.

Let a lattice L be given. A sublattice of L can be defined in two different ways; therefore we shall distinguish algebraic sublattices and set-theoretical ones.

An algebraic sublattice of L is by definition a non-empty subset of L which is closed with respect to the operations of join and meet (i. e. with any two elements it contains also their join and meet).

A set-theoretical sublattice of L is by definition a nonempty subset of L which is a lattice with respect to the ordering induced by the ordering of L.

It can be easily proved that every algebraic sublattice of a lattice L is simultaneously its set-theoretical sublattice. The inverse assertion is not true, as shown in Fig. 1, where the Hasse diagram of the lattice L is drawn, whose elements 0, a, b, I form a set-theoretical sublattice given by the Hasse diagram in Fig. 2. This is evidently a lattice, but this lattice is no algebraic sublattice of L, because it contains the elements a and b, but not the element c which is the join of these elements in L.

Thus we shall distinguish algebraic intersection graphs of lattices and set-theoretical ones. We shall introduce even the third type of intersection graphs of lattices, namely the interval intersection graphs. If $a \leqq b$ in a lattice L, then the interval $\langle a, b\rangle$ is the set of all elements $x \in L$ for which $a \leqq x \leqq b$ holds. The interval $\langle a, b\rangle$ is evidently an algebraic sublattice of L; the inverse assertion is not true, as shown in Fig. 3. Here $\{0, a, b, I\}$ is an algebraic sublattice, but not an interval. The interval intersection graph of the lattice L is by definition the graph whose vertices are intervals $\langle a, b\rangle$ for all pairs a, b of the elements of L for which $a \leqq b$ holds and if 0 and I exist, then either $a \neq 0$, or $b \neq I$ (here and in the following 0 denotes the least, I the greatest element of L) and in which two elements are joined by an edge if and only if the corresponding intervals have a non-empty intersection.

The algebraic intersection graph of the lattice L will be denoted by $GA(L)$, the set-theoretical intersection graph by $GS(L)$, the interval intersection graph by $GI(L)$. The symbols $\vee$ and $\wedge$ denote the join and the meet in L, the symbols $\cup$ and $\cap$ denote the set-theoretical operations of union and intersection.

From the above given definitions it follows that each one-element subset of a lattice L is its algebraic sublattice, set-theoretical sublattice and interval.

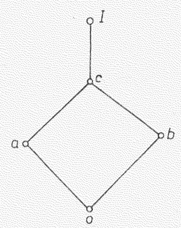

Fig. 1.

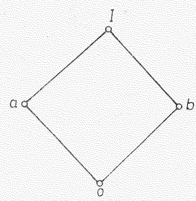

Fig. 2.

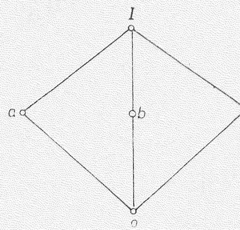

Fig. 3.

Theorem 1. *The system of one-element subsets of a finite lattice L with more than one element is a maximal internally stable* [1] *set in any of the graphs* $GA(L)$, $GS(L)$, $GI(L)$, *while any other internally stable set in any of these graphs has a less number of vertices.*

Proof. Two distinct one-element sets are disjoint, therefore the system of one-element subsets of the lattice L is an internally stable set in $GA(L)$, $GS(L)$ and $GI(L)$. Its number of elements is equal to the number of elements of L. Assume that there exists some other subset of the vertex set of some of these graphs which is internally stable and has the cardinality greater than or equal to the cardinality of L. But there does not exist in any set a system of pairwise disjoint non-empty subsets of the cardinality greater than the cardinality of this set; the system with these properties and of the cardinality equal to the cardinality of the original set is exactly one in a finite set; this is the system of all one-element subsets, which is a contradiction.

Corollary. *The internal stability numbers of the graphs* $GA(L)$, $GS(L)$, $GI(L)$ *for a finite lattice L with more than one element are pairwise equal and are equal to the cardinality of L.*

Theorem 2. *Let the set-theoretical intersection graph* $GS(L)$ *of a finite lattice L with more than one element be given. Then the set of elements of L and the relation of comparability on it can be reconstructed.*

Proof. In the graph $GS(L)$ we find the internally stable set of the greatest cardinality; according to Theorem 1 this is the set of one-element sublattices of the lattice L and therefore its elements correspond in a one-to-one manner to the elements of L. A two-element subset $\{a, b\}$ of L (where $a \neq b$) is a set-theoretical sublattice of the lattice L if and only if the elements a, b are comparable, i. e. if either $a < b$, or $b < a$ holds. To such a sublattice a vertex of $GS(L)$ corresponds which is joined by edges with vertices corresponding to sublattices $\{a\}$, $\{b\}$ and is not joined with any other vertex corresponding to a one-element sublattice. Thus we recognize for any two elements a, b of the lattice L, whether such a vertex exists, and so we reconstruct the relation of comparability on the set of elements of L.

Theorem 3. *Let the algebraic intersection graph $GA(L)$ of a finite lattice L with more than one element be given. Then the set of elements of L and the relation of comparability on it can be reconstructed.*

Proof. A two-element subset $\{a, b\}$ is also an algebraic sublattice of the lattice L if and only if the elements a, b are comparable. Therefore we can proceed in the same way as in the proof of Theorem 2.

Theorem 4. *Let the algebraic intersection graph $GA(L)$ of a finite lattice L with more than one element be given. Then the set of elements of L can be reconstructed and for any two elements a, b of L the set $\{a \wedge b, a \vee b\}$ can be reconstructed.*

Remark. The reconstruction of the set $\{a \wedge b, a \vee b\}$ means finding two elements, one of which is $a \wedge b$, the other is $a \vee b$, but such that in general it is not possible to determine, which of them is $a \wedge b$ and which is $a \vee b$.

Proof. The reconstruction of the set of elements of the lattice L will be performed so as in the proof of Theorem 2 and also the relation of comparability will be determined. If two elements a, b are comparable, then $\{a \wedge b, a \vee b\} = \{a, b\}$. If they are non-comparable, then there exists an algebraic sublattice consisting of the elements $a, b, a \wedge b, a \vee b$ (which are pairwise distinct) and no other four-element algebraic sublattice containing a and b. Thus in the graph $GA(L)$ we find a vertex which is joined with vertices $\{a\}$ and $\{b\}$ and moreover with exactly two further vertices corresponding to one-element sublattices. These two further vertices correspond to the one-element sublattices $\{a \wedge b\}$, $\{a \vee b\}$.

Theorem 5. *Let the interval intersection graph $GI(L)$ of a finite lattice L with more than two elements be given. Then the (undirected) Hasse diagram of L can be reconstructed.*

Remark. Here we speak only about the Hasse diagram as an undirected graph. If this diagram has to determine uniquely the lattice L, it must be drawn in a certain position, which cannot be performed with help of this theorem.

Proof. In the Hasse diagram of the lattice L two elements of L are joined by an (undirected) edge if and only if either a covers b, or b covers a. (We say that a covers b, if $a > b$ and there does not exist any element c such that $a > c > b$.) This is realized if and only if there exists an interval of the lattice L consisting only of the elements a, b. In the graph $GI(L)$ there corresponds to such an interval a vertex joined by edges with one-element intervals $\{a\} = \langle a, a\rangle$, $\{b\} = \langle b, b\rangle$ and not joined with any other one-element interval. (The set of vertices corresponding to one-element intervals can be found similarly as the set of vertices corresponding to one-element sublattices in the proof of Theorem 2.) Thus in the Hasse diagram of the lattice L the elements a, b will be joined by an edge if and only if such a vertex exists in $GI(L)$.

Lemma. *Let a be an element of a finite lattice L with more than two elements. The vertex corresponding to the element a in the Hasse diagram of L is a cut-vertex of this diagram if and only if $a \neq O$, $a \neq I$ and the element a is comparable with all elements of L.*

Proof. Let a be a cut-vertex of the Hasse diagram of the lattice L and let b, c be vertices of this diagram separated by the vertex a (i. e. each path from b to c contains the vertex a). If b, c are non-comparable, there is $b \wedge c \neq b \vee c$. Let C_1 (or C_2 respectively) be the path from b (or from c respectively) to $b \vee c$ corresponding to the saturated chain between these elements. Both these paths have only the vertex $b \vee c$ in common; otherwise there would exist a vertex d so that $b \leqq d < b \vee c$, $c \leqq d < b \vee c$, which is impossible. By C denote the union of these paths. Analogously let C_1' (or C_2' respectively) be the path from $b \wedge c$ to b (or to c respectively) corresponding to the saturated chain between these elements; these two paths have no common vertex either except for $b \wedge c$. Let C' be their union. Assume that the paths C and C' have a common vertex d different from b and c. If d is a common vertex of the paths C_1 and C_1', this means that simultaneously $d < b$ and $d > b$, which is impossible; analogously if d is a common vertex of the paths C_2 and C_2'. If d is a common vertex of the paths C_1 and C_2', then $d > b$, $d < c$ and therefore $b < c$, which is also impossible; analogously if d is a common vertex of C_1' and C_2. Therefore C and C' have no common vertices except for b and c, thus according to Menger's Theorem the connectivity degree of the vertices b, c is at least two and these vertices cannot be separated by a cut-vertex, which is a contradiction. Hence any two vertices separated by a cut-vertex a are comparable. Without loss of generality let $b < c$. In the Hasse diagram of L

there exist a path corresponding to the saturated chain from b to c; this path contains a and therefore $b < a < c$ and a is comparable with both b and c. As b and c were chosen arbitrarily, a is comparable with all elements of L and evidently it is different from both O and I.

Now let a be comparable with all elements of L and different from both O and I. Let $b < a$, $c > a$ and let a path from b to c exist not containing a. Let d be the last vertex of this path (if we go from b to c) which is less than a. The element d is different from c, therefore there exists a vertex e of this path following after d. As a is comparable with all elements of L, there must be $e > a$. But then $d < a < e$ and e does not cover d and d does not cover e, thus d and e are not joined by an edge, which is a contradiction. Therefore each path from b to c contains a and a is a cut-vertex of the Hasse diagram of L.

Theorem 6. *Let the set-theoretical intersection graph $GS(L)$ of a finite lattice L with more than two elements be given and let the interval intersection graph $GI(L)$ of L be marked in it as its subgraph. Then the lattice L is determined uniquely up to the duality.*

Proof. First, according to Theorem 2 from $GS(L)$, we reconstruct the set of elements of the lattice L and the relation of comparability and according tn Theorem 5 from $GI(L)$ we reconstruct the Hasse diagram of L. If there exist exactly two elements comparable with all other elements of the lattice L, then one of them is O, the other is I. If there are more such elements, then O and I are exactly those of them, for which the corresponding one-element intervals do not form cut-vertices of the Hasse diagram of the lattice L (according to Lemma). Therefore let us choose one of these elements to be O; then the other is I. Further, if two different elements a, b are comparable, then $a < b$ if and only if there exists an interval containing O and a and not containing b. To such an interval in $GI(L)$ a vertex corresponds joined by edges with intervals $\langle O, O\rangle$ and $\langle a, a\rangle$ and not joined with the interval $\langle b, b\rangle$ Therefore if there exists such a vertex, then $a < b$, otherwise $b < a$. The unique random step in the whole procedure was the choice of the element O. In the case of the opposite choice we obtain evidently the lattice dual to that obtained in the preceding case. One of these lattice is evidently L.

Theorem 7. *Let the algebraic intersection graph $GA(L)$ of a finite lattice L with more than two elements be given and let the interval intersection graph $GI(L)$ of L be marked in it as its subgraph. Then the lattice L is determined uniquely up to the duality.*

Proof is analogous to the proof of Theorem 6.

Theorem 8. *The graph $GS(L)$ of the lattice L with more than two elements has* *diameter* 2, *if* L *is not isomorphic to the lattice whose Hasse diagram is in Fig.* 2. *The graph* $GS(L)$ *of the lattice whose Hasse diagram is in Fig.* 2 *has diameter* 3.

Proof. Let a, b be two elements of the lattice L. The distance of the vertices $\{a\}$, $\{b\}$ in the graph $GS(L)$ is equal to two, if and only if there exists a proper sublattice of L (set-theoretical) containing the elements a, b. The least (according to the number of elements) sublattice of the lattice L containing a and b has exactly two elements, if a and b are comparable, and is isomorphic to the lattice whose Hasse diagram is in Fig. 2, if they are non-comparable. If such a sublattice is the whole lattice L, then the distance of $\{a\}$ and $\{b\}$ is greater than 2; if it is a proper sublattice, then this distance is equal to 2. Therefore in each lattice which has more than two elements and is not isomorphic to the lattice whose Hasse diagram is in Fig. 2 the distance of arbitrary two one-element set-theoretical sublattices in $GS(L)$ is equal to 2. Now let us have two arbitrary proper sublattices L_1, L_2 of L. If $L_1 \cap L_2 \neq \emptyset$, then their distance is equal to one. If $L_1 \cap L_2 = \emptyset$, then we choose an element $a \in L_1$ and an element $b \in L_2$. There exists a proper sublattice containing a and b and having therefore a non-empty intersection with both L_1 and L_2. The distance of the sublattices L_1 and L_2 is therefore also equal to two and thus also the diameter of the graph $GS(L)$ is equal to two. The second assertion is trivial.

Theorem 9. *The graph* $GA(L)$ *of a lattice* L *with more than two elements has diameter* 2, *if* L *is not isomorphic to the lattice whose Hasse diagram is in Fig.* 2. *The graph* $GA(L)$ *of the lattice whose Hasse diagram is in Fig.* 2 *has diameter* 3.

Proof is analogous to that of Theorem 8.

Theorem 10. *Let* L *be a lattice with more than two elements, let* $GI(L)$ *be its interval intersection graph. Then the diameter of* $GI(L)$ *is equal to* 3 *if and only if* L *has the least and the greatest element. In the reverse case the diameter of* $GI(L)$ *is equal to* 2.

Remark. In our considerations on diameters we admit also infinite lattices.

Proof. If $GI(L)$ has the least element O and the greatest element I, then the unique interval containing these elements is $\langle O, I\rangle$, which is the whole lattice L. Therefore the distance of the intervals $\langle O, O\rangle$ and $\langle I, I\rangle$ in $GI(L)$ is greater than two. As L contains more than two elements, it contains at least one element a different from both O and I. The intervals $\langle O, a\rangle$, $\langle a, I\rangle$ are intervals of L different from $\langle O, I\rangle$, therefore there exist vertices of $GI(L)$ corresponding to them. The interval $\langle O, a\rangle$ has a non-empty intersection with $\langle O, O\rangle$ and with $\langle a, I\rangle$ and $\langle a, I\rangle$ has a non-empty intersection with $\langle O, a\rangle$ and with $\langle I, I\rangle$. The vertices $\langle O, O\rangle$, $\langle O, a\rangle$, $\langle a, I\rangle$, $\langle I, I\rangle$ form a path of length 3 from $\langle O, O\rangle$ to $\langle I, I\rangle$. Now let a, b be two different elements of the lattice L, from which at least one is different from both O and I. If $a = I$, then there exists a path of length 2 from $\langle a, a\rangle$ to $\langle b, b\rangle$; it contains vertices $\langle a, a\rangle = \langle I, I\rangle$, $\langle b, I\rangle$, $\langle b, b\rangle$. Similarly for $b = O$. If $a \neq O$, $b \neq O$, $a \neq I$, $b \neq I$, there exists a path of length 3 from $\langle a, a\rangle$ to $\langle b, b\rangle$ in $GI(L)$ containing vertices $\langle a, a\rangle$, $\langle O, a\rangle$, $\langle O, b\rangle$, $\langle b, b\rangle$. The distance of arbitrary two one-element intervals in $GI(L)$ is therefore at most three. Now if J_1, J_2 are arbitrary two intervals of L different from L, we choose an element $a \in J_1$ and an element $b \in J_2$, find the corresponding path from $\langle a, a\rangle$ to $\langle b, b\rangle$. If we substitute the vertex $\langle a, a\rangle$ by the vertex J_1 and the vertex $\langle b, b\rangle$ by the vertex J_2 in it, we obtain a path of the same or less length from J_1 to J_2, because any interval having a non-empty intersection with $\langle a, a\rangle$ has a non-empty intersection also with J_1 and any interval having a non-empty intersection with $\langle b, b\rangle$ has a non-empty intersection also with J_2. Therefore L has diameter 3. If L has no least element, there exists to arbitrary two different elements a, b of L a path of length 2 joining the vertices $\langle a, a\rangle$, $\langle b, b\rangle$ in $GI(L)$ and containing the vertices $\langle a, a\rangle$, $\langle a \wedge b, a \vee b\rangle$, $\langle b, b\rangle$; the interval $\langle a \wedge b, a \vee b\rangle$ is not equal to L because L must contain an element less than $a \wedge b$. For arbitrary two intervals J_1 and J_2 we proceed then as in the preceding case. We proceed analogously, if L has no greatest element.

Remark. If the lattice L has only two elements, then evidently the diameters of $GS(L)$, $GA(L)$ and $GI(L)$ are all equal to ∞ (these graphs are disconnected).

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