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Matematický časopis, Vol. 23 (1973), No. 3, 209--215

Persistent URL: http://dml.cz/dmlcz/126888

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ON CERTAIN GENERALIZED PROBABILITY DOMAINS

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The σ-field which is used as the domain of a probability measure is in some considerations concerning quantum mechanics substituted by a system of sets which is supposed to be closed under complementation and under forming countable disjoint unions. Such a collection with a probability measure for which the above mentioned system is a domain is called quantum probability space. The notion of the quantum probability space was introduced in [1] and discussed in [2], [3]. It was also studied in [4] and many other papers from the point of view of partially ordered sets. The present paper in its first part adds some remarks to the above mentioned space. Some different proofs of the results of [2], using a result of [3] are given and a certain extension of them is obtained. On the other hand in the second part of the paper the relation between the formulation in the form of systems of sets, and that in the form of partially oredered sets, is discussed.

1. A quantum probability space is a triple (Ω, C, m), where Ω is a non-empty set, C is a collection of its subsets closed under complementation and under forming countable disjoint unions, m is a probability measure defined on C. The collection C is called a σ-class (see [2]).

The result stated in Lemma 1 is simple and its proof will be omitted. Lemma 2 was proved in [2].

Lemma 1. *Let C be a σ-class of subsets of Ω. Then the following holds: If $a, b \in C$, then $a \cup b \in C$ if and only if $a \cap b \in C$.*

Lemma 2. *If A is a collection of sets and C the σ-class generated by A, then either of the following conditions is necessary and sufficient for C to be a σ-field. (The notion of the σ-field coincides with that of the σ-algebra as defined in* [5])

(i) *If $a, b \in A$, then $a - b \in C$,*

(ii) *If $a, b \in A$, then $a \cap b \in C$.*

Note that the condition that C is closed under countable unions of pairwise dijsoint sets may be substituted by the condition that C is closed under finite unions of pairwise disjoint sets. So we shall introduce the notion of the s-class as follows.

C will be called an s-class if

(s_1) For any $a \in C$ the set $\Omega - a \in C$,

(s_2) $a \cup b \in C$ for any two $a, b \in C$ such that $a \cap b = \emptyset$.

There is no difficulty, without changing the conception of the proof in [3], to prove the corresponding analogy of Lemma 2 for s-classes. One can get the formulation of such an assertion by substituing the notion of the σ-class by that of the s-class and that of the σ-field by one of the field.

An important question in the theory of quantum probability spaces is the question whether for a given collection $\{S_t\}$, $(t \in T)$ of sub-σ-fields of a given σ-class there exists a sub-σ-field S such that $\bigcup_{t \in T} S_t \subset S \subset C$. It is well known that it is not true in general. A necessary and sufficient condition for the case of two sub-σ-fields was given in [2] where also a sufficient condition for the general case was given. In what follows we shall give a necessary and sufficient condition for the general case. S_t will be a more general collection than sub-σ-fields.

The notion of compatibility and that of the C-class will be used as in [2]. Thus if C is a σ-class, then $a, b \in C$ are said to be compatible in C if $a \cap b \in C$. If $A \subset C$, then A is said to be compatible in C if $a \cap b \in C$ for any two a, $b \in A$. A is called internally compatible if $a \cap b \in A$ for any $a, b \in A$. If $A_1 \subset C$, $A_2 \subset C$, then A_1, A_2 are called mutually compatible if $a_1 \cap a_2 \in C$ for any $a_1 \in A_1$, $a_2 \in A_2$. The σ-class C is said to be a c-class if for any a, b, $c \in C$ which are mutually compatible, $a \cap (b \cup c) \in C$ holds. Evidently the above notions may be defined also if C is an s-class.

It seems to be useful to introduce the notion of the n-compatibility, where $n \geqq 1$ is a positive integer. If $A \subset C$, then A will be called n-compatible if $a_1 \cap a_2 \cap \ldots a_n \in C$ for any sequence $a_i \in A$, $(i = 1, 2, \ldots, n)$.
The set $A \subset C$ will said to be conditionally n-compatible in C if for any sequence $\{a_i\}$ $(i = 1, 2, \ldots, n)$ of elements belonging to A and such that a_i $(i = 1, 2, \ldots, n)$ are mutually compatible in C, $a_1 \cap a_2 \cap \ldots \cap a_n \in C$ holds.

Theorem 1. *Let C be a σ-class.*

(i) *C is a c-class if and only if it is conditionally 3-compatible.*

(ii) *If C is a c-class and $A \subset C$ is compatible in C, then A is n-compatible in C for any positive integer n.*

Proof. (i) Let C be a c-class. Let a, b, c be mutually compatible in C. Then $a \cap (b \cup c) \in C$ and $a \cap b \in C$, $a \cap c \in C$. Hence $(a \cap b) \cap (a \cap c) = a \cap b \cap \cap c \in C$, according to Lemma 1. On the other hand let C be conditionally 3-compatible and a, b, c be mutually compatible in C. Then $a \cap b \in C$, $a \cap c \in C$ and $a \cap b \cap c = (a \cap b) \cap (a \cap c) \in C$. Again by Lemma 1 $a \cap (b \cup c) \in C$.

(ii) Let us proceed by induction. For $n = 1$, 2 the theorem holds. Let $n \geqq 2$. Suppose the theorem to be true for n. Let $a_i \in A$ for $i = 1, 2, \ldots, n + 1$. Put $a = a_1 \cap \ldots \cap a_{n-1}$, $b = a_2 \cap \ldots \cap a_{n-1} \cap a_{n+1}$, $c = a_n$. By the assumption $a \in C$, $b \in C$, $c \in C$ and a, b, c are mutually compatible. By (i) $a \cap b \cap c = a_1 \cap a_2 \cap \ldots \cap a_{n+1} \in C$.

Theorem 2. *Let C be a σ-class. Let $\{S_t\}$ $(t \in T)$ be any collection of sets such that $S_t \subset C$ for $t \in T$. A necessary and sufficient condition for the existence of a sub-σ-field S of C such that $\bigcup_{t \in T} S_t \subset S \subset C$ is the n-compatibility of $\bigcup_{t \in T} S_t$ in C for any positive integer n.*

Proof. The necessity is obvious. Let $E = \bigcup_{t \in T} S_t$ be n-compatible in C for any positive integer n. Then the set F of all $a_1 \cap \ldots \cap a_n$, where $a_i \in E$ $(i = 1, 2, \ldots, n)$ and n is any positive integer is evidently internally compatible. Let S be the σ-class generated by F. Then $S \subset C$. By a simple corollary of (ii) of Lemma 2, S is a field.

Note that a suitable modified result for s-classes and fields may be also obtained.

Corollary 1. (Theorem 3.1 in [2].) *Let C be a σ-class. Then to any set A compatible in C there exists a σ-field S such that $A \subset S \subset C$ if and only if C is conditionally 3-compatible.*

Proof. Necessity. Let a, b, c be mutually compatible. Then put $A = \{a, b, c\}$. Under the asumption there exists a σ-field S such that $A \subset S \subset C$. Hence $a \cap b \cap c \in S \subset C$.

Sufficiency. Suppose C to be conditionally 3-compatible. Let A be any compatible set in C. Then it is n-compatible for any positive integer n, according to (i) and (ii) of Theorem 1. Thus the result follows from Theorem 2.

Corollary 2. *Let C be a σ-class and S_t $(t \in T)$ a collection of mutually compatible subsets of C each of which is internally compatible. Then there exists a σ-field $S \subset C$ such that $\bigcup_{t \in T} S_t \subset S$.*

Proof. $E = \bigcup_{t \in T} S_t$ is compatible in C, hence the result follows from Corollary 1 and from (i) of Theorem 1.

Corollary 3. (Theorem 2.5 in [2].) *Let C be a σ-class. Let S_1, S_2 be sub-σ-fields of C. A necessary and sufficient condition for the existence of a σ-field S such that $S_1 \cup S_2 \subset S \subset C$ is the mutual compatibility in C of S_1 and S_2.*

Proof. The necessity is trivial. Evidently $S_1 \cup S_2$ is n-compatible for any positive integer n. Thus the result follows from Theorem 2.

2. Many times instead of the σ-class C a partially ordered set T with the first and last elements 0 and 1 respectively is considered. A reasonable general case (see e.g. [6]) is given by the axioms (i)—(iv) which follow. In what follows a' denotes the uniquely determined complement which is supposed to exist for any $a \in T$. The symbols $x \vee y$, $x \wedge y$ stand for $\sup\{x, y\}$, $\inf\{x, y\}$ respectively, if the mentioned elements exist, while $\sum_i a_i$ stand for the $\sup\{a_1, a_2, \ldots\}$, where a_i $(i = 1, 2, \ldots)$ are supposed to be disjoint. Note that a is said to be disjoint with b if $a \leqq b'$.

(i) $(a')' = a$

(ii) $a \leqq b$ implies $b' \leqq a'$

(iii) $a \vee a' = 1$ for all $a \in T$

(iv) $\sum_i a_i \in T$ for any sequence $\{a_i\}$ of mutually disjoint elements $a_i \in T$.

Note that usually some further axioms are added to obtain physically important results (see [6]). We shall restrict our attention to (i)—(iv), noting that (i)—(iii) might be something which could correspond to an s-class while (i)—(iv) resembles a σ-class.

There are results holding for s-classes which have their analogy in the partially ordered sets fulfilling (i)—(iii). But to transfer all the results without any detailed study would be possible only in the case if T were isomorphic to an s-class. This is not true in general. We shall give a necessary and sufficient condition for T satysfiing (i)—(iii) to be isomoprhic to an s-class of sets. In the case when (i)—(iv) are satisfied also a kind of isomorphy with certain systems will be established.

A partially ordered set T with the first and last elements 0 and 1, respectively, is said to be pseudocomplemented provided that to any $a \in T$ there exists an element $a^* \in T$ with the property

(1) $a \wedge x$ exists in T, $a \wedge x = 0$ is equivalent to the assertion $x \leqq a^*$.

The property (1) implies that the pseudocomplement a^* is uniquely determined for any $a \in T$. The following conditions are satisfied in any partially ordered pseudocomplemented set.

(2) $x \geqq y$ implies $x^* \leqq y^*$

(3) $x \leqq x^{**}$

(4) $x^* = x^{***}$.

A bounded partially ordered set T is said to be uniquely complemented if to any $a \in T$ there exists a uniquely determined element $a' \in T$ such that $a \wedge a'$ and $a \vee a'$ exist in T and $a \wedge a' = 0$, $a \vee a' = 1$ hold. Evidently in a pseudocomplemented and uniquely complemented partially ordered set T, $a^* = a'$ holds.

In what follows the MacNeill — Dedekind cut completion of a partially ordered set T will be used. Let us introduce the basic notions. $I^\triangle$ denotes the set of all upper bounds of I, i.e. $I^\triangle = \{x \in T;\ x \geqq y$ for every $y \in I\}$. Analogically $J^\triangledown = \{x \in T;\ x \leqq y$ for every $y \in J\}$. It is known that the system $L(T)$ of all closed sets, i.e. of all sets with the property $I = I^{\triangle\triangledown}$, is a complete lattice with respect to the set inclusion. $L(T)$ is a completion of the partially ordered set T (see [8]). It can be easily found that if $I_1, I_2 \in L(T)$, then $I_1 \cap I_2 \in L(T)$. Hence the lattice-theoretical meet coincides with the intersection of sets, i. e. $I_1 \wedge I_2 = I_1 \cap I_2$.

Lemma 3. *Let T be a uniquely complemented and pseudocomplemented partially ordered set. Then MacNeill — Dedekind cut completion $L(T)$ is a Boolean algebra.*

Proof. As we know $L(T)$ is a complete lattice with the first element $\{0\}$ and the last element T. First we shall prove that it is a pseudocomplemented lattice. Let for $I, K \in L(T)$, $I \cap K = \{0\}$ hold. Then for $x \in K$ and any $y = I$ there exists $x \wedge y$ in T and $x \wedge y = 0$ hold. The last gives $x \leqq y'$ for any $y \in I$. Denote $I' = \{x \in T;\ x = y'$ and $y \in I\}$. We have $K \subset I'^\triangledown$. It is easy to verify that $I'^\triangledown \in L(T)$. On the other hand if $K \subset I'^\triangledown$ for some $K \in L(T)$, then $x \in I \wedge K$ implies $x \leqq x$ and $x \leqq x'$, hence $x = 0$. Thus $I \cap K = \{0\}$ and $I'^\triangledown$ is a pseudocomplement to I in $L(T)$. $L(T)$ is a pseudocomplemented lattice.

Let us prove that $I = I'^{\triangledown\prime\triangledown}$ holds for any $I \in L(T)$. According what was proved we know that $L(T)$ is a pseudocomplemented lattice, $I \cap I'^\triangledown = \{0\}$ and $(I'^\triangledown)'^\triangledown$ is a pseudocomplement to $I'^\triangledown$. Hence $I \subset I'^{\triangledown\prime\triangledown}$. Since $x \leqq y$ implies $x' \geqq y'$ for any $y \in I$ (see (2)), we have $x' \in I'^\triangledown$ for any $x \in I^\triangle$. Thus $I^{\triangle\prime} \subset I'^\triangledown$. The last gives $I^{\triangle\prime\prime} = I^\triangle \subset I'^{\triangle\prime}$. The relation $I = I^{\triangle\triangledown} \supset I'^{\triangledown\prime\triangledown}$ can be verified immediately. Hence $I = I'^{\triangledown\prime\triangledown}$.

It was proved by now that $L(T)$ is a pseudocomplemented lattice and $x = x^{**}$ for any $x \in L(T)$, i.e. any element is "closed". In view of a known Glivenko's theorem (see [9]) the partially ordered set of closed elements is a Boolean algebra. Thus $L(T)$ is a Boolean algebra.

Lemma 4. (M. H. Stone) *To any distributive lattice L there exists a set M*

and a system L of subsets of M such that L under the operation of the union and intersection of sets is a lattice isomorphic with L.

Using the Lemmas 3 and 4 and taking in to account the fact that the condition (1) is satisfied in any s-class, we have the following.

Theorem 3. *A necessary and sufficient condition for a partially ordered set T satisfying the axioms* (i)—(iii) *given at the beginning of section 2 to be isomorphic with an s-class of sets is that T be pseudocomplemented.*

A natural question is whether a partially ordered set satisfying (i)—(iv) and pseudocomplemented is isomorphic to a σ-class of sets. This need not be true since the Stone theorem (Lemma 4) need not be valid even for so called σ-complete Boolean algebras. It means that for a given σ-complete Boolean algebra B there need not exist a σ-field of sets which is isomoprhic with B in the sense that to the $\sup_B x_n$ where $x_n \in B$ ($n = 1, 2, \ldots$) there corresponds the union of the corresponding images of the elements x_n. But a theorem on isomorphy with certain special σ-classes may be obtained.

Let a σ-field M of subsets of a given set be given. Let I be a σ-ideal in M, i.e. $I \subset M$, I is closed under countable unions and for any $x \in I$, $y \subset x$, $y \in M$ the assertion $y \in I$ holds.

Denote by M/I the collection containing as elements those sets $A \subset M$ for which $x \in A$, $y \in A$ implies $(x - y) \cup (y - x) \in I$. It is known that M/I is a Boolean σ-algebra under the natural definition of sums, intersections and complements. This algebra M/I is called a σ-quotient algebra.

If $C \subset M/I$ and C is closed under forming disjoint countable unions and under complementations in M/I, then C will be called a σ-quotient class.

Sikorski [7] proved the following theorem.

Lemma 5. *Every σ-complete Boolean algebra is isomorphic to a σ-quotient algebra.*

Using Lemma 3, Lemma 5 and remembering that (1) is true for any σ-quotient class, we have.

Theorem 5. *A necessary and sufficient condition for a partially ordered set T satisfying* (i)—(iv) *to be isomorphic to a σ-quotient class is that T be pseudocomplemented.*

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Received April 22, 1970

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